More on classical limit of the model with hol. pol. of z_0, z_1

$V = P_N(z_0, z_1)$, V is a Hilbert space (Hermitean) with the metric

$$\langle \tilde{P}_N, P_N \rangle = \int d^2 z_0 \, d^2 z_1 \, \overline{\tilde{P}_N} P_N \exp(-|z_0|^2 - |z_1|^2)$$

In this metric $\left(z_i \frac{\partial}{\partial z_j}\right)^+ = z_j \frac{\partial}{\partial z_i}$

4 Herm. operators: $\mathcal{E} = z_1 \frac{\partial}{\partial z_1} + z_0 \frac{\partial}{\partial z_0}$ equals to N

$\frac{1}{2}\left(z_1 \frac{\partial}{\partial z_1} - z_0 \frac{\partial}{\partial z_0}\right) = T_3$, $\frac{1}{2}\left(z_1 \frac{\partial}{\partial z_0} + z_0 \frac{\partial}{\partial z_1}\right) = T_1$

$$T_2 = \frac{i}{2}\left(z_1 \frac{\partial}{\partial z_0} - z_0 \frac{\partial}{\partial z_1}\right)$$

In physics they are called spin (or momentum) operators. One can show

$$T_1^2 + T_2^2 + T_3^2 = \frac{N}{2}\left(\frac{N}{2} + 1\right)$$

In physics $j = \frac{N}{2}$

$T_a^{cl} = \frac{T_a}{j}$, $[T_a^{cl}, T_b^{cl}] = \frac{i}{j} \epsilon_{abc} T_c^{cl}$ (A)

when $j \to \infty$ ($N \to \infty$) the algebra A tends to commutative algebra, with generators X_a, such that $X_1^2 + X_2^2 + X_3^2 = 1$

this is a sphere S^2

---

let us study how points on this S^2 appear

Recall the concept of dispersion of a stoch. observable $\mathcal{O}$:

$$\mathcal{D}(\mathcal{O}) = \langle \mathcal{O}^2 \rangle - (\langle \mathcal{O} \rangle)^2$$

$$\mathcal{D}(\mathcal{O}) \geq 0 \quad \langle (\mathcal{O} - \langle \mathcal{O} \rangle)^2 \rangle \geq 0$$

$$\langle \mathcal{O}^2 \rangle - \cancel{2} \langle \mathcal{O} \rangle \langle \mathcal{O} \rangle + \cancel{\langle \mathcal{O} \rangle^2} \geq 0$$

$\langle\ \ \rangle$ – taking average depends in QM on the state, over which we are getting the probability distribution.

The rule is $\langle \mathcal{O} \rangle = Tr(\mathcal{O}\, P_\Psi) =$

$$(Av) \qquad = \frac{\langle \Psi, \mathcal{O}\Psi \rangle}{\langle \Psi, \Psi \rangle}$$ (where vector Ψ represents the state $\in P(V)$)

We will look at states of minimal dispersion

<u>Claim</u>: These states are of the form:

$\Psi_\alpha = (\alpha_0 z_0 + \alpha_1 z_1)^N$, i.e. they correspond to the image of the Veronese map from algebraic geometry $\mathbb{C}P^1 \to \mathbb{C}P^N$.

projectivization of space of pol. of degree N

By minimal dispersion I mean total dispersion of 3 observables, T_1, T_2, T_3, i.e.

$$\langle (T_1^2 + T_2^2 + T_3^2) \rangle - (\langle T_1 \rangle)^2 - (\langle T_2 \rangle)^2 - (\langle T_3 \rangle)^2$$

$$T.D. = \frac{N}{2}\left(\frac{N}{2}+1\right) - \langle T_1 \rangle^2 - \langle T_2 \rangle^2 - \langle T_3 \rangle^2$$

I am looking for maximal sum of squares of averages.

Consider examples: Ex. 1. $N=1$

One may show that each state can be transformed by $SU(2)$ rotation (sym. of the problem) to $\begin{pmatrix}1\\0\end{pmatrix}$, $\langle T_2 \rangle_{\binom{1}{0}} = \langle T_1 \rangle_{\binom{1}{0}} = 0$

$$\langle T_3 \rangle_{\binom{1}{0}} = +\frac{1}{2}\cdot 1 + \left(-\frac{1}{2}\right)\cdot 0 = \frac{1}{2}$$

$$\mathcal{D} = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$$

The only possible dispersion

Ex. 2: $N=2$:

$\Psi_\uparrow = z_1^2$ $\qquad \langle T_2 \rangle_\uparrow = \langle T_3 \rangle_\uparrow = 0 \qquad \langle T_1 \rangle_\uparrow = 1$

$T.\mathcal{D}_{\uparrow} = \left(\frac{N}{2}\left(\frac{N}{2}+1\right) = 3\right) - 1 = 2$

$\Psi_{\uparrow\downarrow} = z_1^2 - z_0^2$ ⟷ not of the Veroneze form (not a complete square)

Also

$\langle T_1 \rangle = \langle T_2 \rangle_{\uparrow\downarrow} = 0$

$\langle T_3 \rangle_{\uparrow\downarrow} = (+1) \cdot \frac{1}{2} + (-1) \cdot \frac{1}{2} = 0$

(due to z_1^2) probability probability

How did I computed probabilities?

$\mathcal{O} = \sum_i \lambda_i \, Pr_i$; $\langle \mathcal{O} \rangle_\Psi = \sum_i \lambda_i \, \mathrm{Tr}\, Pr_i \, P_\Psi =$

$= \sum_i \lambda_i \underbrace{\frac{|\langle i, \Psi \rangle|^2}{\langle \Psi, \Psi \rangle}}_{}$ → probability

$\Psi = \underline{\underline{z_1^2 - z_0^2}}$

z_1^2 – eigenvector for eigenvalue 1

z_0^2 – eigenvector for eigenvalue -1

both probabilities are equal and equal to $\frac{1}{2}$.

And Total dispersion now is $3 > 2$.

Let us study large N dispersion of the Veroneze state $(\alpha_0 z_0 + \alpha_1 z_1)^N$

1) By $SU(2)$ rotation we may put it into

$\Psi_{\uparrow N} = (z_1)^N$, then $\langle T_1 \rangle_{\uparrow N} = \langle T_2 \rangle_{\uparrow N} = 0$

Each T_i changes the degree of z_{1N} so it produces state orthogonal to (z_1), (probability is 1)

but $\langle T_3 \rangle_{\uparrow N} = N/2$

The $T\mathcal{D}$. of T is $\frac{N}{2}\left(\frac{N}{2}+1\right) - \left(\frac{N}{2}\right)^2 = \frac{N}{2}$.

Now, $T\mathcal{D}$ of T^{cl} is

T.D. of T^{cl} is $\frac{\text{TD of } T}{(\frac{N}{2})^2} = \frac{2}{N}$

TD of $T^{cl} \to 0$ when $N \to \infty$

At the same time

TD of state of the T $z_1^N - z_0^N = \frac{N}{2}\left(\frac{N}{2}+1\right)$

T.D of T^{cl} of this state $= 1 \not\to 0$

clear, since probability to find $T_3^{cl} = +1$ and $T_3^{cl} = -1$ are equal and

such state is far from give a definite answer on the question what are values of almost commuting observables T_a^{cl}.

Given a minimal disp. state MD,

$\langle T_a \rangle = \frac{\langle MD | T_a | MD \rangle}{\langle MD | MD \rangle}$ — it is a vector in $\mathbb{R}^3$

by su(2) rotation I may turn it to be along the third axis (su(2) rotation of $|MD\rangle$ state)

Thus, I am looking for maximal $(\langle T_3 \rangle)^2$

It can be only $\langle T_3 \rangle = \frac{N}{2}$ or $\langle T_3 \rangle = -\frac{N}{2}$

for states z_1^N or z_0^N that belong to Veronese states □.

Veroneze states tend to orthogonal basis in the space of states when $N \to +\infty$.

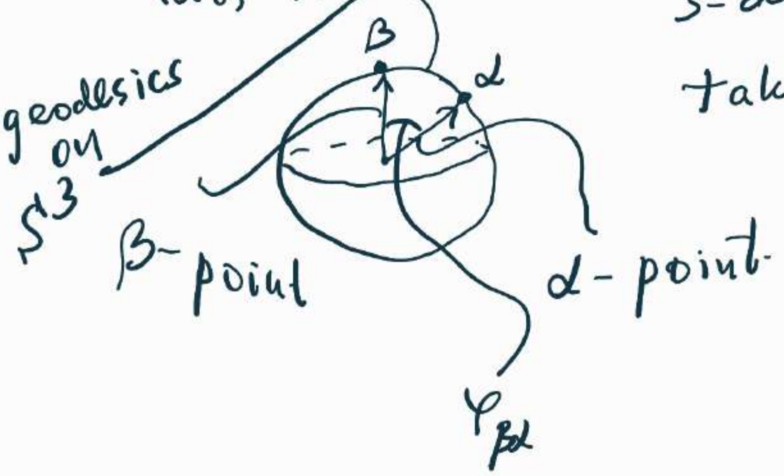
(They do not form such a basis for finite N, really, there are infinitely many of them while the space is finite dim $(N+1)$ for finite N).

Consider two states $(\alpha_0 z_0 + \alpha_1 z_1)^N$ and z_1^N (I can always put the second state

$(\beta_0 z_0 + \beta_1 z_1)^N$ to z_1^N by $SU(2)$ rotation

$\langle z_1^N, (\alpha_0 z_0 + \alpha_1 z_1)^N \rangle = \underline{\cos^N \varphi}$ | I assume that $|\alpha_0|^2 + |\alpha_1|^2 = 1$

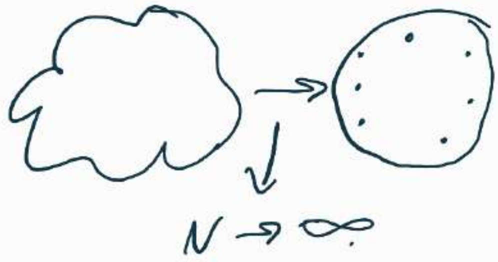
(α_0, α_1) is a point on the 3-dim sphere given by ^.



taking $\beta_g = 1$.

$\langle \beta | \alpha \rangle = \cos^N \varphi_{\beta\alpha} \to 0$ when $N \to +\infty$.

In the $N \to \infty$ limit sveroneze states behaive like points!



<u>Generalization</u> of this example: Hamiltonian reduction

Given a Kahler manifold X with the Kahler form ω consider the holomorphic action of the complex Lie algebra: $\mathfrak{g} \to \mathrm{Vect}\, X$

$t_a \mapsto v_a$, such that compact part of it preserves Kahler form: $(\mathcal{L}_{v_a} - \mathcal{L}_{\bar{v}_a})\,\omega = 0$

In the example we studied we had

$X = \mathbb{C}^2 \quad \mathfrak{g} = \mathbb{C}^* \quad v = z\frac{\partial}{\partial z}, \ \bar{v} = \bar{z}\frac{\partial}{\partial \bar{z}}$

$\omega = dz \wedge d\bar{z} \qquad v - \bar{v} = z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}$

Then there are Hamiltonians H_a:

$$i_{(v_a - \bar{v}_a)}\,\omega = dH_a$$

In our example, $H = |z_0|^2 + |z_1|^2 + \text{const.}$

We may consider so-called Hamiltonian reduction of $\mathcal{X}/\mathcal{Y}$, defined as

$$\{H_a = 0\}/\mathcal{Y}_{compact}$$

In our example we studied S^3

$|z_0|^2 + |z_1|^2 - 1 = 0$ – it was S^3 and we divided by the action of $U(1)$

Choice of the constant(s) could give ~~different~~ different results

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Orbits of $\mathcal{Y}_{noncompact}$ intersect zero level of Hamiltonians at most at one point

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Proof:

Consider $\mathcal{L}_{V_{nonc}} H = \frac{\partial H}{\partial y^i} v^i + \frac{\partial H}{\partial \bar{y}^i} \bar{v}^i =$

$$= 2\frac{\partial H}{\partial y^i}\frac{\partial H}{\partial \bar{y}^j} g^{i\bar{j}} > 0$$

So H is monotoneous along the trajectory and thus can intersect the zero level only once.

$$\boxed{\frac{\mathcal{X} \setminus \text{bad orbits}}{\mathcal{Y}_{complex}} = \frac{\{H=0\}}{\mathcal{Y}\text{-compact}}} \quad \boxed{\begin{array}{c}\text{hol. quotient}\\ \text{is given by}\\ \text{Hamilt. quotient}\end{array}}$$

Most important examples are so-called toric manifolds (generalizations of proj. manifolds)

$$\mathbb{C}^{N+1}/\mathbb{C}^* \quad \text{where} \quad \mathbb{C}^*: z_i \to e^{i\varphi} z_i$$

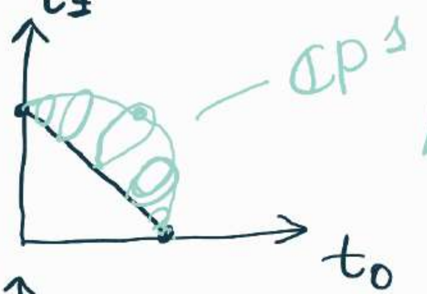
$$H = |z_0|^2 + \dots + |z_N|^2 - C$$

It is conv. to write everything in coord.

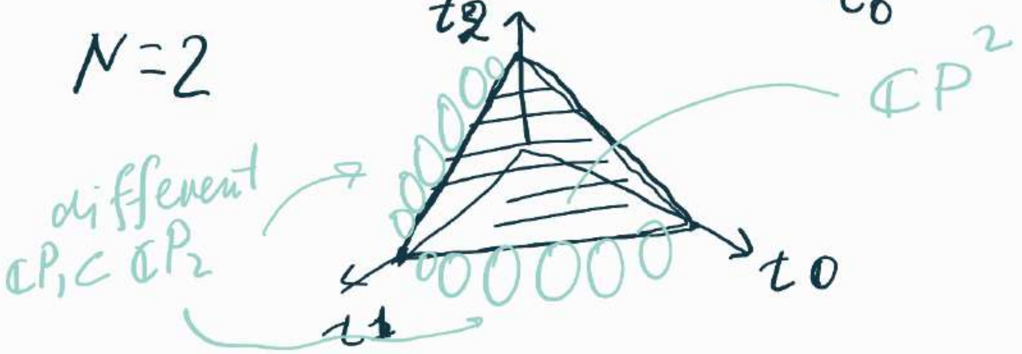
$t_i = |z_i|^2$ $\quad 0 \le t_i$

0 - phase of one of two phases z_0, z_1

$N = 1$



$N = 2$

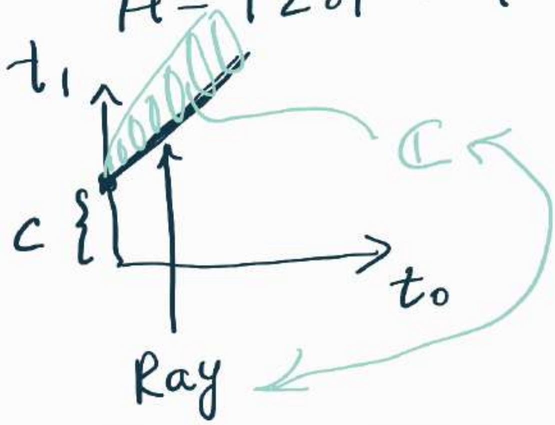


$\mathbb{C}^2$: $\quad \mathbb{C}^*$: $\quad z_0 \to e^{i\varphi} z_0$

$\qquad\qquad\qquad z_1 \to e^{-i\varphi} z_0$

$$H = |z_0|^2 - |z_1|^2 + c$$

$c > 0$

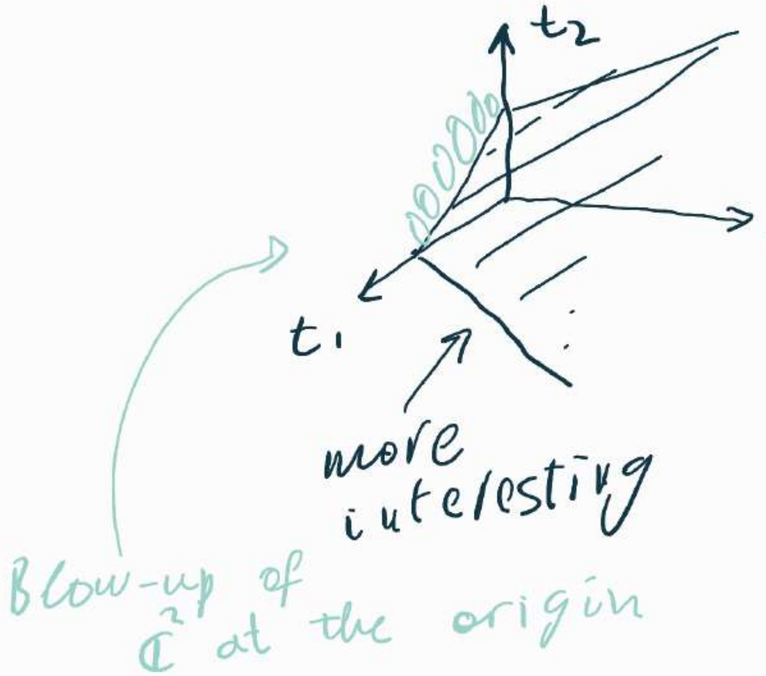


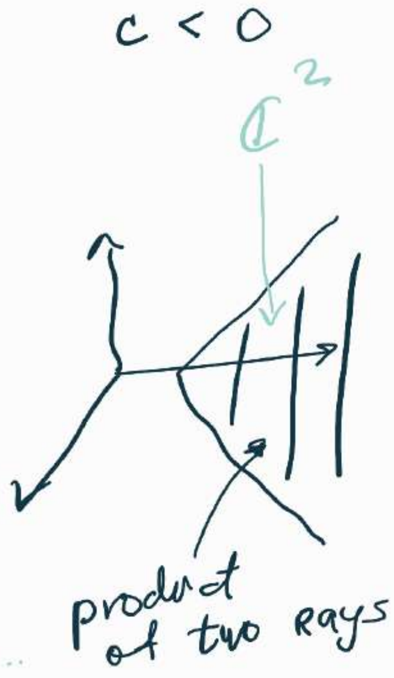
$\mathbb{C}^3$: $\quad \mathbb{C}^*$: $\quad z_0 \to e^{-i\varphi} z_0$

$\qquad\qquad\qquad z_1 \to e^{i\varphi} z_1$

$\qquad\qquad\qquad z_2 \to e^{i\varphi} z_2$

$$H = |z_1|^2 + |z_2|^2 - |z_0|^2 + c$$





<u>Next class on 9-th of January</u>

QFT with and without $\hbar$
QFT with $\hbar$ means Functional integral.
M.-Q. integral (in instantonic theories)
theory and BV
Batalin-Vilkovissky integrals.

Gauge theories – that describe our world – are a part. example of BV integral.
BV integral plays a prominent role in math. physics.

BV theory and later I will come back to our models of QM like $P_N(z_0, z_1)$ and generalizations treated as gauge Q. Mechanics (not written this way in books).

Inputs.

1.) Concept of De-Rham closed differential forms and their direct image: $F \to X$, $\pi: X \to B$

$\pi_* \omega \doteq \int_F \omega$ or $\int_{C\subset F} \omega$ $\quad \omega \in \Omega^\bullet_X$ differential forms on X

π_* depends on C $\quad$ cycle in the fiber F

This will turn into a BV integral

2.) "Odd" Fourier transform

$$\Omega^\bullet_X \longleftrightarrow \text{Polyvectors on } X$$

if we include $\hbar$:

1) + 2) $\Leftrightarrow$ BV theory

3.) $\quad$ Polyvector $= e^{\frac{1}{\hbar} S_{BV}(\hbar)}$

S_{BV} would be polyvector on X taking values in $\mathbb{C}[[\hbar]]$.

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1) Differential form ω on X may be considered as a function $\omega(x^\mu, \psi^\mu)$, x^μ is even, ψ^μ is odd

or even in the supercase parity x^μ = parity of $\psi^\mu + 1$ (mod $\mathbb{Z}_2$)

De Rham differential

$$d = \psi^\mu \frac{\partial}{\partial x^\mu}, \quad d^2 = 0$$

$d\omega = 0 \Leftrightarrow \omega$ - is closed.

$$F \to X, \quad X \xrightarrow{} B$$

locally

$$x^\mu = (x_f^\mu, x_B^a)$$

$$\psi^\mu = (\psi_f^\mu, \psi_B^a)$$

$d = d_f + d_B$

Take closed form $\omega \in \Omega^{\bullet}_X$

Take a cycle $C \subset F$

$$\underline{I}_C = \int_{C \subset F} \omega \in \mathrm{Fun}(x^a_B, \psi^a_B)$$

$$\qquad\qquad\qquad\qquad \cap \; \Omega^{\bullet}_B$$

$$d_B \underline{I}_C = \int_{C \subset F} d_B \omega = \int_{\underline{\underline{C}}} - \underline{\underline{d_F}} \omega = 0$$

since $\partial C = 0$

C is a cycle

This operation is called direct image: π_*: closed forms on $X \xrightarrow{C}$ closed forms on B.

2) "Odd" Fourier transform – known mostly in the case when X is purely bosonic.

2a). We assume that X is equipped with nonvanishing top form Ω_0

$\Omega_0 = \rho(x) \psi^1 \dots \psi^n$ $\quad n$ is $\dim X$.

Remind that polyvector is a function on $T^*[1]X$; i.e. a function of x^μ and $\theta_\mu \leftarrow$ coordinates in the fiber of $T^*[1]X$

(odd – in the case when X is purely bosonic (even), or in general parity θ_μ = parity of $x^\mu + 1$)

In classical dif. geometry there is an operation of contraction of polyvector and dif. form:

(i) $\quad i\, P(X, \theta)\, \omega(X, \psi) = P(X, \frac{\partial}{\partial \psi})\, \omega(X, \psi)$

If $P(X, \theta)$ represents a vector field $P = V^\mu(x)\theta_\mu$

Then $\iota_{v^\mu(x)\theta_\mu} = v^\mu \frac{\partial}{\partial\psi^\mu}\,\omega(x,\psi)$ — see in most of the text books.

Like in Cartan formula
$$\mathcal{L}_v = \{\iota_v, d\}.$$
ι is the generalization to polyvectors
$$p(x,\theta) = \sum_{k=0}^{n} p_k^{\mu_1\ldots\mu_k}(x)\,\theta_{\mu_1}\ldots\theta_{\mu_k}$$

$p \underset{F^{-1}}{\overset{F}{\rightleftarrows}} \omega$ correspondence:
$$\omega_p \overset{F^{-1}}{=} \iota_p\, \omega_0$$
This is an isomorphysm, I can invert it p_ω, in particular $p_{\omega_0} = 1$

$$\Delta_{BV} = F\circ d\circ F^{-1}$$
Δ_{BV} is Batalin-Vilkovissky operator.

$$\Delta_{BV}: Pol \to Pol.$$

$$\begin{tikzcd} Pol & \Omega_X \arrow[l, "F"'] \arrow[loop above, "d"] & Pol \arrow[l, "F^{-1}"'] \end{tikzcd}$$

From the definition it is clear that
$$\Delta_{BV}^2 = 0:\ F\, d\circ \underbrace{F^{-1}\circ F}_{id}\circ d\circ F^{-1} = 0$$

Let us for siml. compute it in coordinates when $\rho(x) = 1$

<u>Claim</u>: $\Delta_{BV} = \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial\theta_\mu}$

Reason: F is an odd version of Fourier transform, it takes $\psi^\mu \to \frac{\partial}{\partial\theta_\mu}$
$$\theta_\mu \to \frac{\partial}{\partial\psi^\mu}$$

Another way to write $P \to \omega_P$ is

$$\omega_P(x,\psi) = \rho(x)\int d\theta_1 \dots d\theta_n \, \exp\left(\underline{\underline{\psi^\mu \theta_\mu}}\right) \cdot P(\theta_1,\dots,\theta_n) \qquad (\mathrm{OF})$$

(in the case $\rho(x) = 1$)

what happens if $P = 1$

$$\rho(x)\int \underbrace{\underline{\underline{d\theta_1}} \dots \underline{\underline{d\theta_n}}}_{\text{Ber.}} \, \frac{\overbrace{(\psi\theta)\dots(\psi\theta)}^{n\text{-times}}}{n!} = \psi^1 \dots \psi^n$$

I call it "odd" Fourier transform since in standard Fourier transform

$$\tilde{f}(K) = \int \underline{\underline{dy}} \, \exp(iKy) f(y) \qquad (\mathrm{F})$$

↖standard integral

In the standard FT

$$K \tilde{f}(K) = \frac{1}{i}\int dy \, \exp(iKy) \frac{\partial}{\partial y} f(y)$$

Similarly, in the odd F.T.

$$\psi^\mu \omega_P(x,\psi) = \int d\theta_1 \dots d\theta_n \, \exp(\psi\theta) \frac{\partial}{\partial \theta_\mu} P$$

$\Rightarrow$ For $\rho = 1$ $\quad \Delta = \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial \theta_\mu}$

Combine 1) and 2) together.

A) Go to supercase and rethink what is

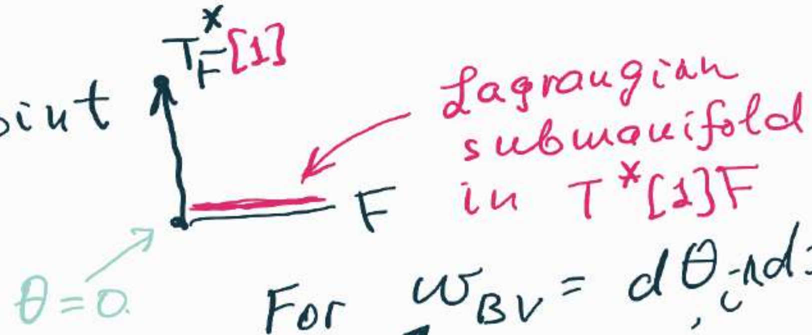
$$\int_{F'} \omega \underset{\text{locally}}{\overset{\uparrow}{=}} \int_{F} \prod_i \underline{\underline{d\psi_f^i}} \, dx_f^i \; \omega(\underline{\underline{\psi_f^i}}, x_f^i, \psi_b^a, x_b^a) \qquad (\mathrm{F.I})$$

Apply odd Fourier transform

$$\int \prod_i dx_f^i \; P(x_f, 0; \psi_b^a, x_b^a)$$

And later I may perform odd F.T. on the base.

Important point



For $\omega_{BV} = d\theta_i \wedge dx_f^i$

odd simplectic structure

or BV simplectic structure.

(really, ω is symmetric)

Consider a cycle in the Fiber, say $x_f^1 = 0$

$$\int dx_f^2 \, d\psi_f^2 \ldots dx_f^n \, d\psi_f^n$$

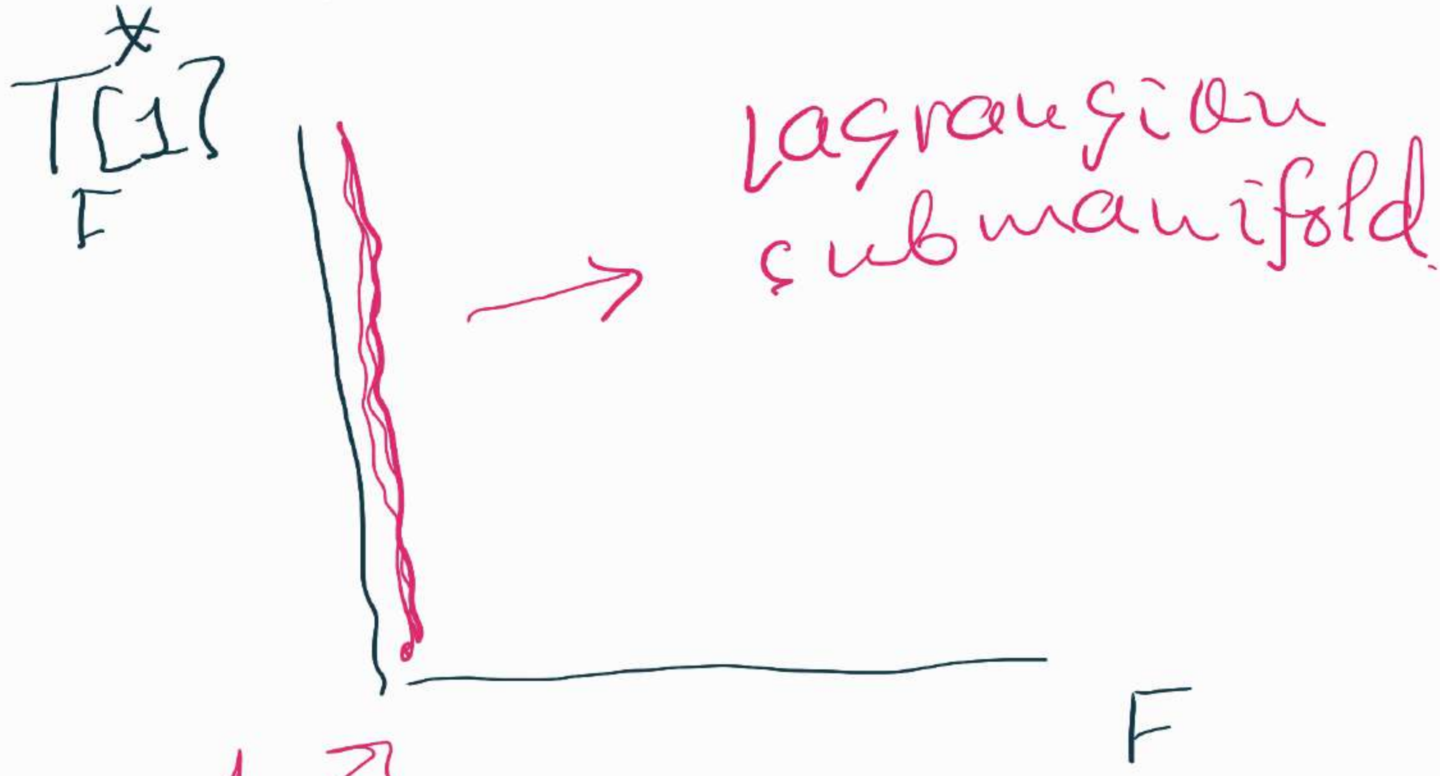
$$\omega \Big|_{x_f^1 = 0,\ \psi_f^1 = 0} (x_f, \psi_f; x_b, \psi_b) =$$

$$= \int dx_f^2 \, \underline{d\psi_f^2} \ldots dx_f^n \, \underline{d\psi_f^n}$$

$$\int d\theta_1 \, \underline{d\psi_f^1} \; \exp(\underline{\theta_1 \psi_f^1}) \; \omega \Big|_{x_f^1 = 0} =$$

$$\int d\theta_1 \, dx_f^2 \cdots dx_f^n$$

$$\underline{P}(0, x_f^2, \ldots, x_f^n, \theta_1, 0, \ldots, 0; x^b, t^b)$$



$x_f^1 = 0$, $\theta_{1,f}$ — unrestricted

Generalizing →
Integral over the cycle in Π_A
goes into integral over
Lagrangian submanifold.
(BV integral)

$$\int_{C\subset F} \omega \longrightarrow \int_{\mathcal{L}\subset T^*[1]F} P_\omega$$

In particular, if $d\omega = 0 \Leftrightarrow \Delta_{BV} P_\omega = 0$ then

$$\Delta_{BV} = \frac{\partial}{\partial \theta_{b,a}} \frac{\partial}{\partial x_b^a} \left(\int_{\mathcal{L}\subset T^*[1]F} P_\omega \right) = 0$$

Theorem for direct images for BV integrals.

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(3) $P = e^{\frac{1}{\hbar} S(\hbar)}$ $\quad S(\hbar)$ is a polyvector with values in $\mathbb{C}[[\hbar]]$

$\Delta_{BV} P = 0$, it corresponds to

$$0 = \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial \theta_\mu} e^{\frac{i}{\hbar} S(\hbar)} =$$

$$= \frac{\partial}{\partial x^\mu} \left(\frac{1}{\hbar} \frac{\partial S}{\partial \theta_\mu} e^{\frac{i}{\hbar} S(\hbar)} \right) =$$

$$= \left[\frac{1}{\hbar} \Delta S + \frac{1}{\hbar^2} \left(\frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial \theta_\mu} \right) \right] e^{\frac{i}{\hbar} S(\hbar)} = 0$$

<u>Quantum master equation</u>

$$\hbar \Delta S(\hbar) + \frac{\partial S(\hbar)}{\partial x^\mu} \frac{\partial S(\hbar)}{\partial \theta_\mu} = 0$$

taking $\hbar = 0$ we get
classical master equation

$$\frac{\partial S^{(0)}}{\partial x^\mu} \frac{\partial S^{(0)}}{\partial \theta_\mu} = 0 \qquad \{S, S\}_{BV} = 0$$

Many concepts of mathematics have the form $\{S, S\} = 0$ and quantum master equation is a uniform way to "quantize" them.

$\boxed{\text{Next class will be on 9 of January.}}$