Make Schubert calculus calculable

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Institute of Mathematics, Chinese Academy of Sciences

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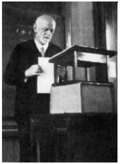
Yau Mathematical Sciences Center, 2024

The contents of the talk

In the year 1900, Hilbert proposed 23 problems. The 15th one was about the enumerative geometry of the 19th century, entitled

Problem 15: Rigorous Foundation of Schubert's Enumerative Calculus



15. RIGOROUS FOUNDATION OF SCHUBERT'S ENUMERATIVE CALCULUS.

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Although the algebra of to-day guarantees, in principle, the possibility of carrying out the processes of elimination, yet for the proof of the theorems of enumerative geometry decidedly more is requisite, namely, the actual carrying out of the process of elimination in the case of equations of special form in such a way that the degree of the final equations and the multiplicity of their solutions may be foreseen.

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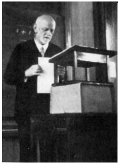

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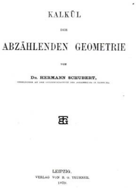
The plan of the talk:

1. The background of Problem 15;
2. Studies before 1960: Schubert's problem of characteristics;
3. Solution to the problem of characteristics (Duan+Zhao).

1. Background of Problem 15

In 1879 H. Schubert published the book "**Calculus of Enumerative Geometry**" that represents the summit of the intersection theory in the 19th century:


KALKÜL

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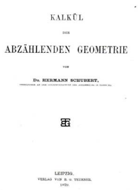
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DER

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VON

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In the course of developing intersection theory, he demonstrated amazing applications to enumerative geometry, such as

- The number of conics tangent to 8 quadrics in space is 4,407,296.
- The number of quadrics tangent to 9 quadrics in space is 666,841,088.
- The number of twisted cubic curves tangent to 12 quadrics in space is 5,819,539,783,680.

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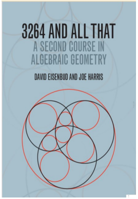
These results are great extensions of the earlier works of enumerative geometry:

Apollonius (300BC): The number of circles tangent to 3 general circles in the plane is 8.

Chasles (1864): The number of conics tangent to 5 general conics in the plane is 3264.



Remark: The original manuscript of Apollonius was lost. A report of the result by Pappus dated in the 4th century survived. During the Renaissance, different proofs of the result were founded by Viete, Roomen, Gergonne and Newton.

1. Background of Problem 15

On the other hand, Schubert's works was controversial at his time:

- He made extensive use of Poncelet's **principle of continuity**, which was attached bitterly by Cauchy in 1816;
- To circumvent the prejudice, Schubert renamed the principle as
  "**the principle of special position**" in 1874;
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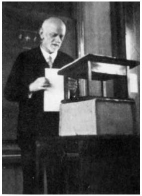
  "**the principle of conservation of numbers**" in 1876.
- Van der Waerden (1992) recalled that: Schubert gave "no definition of intersection multiplicites, no way to find it nor to calculate it".

1. Kleiman S., Problem 15: Rigorous foundation of Schubert's enumerative calculus, 1976.
2. Yvonne, D. S., Interview with Bartel Leendert van der Aaerden,1997.

1. Background of Problem 15

In Problem 15, Hilbert asked for a rigorous of Schubert's enumerative calculus:



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where he expressed his interests in Schubert's work:

- to foresee the final degree of a polynomial system before carrying out the process of elimination.

2. Schubert's problem of characteristics

To access the central part of Schubert's computation, we take two tables of computation from his book (1879):

Tabelle zusammengestellten Zahlen, und zwar alle diejenigen 2 mal, welche sowohl ν als ϱ zum Faktor haben.

Tabelle der Kegelschnittzahlen $\mu^m \nu^n \varrho^{8-m-n}$.

$\mu^3\nu^5 = 1$	$\mu^2\nu^6 = 8$	$\mu\nu^7 = 34$	$\nu^8 = 92$
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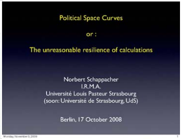
Schubert himself claimed that "**the problem of characteristics is the fundamental one** of the enumerative geometry."

However, it took 60 years for mathematicians to make the problem precise.

1. Schubert H., Zur Theorie der Charakteristike, Celles Journ. 1870.
2. Schubert H., a Losung des Characteristiken-Problems fur lineare Raume beliebiger Dimension, Mitteilungen der Mathematische Gesellschaft in Hamburg, 1886.

2. Schubert's problem of characteristics

The Italian school: The study of Problem 15 began with the Italian school:



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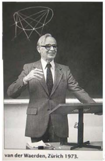
Van der Waerden (1971) commented that " *They erected an admirable structure, but their logical foundation was shaky, the notions were not well-defined, the proofs were insufficient*."

- van der Waerden B L., The foundation of algebraic geometry from Severi to André Weil, 1971

2. The Schubert's problem of characteristics

The Gottingen school: Van der Waerden propose to study Problem 15 using cohomology theory developed by Lefschetz:


van der Waerden, Zürich 1973.


Topologische Begründung des Kalküls der abzählenden Geometrie.

Von

Bartel L. van der Waerden in Groningen (Niederlande).

§ 1.

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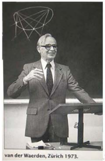
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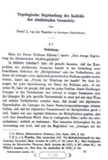
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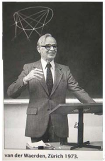
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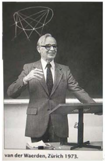


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- Each Schubert's symbolic equation is a homological relation in some projective manifold;
- The solvability of the characteristic problem depends on a **finite basis** of the homology of the relevant projective manifold.
- The common goal of all enumerative methods is the intersection products in the cohomology theory.

- van der Waerden B L., Topologische Begrundung des Kalkuls der abzahlenden Geometrie, Math. Ann 1930

2. Schubert's problem of characteristics

The Bourbaki: C. Ehresmann (1934) went two important steps further. He discovered that

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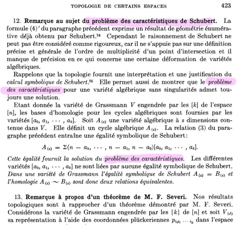
- The parameter spaces of the geometric figures concerned by Schubert are essentially certain cases of "**flag manifolds** G/P", where G is a Lie group and P is a parabolic subgroup;
- For the Grassmannian $G_{n,k}$ of k planes on the n-space $\mathbb{C}^n$, the set of **Schubert's symbols** form exactly **a basis** of the cohomology $H^*(G_{n,k})$,

where he emphasized the relevance of his work with the problem of characteristics:



TOPOLOGIE DE CERTAINS ESPACES 423

12. **Remarque au sujet du problème des caractéristiques de Schubert.** La formule (4)′ du paragraphe précédent exprime un résultat de géométrie énumérative déjà obtenu par Schubert.[24] Cependant le raisonnement de Schubert ne peut pas être considéré comme rigoureux, car il ne s'appuie pas sur une définition précise et générale de l'ordre de multiplicité d'un point d'intersection et il manque de précision en ce qui concerne une certaine déformation de variétés algébriques.

Rappelons que la topologie fournit une interprétation et une justification du *calcul symbolique de Schubert*.[25] Elle permet aussi de montrer que le *problème des caractéristiques* pour une variété algébrique sans singularités admet toujours une solution.

Etant donnée la variété de Grassmann V engendrée par les $[k]$ de l'espace $[n]$, les bases d'homologie pour les cycles algébriques sont fournies par les variétés $[a_0, a_1, \cdots, a_k]$. Soit $A_{(s)}$ une variété algébrique à s dimensions contenue dans V. Elle définit un cycle algébrique $A_{(s)}$. La relation (3) du paragraphe précédent entraîne une égalité symbolique de Schubert:

$$A_{(s)} = \Sigma(n - a_k, \cdots, n - a_1, n - a_0)[a_0, a_1, \cdots, a_k].$$

Cette égalité fournit la solution du problème des caractéristiques. Les différentes variétés $[a_0, a_1, \cdots, a_k]$ ne sont liées par aucune égalité symbolique de Schubert. *Dans une variété de Grassmann l'égalité symbolique de Schubert $A_{(s)} = B_{(s)}$ et l'homologie $A_{(s)} \sim B_{(s)}$ sont donc deux relations équivalentes.*

13. **Remarque à propos d'un théorème de M. F. Severi.** Nos résultats topologiques sont à rapprocher d'un théorème démontré par M. F. Severi. Considérons la variété de Grassmann engendrée par les $\{k\}$ de $[n]$ et soit $V_{(d)}$ sa représentation à l'aide des coordonnées plückeriennes $p_{i_0 i_1 \cdots i_k}$ dans l'espace

- C. Ehresmann, Sur la topologie de certains espaces homogenes, Ann. of Math. 1934.

2. Schubert's problem of characteristics

Carrying on the work of Ehressman, Chevalley(1958), Bernstein-Gel'fand-Gel'fand (1973) obtained "**the basis theorem of Schubert calculus**" in the natural generalities. Let W_G denotes the Weyl group of a Lie group G.

Theorem 1(Basis Theorem): For each flag manifold G/P, the set of Schubert classes $\{s_w, w \in W_G/W_P\}$ on G/P is a basis of the cohomology $H^*(G/P)$.

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Theorem 1(Basis Theorem): For each flag manifold G/P, the set of Schubert classes $\{s_w, w \in W_G/W_P\}$ on G/P is a basis of the cohomology $H^*(G/P)$.

Proof. Every flag manifold G/P admits a cell-decomposition into the Schubert varieties with even dimension

$$G/P = \cup_{w\in W_G/W_P} X_w, \ \dim X_w = 2 \cdot l(w),$$

where $l : W_G \to \mathbb{Z}$ is the length function on the Weyl group W_G.$\square$

1. Chevalley C. Sur les d'ecompositions cellulaires des Espaces G/B, 1958.
2. Bernstein I N, Gel'fand I M, Gel'fand S I. Schubert cells and cohomology of the spaces G/P. Russian Math Surveys, 1973.

2. Schubert's problem of characteristics

Granted with the basis theorem, the problem of characteristics has a concise statement

The problem of characteristics: Given a set $\{s_{u_1}, \cdots, s_{u_k}\}$ of Schubert classes on G/P, express their products in term of the basis elements linearly:

$$s_{u_1} \cdots s_{u_k} = \sum c^{w}_{u_1,\cdots u_k} \cdot s_w, \quad c^{w}_{u_1,\cdots u_k} \in \mathbb{Z},$$

where the coefficients $c^{w}_{u_1,\cdots u_k} \in \mathbb{Z}$ are the **Schubert characteristics**.

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Remark: Coolidge J.L.(1940) recalled that:

"The fundamental problem which occupies Schubert is to express the product of these symbols in terms of others linearly. He succeeds in part."

1. Coolidge J.L., A history of geometrical methods, Oxford Univ. press,1940

2. Schubert's problem of characteristics

The characteristics play a fundamental role in geometry, algebra, topology and representation theory, where they were also termed, respectively, as

- **the degree of the final equation of a system** by Hilbert;
- **the intersection multiplicities** by Van der Waerden, Weil, Chevalley, Samuel, Serre, etc.;

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- **the degree of the final equation of a system** by Hilbert;
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and for the special case $k = 2$,

- **the structure constants** of the flag manifolds G/P by topologists;
- **the Littlewood-Richardson coefficients** in representation theory.

2. Schubert's problem of characteristics

Serre formula: Based on Weil's definition on the intersection multiplicities in 1946, Serre (1965) obtained "**an elegant formula**" by which:

$$s_{u_1} \cdot s_{u_2} = \sum c_{u_1,u_2}^{w} \cdot s_w, \text{ where } c_{u_1,u_2}^{w} = \sum_{k\geq 0} (-1)^k L(\underset{k}{\overset{A}{Tor}}(A/\mathfrak{a}_1, A/\mathfrak{a}_2)),$$

and where A is the local ring $\mathcal{O}(G/P, X_w), \mathfrak{a}_i$ is ideal of the Schubert varieties X_{u_i}, and L is the length of the A-modules.

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Unfortunately, this formula is not computable, because it uses the defining polynomials f_w, f_{u_i} of the Schubert varieties $X_w, X_{u_i} \subset G/P$ as input.

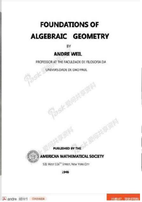
Remark: Nowadays, algebraic geometers use "**intersection multiplicities**" instead of "**characteristics**", e.g. the survey articles:

1. Pierre Samuel, Sur l'histoire du quinzième problème de Hilbert, Gaz.Math. Soc. Math. Fr. 1974.
2. W. Fulton, R.D. MacPherson, "Defining algebraic intersections", LNM. 687, 1978.
3. J.P.Serre, Algebre Locale, Multiplicites, LNM, 11, 1965.

2. The Schubert's problem of characteristics

Weil Problem: In the momentous treatise "Foundations of Algebraic Geometry", A. Weil completed the definition of "the intersection multiplicities" for the first time in the history, and made the task of Problem 15 precise:

"The classical Schubert calculus amounts to the determination of the intersection rings of flag manifolds G/P."



FOUNDATIONS OF
ALGEBRAIC GEOMETRY

BY

ANDRE WEIL

PROFESSOR AT THE FACULDADE DE FILOSOFIA DA
UNIVERSIDADE DE DAO PAUL

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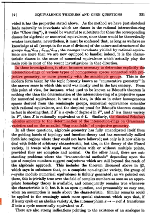
1946



[9] EQUIVALENCE THEORIES AND OPEN QUESTIONS 331

vided it has the properties stated above. As the method we have just sketched leads naturally to invariants which are classes in the rational intersection-ring (the "Chow ring"), it would be wasteful to substitute for these the corresponding classes for algebraic or numerical equivalence, since these would be theoretically weaker invariants; nevertheless, it must be confessed that, as long as we have no knowledge at all (except in the case of divisors) of the nature and structure of the groups $\mathfrak{G}_{alg}/\mathfrak{G}_{rat}$, $\mathfrak{G}_{num}/\mathfrak{G}_{rat}$, the stronger invariants yielded by rational equivalence are more than we are now equipped to handle; indeed, it is the characteristic classes in the sense of numerical equivalence which actually play the main role in most of the recent investigations in that direction.

In these investigations, it often becomes necessary to determine explicitly the intersection-rings of various types of homogeneous spaces connected with projective geometry, or more generally with the semisimple groups. This is the modern form taken by the topic formerly known as "enumerative geometry" in the narrow sense in which this word was mostly used in the last century. From this point of view, for instance, what used to be known as Bézout's theorem is nothing else than the determination of the intersection-ring of a projective space for numerical equivalence; it turns out that, in this as in other homogeneous spaces derived from the semisimple groups, numerical equivalence coincides with rational equivalence, and the simplest proof for Bézout's theorem consists in fact in showing that, if X^r is a cycle of degree d in P^n, and L^r is a linear variety in P^n, then X is rationally equivalent to $d \cdot L$. Similarly, the classical Schubert calculus amounts to the determination of the intersection-rings on Grassmann varieties and on the so-called "flag manifolds" of projective geometry.

In all these questions, algebraic geometry has fully emancipated itself from the guiding hands of topology and function-theory and has successfully sallied forth into regions where they could not have followed; not only is it now able to deal with fields of arbitrary characteristic, but also, in the theory of the Picard variety, it treats with equal ease varieties with or without multiple points, provided they are complete and normal. On the other hand, there are outstanding problems where the "transcendental methods" depending upon the use of complex numbers suggest conjectures which are still beyond the reach of the algebraic approach. This includes the theorem discussed above in §2, which says in substance that, on a complete non-singular variety, the group of r-cycles modulo numerical equivalence is finitely generated; as we pointed out there, this is trivially true over the field of complex numbers as soon as an appropriate homology theory is available, and consequently remains true whenever the characteristic is 0, but it is an open question, and presumably no easy one, when no assumption is made about the characteristic. Similar remarks could be applied to the seemingly much more special statement which says that, if X is any cycle on an abelian variety A, the automorphism $x \to -x$ of A transforms X into a cycle numerically equivalent to X.

There are also strong indications pointing to the existence of an analogue in

2. Schubert's problem of characteristics

Weil commented that his problem as "the modern form taken by the topic formerly known as enumerative geometry in the last century". We show that

Theorem 2: Weil problem is equivalent to the problem of characteristics.

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Proof. A **ring** is an abelian group R that is furnished with a product:

$$R \times R \to R.$$

By the basis theorem, the cohomology $H^*(G/P)$ is a free abelian group with a basis consisting of Schubert classes.

Theorefore, the product on the ring $H^*(G/P)$ is determined uniquely by the product among the basis elements (i.e. the Schubert symbols), which is handled by the problem of characteristics.□

Solution to the problem of characteristics (Duan and Zhao)

Summarizing the earlier studies on Problem 15 by 1960 there two problems remain. For a flag manifold G/P

1. **Schubert:** Compute all the characteristics numbers $c^{w}_{u_1,\cdots,u_k}$;
2. **Weil:** Determine the cohomology ring $H^*(G/P)$.

Solution to the problem of characteristics

The difficulties that one encounters with characteristics are fairly transparent:

- The simply-connected simple Lie groups G consist of the three infinite families of the classical groups $Spin(n), Sp(n), SU(n)$, as well as the five exceptional ones G_2, F_4, E_6, E_7, E_8;

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- For each Lie group G with rank n, there are precisely $2^n - 1$ parabolic subgroups P.
- For each flag manifold G/P, the number of Schubert classes on G/P is equal to the Euler characteristic $\chi(G/P)$, which is normally very large:

G	G_2	F_4	E_6	E_7	E_8
$\chi(G/T)$	12	1152	$2^7 \cdot 3^4 \cdot 5$	$2^{10} \cdot 3^4 \cdot 5 \cdot 7$	$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$

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Summarizing, it is impossible to solve the problem of characteristics **case by case.**

Solution to the problem of characteristics

On the other hand, by a fundamental contribution of E. Cartan, the simply-connected Lie groups are classified by their Cartan matrices

$$G_2 : \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \quad F_4 : \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \quad E_6 : \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$E_7 : \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} \quad E_8 : \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

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The cosmological constants by which all flag manifolds can be classified!

Question

Can we compute all the characteristics numbers, or construct the Chow ring of a flag manifold G/P, merely from the Cartan matrix of the Lie group G?

We realize this expectation.

Solution to the problem of characteristics

Thus, let $C = (c_{i,j})_{n\times n}$ be the Cartan matrix of some compact Lie group G. Let $\mathbb{R}^n$ be the n-dimensional real vector space with basis $\{\omega_1, \cdots, \omega_n\}$. Define in term of C the automorphisms $\sigma_i \in Aut(\mathbb{R}^n), 1 \leq i \leq n$, by the formula

$$\sigma_i(\omega_k) = \begin{cases} \omega_k \text{ if } i \neq k; \\ \omega_k - (c_{k,1}\omega_1 + c_{k,2}\omega_2 + \cdots + c_{k,n}\omega_n) \text{ if } i = k. \end{cases}$$

Theorem 3. The subgroup W of $Aut(\mathbb{R}^n)$ generated by the σ_i's is **the Weyl group** of G.

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Theorem 3. The subgroup W of $Aut(\mathbb{R}^n)$ generated by the σ_i's is **the Weyl group** of G.

Definition. For a Weyl group element $w \in W$ with a minimized decomposition

$$w = \sigma_{i_1} \circ \sigma_{i_2} \circ \cdots \circ \sigma_{i_m}, \ 1 \leq i_1, i_2, \cdots, i_m \leq n,$$

the structure matrix of w is $A_w = (a_{s,t})_{m\times m}$, where

$$a_{s,t} = 0 \ \text{if } s \geq t; \quad a_{s,t} = -c_{i_s,i_t} \ \text{if } s < t.$$

Solution to the problem of characteristics

Example: The Cartan matrix of the exceptional Lie group G_2 is

$$C = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix},$$

by which we get two generators σ_1, σ_2 of the Weyl group W of G_2.

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Consider the following elements of W with length 4:

$$u = \sigma_1 \circ \sigma_2 \circ \sigma_1 \circ \sigma_2 \text{ and } v = \sigma_2 \circ \sigma_1 \circ \sigma_2 \circ \sigma_1.$$

From the Cartan matrix C one reads the structure matrices of u, v, respectively,

$$A_u = \begin{pmatrix} 0 & 1 & -2 & 1 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } A_v = \begin{pmatrix} 0 & 3 & -2 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Solution to the problem of characteristics

Definition. Given a strictly upper triangular matrix $A = (a_{i,j})_{m\times m}$ **the triangular operator** T_A associated to A is the linear map

$$T_A : \mathbb{Z}[x_1, \dots, x_m]^{(m)} \to \mathbb{Z}[x_1, \dots, x_{m-1}]^{(m-1)} \to \cdots \to \mathbb{Z}$$

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defined recursively by the following elimination rules:

$$T_A(x_1^{r_1} \dots x_m^{r_m}) = 0 \text{ if } r_m = 0;$$
$$T_A(x_1^{r_1} \dots x_m^{r_m}) = T_{A_1}(x_1^{r_1} \dots x_{m-1}^{r_{m-1}}(a_{1,m}x_1 + \cdots + a_{m-1,m}x_{m-1})^{r_m - 1})$$

if $r_m > 0$, where A_1 is obtained from A by deleting the last column and row.

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if $r_m > 0$, where A_1 is obtained from A by deleting the last column and row.

Summarizing, starting barely from the Cartan matrix $C = (c_{i,j})_{n\times n}$ of G, we have constructed:

- The Weyl group W of G;
- Strictly upper-triangular matrices $\{A_w,\ w \in W\}$;
- Linear maps $\{T_{A_w} : \mathbb{Z}[x_1, \dots, x_m]^{(m)} \to \mathbb{Z},\ w \in W\}$.

Solution to the problem of characteristics

Applying Morse theory to the canonical embeddings $G/P \hookrightarrow L(G)$ of the flag manifold G/P into the Lie algebra $L(G)$, we have obtained the following formula.

Theorem 4 (The Characteristic Formula): Let A_w be the structure matrix of w associated the minimized decomposition $w = \sigma_{i_1} \circ \sigma_{i_2} \circ \cdots \circ \sigma_{i_m}$, then the characteristic $c^w_{u_1,\cdots u_k}$ is given by

$$c^w_{u_1,\cdots,u_k} = T_{A_w}\left(\prod_{i=1,\dots,k}\left(\sum_{I\subseteq\{1,\dots,m\},|I|=l(u_i),\sigma_I=u_i} x_I\right)\right)$$

where for a multi-index $I = \{s_1, \cdots, s_t\} \subseteq \{1, \cdots, m\}$

$$|I| = t,\ \sigma_I = \sigma_{i_{s_1}} \circ \cdots \circ \sigma_{i_{s_t}},\ x_I = x_{s_1} \cdots x_{s_t}.\square$$

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Remark: The formula depends only on the Cartan matrix C of G, and applies uniformly

- to all flag manifolds G/P;
- to any monomial $s_{u_1} \cdots s_{u_k}$ in the Schubert basis of G/P.

Solution to the problem of characteristics

Based on the formula, a packages entitled "*CHARACTERISTICS*" have been composed in the works

- Duan and Zhao, Multiplicative rule of Schubert classes (2005)
- Duan and Zhao, Algorithm for multiplying Schubert classes (2006)
- Duan and Zhao, Schubert presentations of complete flag manifolds G/T (2015)
- Duan and Zhao, On Schubert's Problem of Characteristics (2020).

whose function can be described as follows:

Algorithm

Input: *The Cartan matrix* $C = (c_{ij})_{n\times n}$ *of the Lie group* G*, and a subset* $I \subseteq \{1, 2, \cdots, n\}$ *to specify a parabolic subgroup* $P \subset G$.

Output: *The characteristic numbers* $c^{w}_{u_1,\cdots u_k}$ *of* G/P.

Solution to the problem of characteristics

Example

The characteristics of the Grassmannian $G_{9,4}(\mathbb{C})$ (the universal Chern numbers):

$c_4^5 = 1$	$c_3^4 c_4^2 = 1$	$c_2 c_3^2 c_4^3 = 1$	$c_2 c_3^6 = 9$
$c_2^2 c_4^4 = 1$	$c_2^2 c_3^4 c_4 = 6$	$c_2^3 c_3^2 c_4^2 = 4$	$c_2^4 c_4^3 = 3$
$c_2^4 c_3^4 = 45$	$c_2^5 c_3^2 c_4 = 26$	$c_2^6 c_4^2 = 16$	$c_2^7 c_3^2 = 231$
$c_2^8 c_4 = 126$	$c_2^{10} = 1296$	$c_1 c_3 c_4^4 = 1$	$c_1 c_3^5 c_4 = 4$
$c_1 c_2 c_3^3 c_4^2 = 3$	$c_1 c_2^2 c_3 c_4^3 = 2$	$c_1 c_2^2 c_3^5 = 29$	$c_1 c_2^3 c_3^3 c_4 = 17$
$c_1 c_2^4 c_3 c_4^2 = 10$	$c_1 c_2^5 c_3^3 = 141$	$c_1 c_2^6 c_3 c_4 = 76$	$c_1 c_2^8 c_3 = 756$
$c_1^2 c_3^2 c_4^3 = 2$	$c_1^2 c_3^6 = 19$	$c_1^2 c_2 c_4^4 = 1$	$c_1^2 c_2 c_3^4 c_4 = 12$
$c_1^2 c_2^2 c_3^2 c_4^2 = 7$	$c_1^2 c_2^3 c_4^3 = 4$	$c_1^2 c_2^3 c_3^4 = 89$	$c_1^2 c_2^4 c_3^2 c_4 = 48$
$c_1^2 c_2^5 c_4^2 = 26$	$c_1^2 c_2^6 c_3^2 = 451$	$c_1^2 c_2^7 c_4 = 231$	$c_1^2 c_2^9 = 2556$
$c_1^3 c_3^3 c_4^2 = 6$	$c_1^3 c_2 c_3 c_4^3 = 3$	$c_1^3 c_2 c_3^5 = 59$	$c_1^3 c_2^2 c_3^3 c_4 = 32$
$c_1^3 c_2^3 c_3 c_4^2 = 17$	$c_1^3 c_2^4 c_3^3 = 276$	$c_1^3 c_2^5 c_3 c_4 = 141$	$c_1^3 c_2^7 c_3 = 1491$
$c_1^4 c_4^4 = 1$	$c_1^4 c_3^4 c_4 = 24$	$c_1^4 c_2 c_3^2 c_4^2 = 12$	$c_1^4 c_2^2 c_4^3 = 6$
$c_1^4 c_2^2 c_3^4 = 175$	$c_1^4 c_2^3 c_3^2 c_4 = 89$	$c_1^4 c_2^4 c_4^2 = 45$	$c_1^4 c_2^5 c_3^2 = 886$
$c_1^4 c_2^6 c_4 = 436$	$c_1^4 c_2^8 = 5112$	$c_1^5 c_3 c_4^3 = 4$	$c_1^5 c_3^5 = 119$
$c_1^5 c_2 c_3^3 c_4 = 59$	$c_1^5 c_2^2 c_3 c_4^2 = 29$	$c_1^5 c_2^3 c_3^3 = 539$	$c_1^5 c_2^4 c_3 c_4 = 264$
$c_1^5 c_2^6 c_3 = 2962$	$c_1^6 c_3^2 c_4^2 = 19$	$c_1^6 c_2 c_4^3 = 9$	$c_1^6 c_2 c_3^4 = 339$
$c_1^6 c_2^2 c_3^2 c_4 = 164$	$c_1^6 c_2^3 c_4^2 = 79$	$c_1^6 c_2^4 c_3^2 = 1744$	$c_1^6 c_2^5 c_4 = 832$
$c_1^6 c_2^7 = 10302$	$c_1^7 c_3^3 c_4 = 104$	$c_1^7 c_2 c_3 c_4^2 = 49$	$c_1^7 c_2^2 c_3^3 = 1047$
$c_1^7 c_2^3 c_3 c_4 = 496$	$c_1^7 c_2^5 c_3 = 5912$	$c_1^8 c_4^3 = 14$	$c_1^8 c_3^4 = 641$
$c_1^8 c_2 c_3^2 c_4 = 300$	$c_1^8 c_2^2 c_4^2 = 140$	$c_1^8 c_2^3 c_3^2 = 3437$	$c_1^8 c_2^4 c_4 = 1600$
$c_1^8 c_2^6 = 20887$	$c_1^9 c_3 c_4^2 = 84$	$c_1^9 c_2 c_3^3 = 2025$	$c_1^9 c_2^2 c_3 c_4 = 936$
$c_1^9 c_2^4 c_3 = 11853$	$c_1^{10} c_3^2 c_4 = 552$	$c_1^{10} c_2 c_4^2 = 252$	$c_1^{10} c_2^2 c_3^2 = 6792$
$c_1^{10} c_2^3 c_4 = 3102$	$c_1^{10} c_2^5 = 42597$	$c_1^{11} c_3^3 = 3927$	$c_1^{11} c_2 c_3 c_4 = 1782$
$c_1^{11} c_2^3 c_3 = 23892$	$c_1^{12} c_4^2 = 462$	$c_1^{12} c_2 c_3^2 = 13497$	$c_1^{12} c_2^2 c_4 = 6072$
$c_1^{12} c_2^4 = 87417$	$c_1^{13} c_3 c_4 = 3432$	$c_1^{13} c_2^2 c_3 = 48477$	$c_1^{14} c_3^2 = 27027$
$c_1^{14} c_2 c_4 = 12012$	$c_1^{14} c_2^3 = 180609$	$c_1^{15} c_2 c_3 = 99099$	$c_1^{16} c_4 = 24024$
$c_1^{16} c_2^2 = 375804$	$c_1^{17} c_3 = 204204$	$c_1^{18} c_2 = 787644$	$c_1^{20} = 1662804$

Table 2. The characteristics of the Grassmaniann $G_{9,4}$

Solution to the problem of characteristics

Example

The characteristics of the exceptional flag manifold $E_6/S^1 \times SU(6)$

$y_3y_6^3 = 3$	$y_3y_4^3y_6 = 3$	$y_3^3y_6^2 = 21$	$y_3^3y_4^3 = 21$	$y_3^5y_6 = 156$
$y_3^7 = 1158$	$y_1y_4^2y_6^2 = 2$	$y_1y_4^5 = 2$	$y_1y_3^2y_4^2y_6 = 14$	$y_1y_3^4y_4^2 = 100$
$y_1^2y_3y_4y_6^2 = 9$	$y_1^2y_3y_4^4 = 9$	$y_1^2y_3^3y_4y_6 = 66$	$y_1^2y_3^5y_4 = 483$	$y_1^3y_6^3 = 6$
$y_1^3y_4^3y_6 = 6$	$y_1^3y_3^2y_6^2 = 42$	$y_1^3y_3^2y_4^3 = 42$	$y_1^3y_3^4y_6 = 312$	$y_1^3y_3^6 = 2328$
$y_1^4y_3y_4^2y_6 = 28$	$y_1^4y_3^3y_4^2 = 201$	$y_1^5y_4y_6^2 = 18$	$y_1^5y_4^4 = 18$	$y_1^5y_3^2y_4y_6 = 132$
$y_1^5y_3^4y_4 = 972$	$y_1^6y_3y_6^2 = 84$	$y_1^6y_3y_4^3 = 84$	$y_1^6y_3^3y_6 = 624$	$y_1^6y_3^5 = 4677$
$y_1^7y_4^2y_6 = 56$	$y_1^7y_3^2y_4^2 = 404$	$y_1^8y_3y_4y_6 = 264$	$y_1^8y_3^3y_4 = 1956$	$y_1^9y_6^2 = 168$
$y_1^9y_4^3 = 168$	$y_1^9y_3^2y_6 = 1248$	$y_1^9y_3^4 = 9390$	$y_1^{10}y_3y_4^2 = 813$	$y_1^{11}y_4y_6 = 528$
$y_1^{11}y_3^2y_4 = 3936$	$y_1^{12}y_3y_6 = 2496$	$y_1^{12}y_3^3 = 18837$	$y_1^{13}y_4^2 = 1638$	$y_1^{14}y_3y_4 = 7917$
$y_1^{15}y_6 = 4992$	$y_1^{15}y_3^2 = 37752$	$y_1^{17}y_4 = 15912$	$y_1^{18}y_3 = 75582$	$y_1^{21} = 151164$

Table 3. The characteristics of the flag manifold $E_6/S^1 \cdot SU(6)$.

Example

The characteristics of the exceptional flag manifold $E_7/S^1 \times E_6$

$y_9^3 = 10$	$y_1^2y_5^5 = 184$	$y_1^3y_5^3y_9 = 92$	$y_1^4y_5y_9^2 = 46$	$y_1^7y_5^4y_9 = 432$
$y_1^8y_5^2y_9 = 216$	$y_1^9y_9^2 = 108$	$y_1^{12}y_5^3 = 1014$	$y_1^{13}y_5y_9 = 507$	$y_1^{17}y_5^2 = 2380$
$y_1^{18}y_9 = 1190$	$y_1^{22}y_5 = 5586$	$y_1^{27} = 13110$		

Table 4. The characteristics of the flag manifold $E_7/S^1 \cdot E_6$

Solution to Weil problem

Turning to the Weil problem we show that:

Theorem 5. For a flag manifold G/P, there exist a minimal system of Schubert classes $\{x_1, \cdots, x_n\}$, and polynomials $f_1, \cdots, f_m \in \mathbb{Z}[x_1, \cdots, x_n]$, such that

$$H^*(G/P) = \frac{\mathbb{Z}[x_1, \cdots, x_n]}{\langle f_1, \cdots, f_m \rangle}.$$

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Proof. Since the flag manifold G/P is finite dimensional, the quotient group

$$H^+(G/P)/H^+(G/P) \cdot H^+(G/P)$$

is finitely generated. By the basis theorem of Schubert calculus, there exists a set of Schubert classes $\{x_1, \cdots, x_n\}$ on G/P that correspond to a basis of the quotient group $H^+(G/P)/H^+(G/P) \cdot H^+(G/P)$.
It follows that the inclusion $x_1, \cdots, x_n \in H^*(G/P)$ induces an epimorphism

$$h : \mathbb{Z}[x_1, \cdots, x_n] \to H^*(G/P).$$

By the Hilbert's basis theorem, there exist finite set of polynomials $\{f_1, \cdots, f_m\}$ in $x_1, \cdots, x_n$ such that $\ker h = \langle f_1, \cdots, f_m \rangle$.□

Solution to Weil problem

Combining "*CHARACTERISTICS*" with the proof of Theorem 5, we have composed a package entitled "*CHOWRING*" in the works:

- The Chow rings of generalized Grassmannians (2010);
- Schubert presentations of complete flag manifolds G/T (2015),

Solution to Weil problem

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whose function can be described as follows:

Algorithm

Input: *The Cartan matrix* $C = (c_{ij})_{n\times n}$ *of the Lie group* G*, and a subset* $I \subseteq \{1, 2, \cdots, n\}$ *to specify a parabolic subgroup* $P \subset G$*.*

Output: *A presentation of the Chow ring* $H^*(G/P) = A^*(G/P)$*.*

Solution to Weil problem

Example

The Chow rings $A^*(G/P)$ of the flag manifolds

$$G/P = \frac{F_4}{Sp(3)\cdot S^1}, \frac{F_4}{Spin(3)\cdot S^1}, \frac{E_6}{SU(6)\cdot S^1}, \frac{E_6}{Spin(10)\cdot S^1}, \frac{E_7}{E_6\cdot S^1}, \frac{E_7}{Spin(12)\cdot S^1}, \frac{E_8}{E_7\cdot S^1}$$

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Table 1 Some generalized Grassmannians and their corresponding parabolic subgroups

G	F_4	F_4	E_6	E_6	E_7	E_7	E_8
ω	ω_1	ω_4	ω_2	ω_6	ω_7	ω_1	ω_8
H	$C_3\cdot S^1$	$B_3\cdot S^1$	$A_5\cdot S^1$	$D_5\cdot S^1$	$D_6\cdot S^1$	$E_6\cdot S^1$	$E_7\cdot S^1$
H_ω	C_3	B_3	A_5	D_5	D_6	E_6	E_7

As applications of the methods developed in this paper, calculation is carried out for the seven Grassmannians G/H (resp. rank 1 homogeneous spaces G/H_ω) specified above. More precisely, provided with the Weyl coordinates for Schubert classes (i.e., our indices for Schubert classes; see Sect. 2.2), the following results are established.

Given a subset $\{f_1, \ldots, f_m\}$ in a ring, write $\langle f_1, \ldots, f_m\rangle$ for the ideal generated by $f_1, \ldots, f_m$.

Theorem 1 *Let y_1, y_3, y_4, y_6 be the Schubert classes on $F_4/C_3\cdot S^1$ with Weyl coordinates $\sigma[1], \sigma[3,2,1], \sigma[4,3,2,1], \sigma[3,2,4,3,2,1]$ respectively. Then*

$$A^*(F_4/C_3\cdot S^1) = \mathbb{Z}[y_1, y_3, y_4, y_6]/\langle r_3, r_6, r_8, r_{12}\rangle,$$

where

$$r_3 = 2y_3 - y_1^3;$$
$$r_6 = 2y_6 + y_3^2 - 3y_1^2 y_4;$$
$$r_8 = 3y_4^2 - y_1^2 y_6;$$
$$r_{12} = y_6^2 - y_4^3.$$

Theorem 2 *Let y_1, y_4 be the Schubert classes on $F_4/B_3\cdot S^1$ with Weyl coordinates $\sigma[4], \sigma[3,2,3,4]$ respectively. Then*

$$A^*(F_4/B_3\cdot S^1) = \mathbb{Z}[y_1, y_4]/\langle r_8, r_{12}\rangle,$$

where

$$r_8 = 3y_4^2 - y_1^8;$$
$$r_{12} = 26y_4^3 - 5y_1^{12}.$$

Theorem 3 *Let y_1, y_3, y_4, y_6 be the Schubert classes on $E_6/A_5\cdot S^1$ with Weyl coordinates $\sigma[2], \sigma[5,4,2], \sigma[6,5,4,2], \sigma[1,3,6,5,4,2]$ respectively. Then*

$$A^*(E_6/A_5\cdot S^1) = \mathbb{Z}[y_1, y_3, y_4, y_6]/\langle r_6, r_8, r_9, r_{12}\rangle,$$

where

$$r_6 = 2y_6 + y_3^2 - 3y_1^2 y_4 + 2y_1^3 y_3 - y_1^6;$$
$$r_8 = 3y_4^2 - 6y_1 y_3 y_4 - y_1^2 y_6 + 5y_1^2 y_3^2 - 2y_1^5 y_3;$$

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$$r_9 = 2y_3 y_6 - y_1^3 y_6;$$
$$r_{12} = y_4^3 - y_6^2.$$

Theorem 4 *Let y_1, y_4 be the Schubert classes on $E_6/D_5\cdot S^1$ with Weyl coordinates $\sigma[6], \sigma[2,4,5,6]$ respectively. Then*

$$A^*(E_6/D_5\cdot S^1) = \mathbb{Z}[y_1, y_4]/\langle r_9, r_{12}\rangle,$$

where

$$r_9 = 2y_1^9 + 3y_1 y_4^2 - 6y_1^5 y_4;$$
$$r_{12} = y_4^3 - 6y_1^4 y_4^2 + y_1^{12}.$$

Theorem 5 *Let y_1, y_5, y_9 be the Schubert classes on $E_7/E_6\cdot S^1$ with Weyl coordinates $\sigma[7], \sigma[2,4,5,6,7], \sigma[1,5,4,2,3,4,5,6,7]$ respectively. Then*

$$A^*(E_7/E_6\cdot S^1) = \mathbb{Z}[y_1, y_5, y_9]/\langle r_{10}, r_{14}, r_{18}\rangle,$$

where

$$r_{10} = y_5^2 - 2y_1 y_9;$$
$$r_{14} = 2y_5 y_9 - 9y_1^4 y_5^2 + 6y_1^9 y_5 - y_1^{14};$$
$$r_{18} = y_9^2 + 10y_1^3 y_5^3 - 9y_1^8 y_5^2 + 2y_1^{13} y_5.$$

Theorem 6 *Let y_1, y_4, y_6, y_9 be the Schubert classes on $E_7/D_6\cdot S^1$ with Weyl coordinates $\sigma[1], \sigma[2,4,3,1], \sigma[2,6,5,4,3,1], \sigma[3,4,2,7,6,5,4,3,1]$ respectively. Then*

$$A^*(E_7/D_6\cdot S^1) = \mathbb{Z}[y_1, y_4, y_6, y_9]/\langle r_9, r_{12}, r_{14}, r_{18}\rangle,$$

where

$$r_9 = 2y_9 + 3y_1 y_4^2 + 4y_1^3 y_6 + 2y_1^5 y_4 - 2y_1^9;$$
$$r_{12} = 3y_6^2 - y_4^3 - 3y_1^4 y_4^2 - 2y_1^6 y_6 + 2y_1^8 y_4;$$
$$r_{14} = 3y_4^2 y_6 + 3y_1^2 y_6^2 + 6y_1^2 y_4^3 + 6y_1^4 y_4 y_6 + 2y_1^5 y_9 - y_1^{14};$$
$$r_{18} = 5y_9^2 + 29y_6^3 - 24y_1^6 y_6^2 + 45y_1^2 y_4 y_6^2 + 2y_1^9 y_9.$$

Theorem 7 *Let y_1, y_6, y_{10}, y_{15} be the Schubert classes on $E_8/E_7\cdot S^1$ with Weyl coordinates $\sigma[8], \sigma[3,4,5,6,7,8], \sigma[1,5,4,2,3,4,5,6,7,8], \sigma[5,4,3,1,7,6,5,4,2,3,4,5,6,7,8]$ respectively. Then*

$$A^*(E_8/E_7\cdot S^1) = \mathbb{Z}[y_1, y_6, y_{10}, y_{15}]/\langle r_{15}, r_{20}, r_{24}, r_{30}\rangle,$$

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where

$$r_{15} = 2y_{15} - 16y_1^5 y_{10} - 10y_1^3 y_6^2 + 10y_1^9 y_6 - y_1^{15};$$
$$r_{20} = 3y_{10}^2 + 10y_1^2 y_6^3 + 18y_1^4 y_6 y_{10} - 2y_1^5 y_{15} - 8y_1^8 y_6^2 + 4y_1^{10} y_{10} - y_1^{14} y_6;$$
$$r_{24} = 5y_6^4 + 30y_1^2 y_6^2 y_{10} + 15y_1^4 y_{10}^2 - 2y_1^9 y_{15} - 5y_1^{12} y_6^2 + y_1^{14} y_{10};$$
$$r_{30} = y_{15}^2 - 8y_{10}^3 + y_6^5 - 2y_1^3 y_6^2 y_{15} + 3y_1^4 y_6 y_{10}^2 - 8y_1^5 y_{10} y_{15} + 6y_1^9 y_6 y_{15} - 9y_1^{10} y_{10}^2 - y_1^{12} y_6^3 - 2y_1^{14} y_6 y_{10} - 3y_1^{15} y_{15} + 8y_1^{20} y_{10} + y_1^{24} y_6 - y_1^{30}.$$

Traditionally, Schubert calculus deals with intersection theory on flag varieties. The algorithms in Sect. 4.3 and the proofs of Theorems 8–14 in Sect. 5 demonstrate how this calculation is extended to homogeneous spaces of other types.

This paper is arranged as follows. Section 2 contains a brief introduction to what we need from Schubert calculus; Section 3 develops some algebraic results concerning computation in the quotient of a polynomial ring. Resorting to the Gysin sequence of circle bundles, the relationship between cohomologies of a Grassmannian G/H and its allied space G/H_ω is formulated in Sect. 4. With these preliminaries, Theorems 1–7 (resp. Theorems 8–14) are established in a unified pattern in Sect. 6 (resp. Sect. 5).

Historically, the problem of computing the Chow ring of a flag variety (resp. the integral cohomology of a homogeneous space) has been studied by many authors. Comparisons between our method and the classical means are made in Sect. 7, where mistakes occurring in the earlier computations are corrected in Sect. 7.5.

Certain theoretical notions and results of this paper are also algorithmic in nature. Their effective computability is emphasized by referring to appropriate sections of [13], where intermediate data facilitating our calculations are given in detail. To make the present work self-contained, the most relevant data from [13] are summarized and tabulated in the proofs of Theorems 8–14 in Sect. 5.

2 Elements of Schubert Calculus

Assume throughout that the Lie group G under consideration is compact and 1-connected. Fix a maximal torus T in G and equip the Lie algebra $L(G)$ with an inner product $(\,,)$, so that the adjoint representation acts as isometries of $L(G)$. Let $\Phi = \{\beta_1, \ldots, \beta_n\} \subset L(T)$ be a *set of simple roots* of G [20, p. 47]. The *Cartan matrix* of G is $C = (c_{ij})_{n\times n}$, where

$$c_{ij} := 2(\beta_i, \beta_j)/(\beta_j, \beta_j), \quad 1 \le i, j \le n \text{ [20, p. 55]}.$$

We recall two algorithms "*Decomposition*" and "*L–R coefficients*" developed in [11]. The first presents the Weyl group of G by the minimized decompositions of its elements, in terms of which the Schubert varieties on G/H can be constructed. The second expands a polynomial in the Schubert classes as the linear combination of the Schubert basis.

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Solution to Weil problem

Example

The Chow rings $A^*(G/T)$ of the complete flag manifolds: $G = G_2, F_4, E_6, E_7$:

THEOREM 5.1. *For each exceptional Lie group G, the cohomology ring $H^*(G/T)$ has the following presentation:*

$$\begin{aligned}
&H^*(G_2/T) = \mathbb{Z}[\omega_1, \omega_2, y_3]/\langle \rho_2, r_3, r_6 \rangle, \text{ where} \\
&\rho_2 = 3\omega_1^2 - 3\omega_1\omega_2 + \omega_2^2; \\
&r_3 = 2y_3 - \omega_1^3; \\
&r_6 = y_3^2.
\end{aligned} \tag{5.1}$$

$$\begin{aligned}
&H^*(F_4/T) = \mathbb{Z}[\omega_1, \omega_2, \omega_3, \omega_4, y_3, y_4]/\langle \rho_2, \rho_4, r_3, r_6, r_8, r_{12} \rangle \text{ where} \\
&\rho_2 = c_2 - 4\omega_1^2; \\
&\rho_4 = 3y_4 + 2\omega_1 y_3 - c_4; \\
&r_3 = 2y_3 - \omega_1^3; \\
&r_6 = y_3^2 + 2c_6 - 3\omega_1^2 y_4; \\
&r_8 = 3y_4^2 - \omega_1^2 c_6; \\
&r_{12} = y_4^3 - c_6^2.
\end{aligned} \tag{5.2}$$

$$\begin{aligned}
&H^*(E_6/T) = \mathbb{Z}[\omega_1, \dots, \omega_6, y_3, y_4]/\langle \rho_2, \rho_3, \rho_4, \rho_5, r_6, r_8, r_9, r_{12} \rangle, \text{ where} \\
&\rho_2 = 4\omega_2^2 - c_2; \\
&\rho_3 = 2y_3 + 2\omega_2^3 - c_3; \\
&\rho_4 = 3y_4 + \omega_2^4 - c_4; \\
&\rho_5 = 2\omega_2^2 y_3 - \omega_2 c_4 + c_5; \\
&r_6 = y_3^2 - \omega_2 c_5 + 2c_6; \\
&r_8 = 3y_4^2 - 2c_5 y_3 - \omega_2^2 c_6 + \omega_2^3 c_5; \\
&r_9 = 2y_3 c_6 - \omega_2^3 c_6; \\
&r_{12} = y_4^3 - c_6^2.
\end{aligned} \tag{5.3}$$

$$\begin{aligned}
&H^*(E_7/T) = \mathbb{Z}[\omega_1, \dots, \omega_7, y_3, y_4, y_5, y_9]/\langle \rho_2, \rho_3, \rho_4, \rho_5, r_i \rangle, \\
&\quad \text{where } i \in \{6, 8, 9, 10, 12, 14, 18\} \text{ and where} \\
&\rho_2 = 4\omega_2^2 - c_2; \\
&\rho_3 = 2y_3 + 2\omega_2^3 - c_3; \\
&\rho_4 = 3y_4 + \omega_2^4 - c_4; \\
&\rho_5 = 2y_5 - 2\omega_2^2 y_3 + \omega_2 c_4 - c_5; \\
&r_6 = y_3^2 - \omega_2 c_5 + 2c_6; \\
&r_8 = 3y_4^2 + 2y_3 y_5 - 2y_3 c_5 + 2\omega_2 c_7 - \omega_2^2 c_6 + \omega_2^3 c_5; \\
&r_9 = 2y_9 + 2y_4 y_5 - 2y_3 c_6 - \omega_2^2 c_7 + \omega_2^3 c_6; \\
&r_{10} = y_5^2 - 2y_3 c_7 + \omega_2^3 c_7; \\
&r_{12} = y_4^3 - 4y_5 c_7 - c_6^2 - 2y_3 y_9 - 2y_3 y_4 y_5 + 2\omega_2 y_5 c_6 + 3\omega_2 y_4 c_7 + c_5 c_7; \\
&r_{14} = c_7^2 - 2y_5 y_9 + 2y_3 y_4 c_7 - \omega_2^3 y_4 c_7; \\
&r_{18} = y_9^2 + 2y_5 c_6 c_7 - y_4 c_7^2 - 2y_4 y_5 y_9 + 2y_3 y_5^3 - 5\omega_2 y_5^2 c_7.
\end{aligned} \tag{5.4}$$

Solution to Weil problem

For a compact Lie group G and a closed subgroup H, the quotient space G/H is smooth manifold, called a **homogeneous space** of G.

A classical problem in topology, starting with the works of H. Cartan, A. Borel, P. Baum, H. Toda (and so forth), is to express the cohomology ring $H^*(G/H)$ by a minimal system of **explicit generators** and **relations**.
To study the problem, various spectral sequence techniques were developed for certain fibrations associated with G/H, such as

- Leray-Serre spectral sequences;
- Eilenberg-Moore spectral sequences;
- Bockstein spectral sequences;
- $\cdots$.

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- Leray-Serre spectral sequences;
- Eilenberg-Moore spectral sequences;
- Bockstein spectral sequences;
- $\cdots$.

But the calculation encounters the same difficulties when the cohomology $H^*(G)$ contains torsion elements, in particular, if G is an exceptional Lie group.

Solution to Weil problem (Duan and Zhao)

Example

Schubert calculus makes the cohomology theory of homogeneous spaces appearing in a new light: $(G,H)=(E_6,SU(6)),(E_7,Spin(12)),(E_8,E_7)$

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Table 4 The ring structure of $H^*(E_6/A_5)$

Nontrivial H^i	Basis elements	Relations
$H^6\cong\mathbb{Z}$	$y_{3,2}$	
$H^8\cong\mathbb{Z}$	$y_{4,3}$	
$H^{12}\cong\mathbb{Z}$	$y_{6,1}$	$-2y_{6,1}=y_{3,2}^2$
$H^{14}\cong\mathbb{Z}$	$y_{7,1}$	$\bar y_{3,2}\bar y_{4,3}$
$H^{16}\cong\mathbb{Z}_3$	$y_{8,1}$	$\bar y_{4,3}^2$
$H^{18}\cong\mathbb{Z}_2$	$y_{9,1}$	$\bar y_{3,2}\bar y_{6,1}$
$H^{20}\cong\mathbb{Z}$	$y_{10,1}$	$-\bar y_{4,3}\bar y_{6,1}$
$H^{22}\cong\mathbb{Z}_3$	$y_{11,1}$	$\bar y_{4,3}^2\bar y_{3,2}$
$H^{26}\cong\mathbb{Z}_2$	$y_{13,2}$	$\bar y_{3,2}\bar y_{4,3}\bar y_{6,1}$
$H^{28}\cong\mathbb{Z}_3$	$y_{14,1}$	$-\bar y_{4,3}^2\bar y_{6,1}$
$H^{23}\cong\mathbb{Z}$	$d_{23}=\beta^{-1}(s_{11,1}-s_{11,2}-s_{11,3}+s_{11,4}-s_{11,5}+s_{11,6})$	
$H^{29}\cong\mathbb{Z}$	$d_{29}=\beta^{-1}(-s_{14,1}+s_{14,2}+s_{14,4}-s_{14,5})$	$2d_{29}=\pm y_{3,2}d_{23}$
$H^{31}\cong\mathbb{Z}$	$d_{31}=\beta^{-1}(s_{15,1}-2s_{15,2}+s_{15,3}-s_{15,4})$	$\pm y_{4,3}d_{23}$
$H^{35}\cong\mathbb{Z}$	$d_{35}=\beta^{-1}(-s_{17,1}+s_{17,2}+s_{17,3})$	$\pm y_{6,1}d_{23}$
$H^{37}\cong\mathbb{Z}$	$d_{37}=\beta^{-1}(-s_{18,1}+s_{18,2})$	$\pm y_{4,3}d_{29}$
$H^{43}\cong\mathbb{Z}$	$d_{43}=\beta^{-1}(s_{22,1})$	$\pm y_{4,3}y_{6,1}d_{23}$

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Table 7 The ring structure of $H^*(E_7/D_6)$

Nontrivial H^i	Basis elements	Relations
$H^8\cong\mathbb{Z}$	$\bar y_{4,1}$	
$H^{12}\cong\mathbb{Z}$	$\bar y_{6,1}$	
$H^{16}\cong\mathbb{Z}$	$\bar y_{8,1}$	$\bar y_{4,1}^2$
$H^{18}\cong\mathbb{Z}_2$	$\bar y_{9,2}$	
$H^{20}\cong\mathbb{Z}$	$\bar y_{10,1}$	$\bar y_{4,1}\bar y_{6,1}$
$H^{24}\cong\mathbb{Z}$	$\bar y_{12,2}$	$\bar y_{12,2}=\bar y_{6,1}^2;\ 3\bar y_{12,2}=\bar y_{4,1}^3$
$H^{26}\cong\mathbb{Z}_2$	$\bar y_{13,1}$	$\bar y_{4,1}\bar y_{9,2}$
$H^{28}\cong\mathbb{Z}_3$	$\bar y_{14,1}$	$-\bar y_{4,1}^2\bar y_{6,1}$
$H^{30}\cong\mathbb{Z}_2$	$\bar y_{15,1}$	$\bar y_{6,1}\bar y_{9,2}$
$H^{32}\cong\mathbb{Z}$	$\bar y_{16,1}$	$\bar y_{4,1}\bar y_{6,1}^2$
$H^{34}\cong\mathbb{Z}_2$	$\bar y_{17,2}$	$\bar y_{4,1}^2\bar y_{9,2}$
$H^{38}\cong\mathbb{Z}_2$	$\bar y_{19,2}$	$\bar y_{4,1}\bar y_{6,1}\bar y_{9,2}$
$H^{40}\cong\mathbb{Z}_3$	$\bar y_{20,1}$	$\bar y_{4,1}^2\bar y_{6,1}^2$
$H^{42}\cong\mathbb{Z}_2$	$\bar y_{21,3}$	$\bar y_{4,1}^3\bar y_{9,2}$
$H^{50}\cong\mathbb{Z}_2$	$\bar y_{25,1}$	$\bar y_{4,1}^4\bar y_{9,2}$
$H^{35}\cong\mathbb{Z}$	$d_{35}=\beta^{-1}(s_{17,1}-s_{17,2}-s_{17,3}+s_{17,4}-s_{17,5}+s_{17,6}-s_{17,7})$	
$H^{43}\cong\mathbb{Z}$	$\beta^{-1}(s_{21,1}-2s_{21,2}+s_{21,3}-3s_{21,4}+2s_{21,5}-s_{21,6})$	$\pm\bar y_{4,1}d_{35}$
$H^{47}\cong\mathbb{Z}$	$\beta^{-1}(2s_{23,1}-s_{23,2}+s_{23,3}-s_{23,4}+s_{23,5})$	$\pm\bar y_{6,1}d_{35}$
$H^{51}\cong\mathbb{Z}$	$d_{51}=\beta^{-1}(s_{25,1}-s_{25,2}-s_{25,4})$	$3d_{51}=\pm\bar y_{4,1}^2d_{35}$
$H^{55}\cong\mathbb{Z}$	$\beta^{-1}(s_{27,1}+s_{27,2}-s_{27,3})$	$\pm\bar y_{4,1}\bar y_{6,1}d_{35}$
$H^{59}\cong\mathbb{Z}$	$\beta^{-1}(s_{29,1}-s_{29,2})$	$\pm\bar y_{6,1}^2d_{35},\ \pm\bar y_{4,1}d_{51}$
$H^{67}\cong\mathbb{Z}$	$\beta^{-1}(s_{33,1})$	$\bar y_{4,1}\bar y_{6,1}^2d_{35}=\pm\bar y_{4,1}^2d_{51}$

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Table 8 The ring structure of $H^*(E_8/E_7)$

Nontrivial H^i	Basis elements	Relations
$H^{12}\cong\mathbb{Z}$	$\bar y_{6,2}$	
$H^{20}\cong\mathbb{Z}$	$\bar y_{10,1}$	
$H^{24}\cong\mathbb{Z}$	$\bar y_{12,1}$	$\pm\bar y_{6,2}^2$
$H^{30}\cong\mathbb{Z}_2$	$\bar y_{15,4}$	
$H^{32}\cong\mathbb{Z}$	$\bar y_{16,1}$	$\pm\bar y_{6,2}\bar y_{10,1}$
$H^{36}\cong\mathbb{Z}$	$\bar y_{18,2}$	$\pm\bar y_{6,2}^3$
$H^{40}\cong\mathbb{Z}_3$	$\bar y_{20,1}$	$\pm\bar y_{10,1}^2$
$H^{42}\cong\mathbb{Z}_2$	$\bar y_{21,3}$	$\pm\bar y_{6,2}\bar y_{15,4}$
$H^{44}\cong\mathbb{Z}$	$\bar y_{22,1}$	$\pm\bar y_{6,2}^2\bar y_{10,1}$
$H^{48}\cong\mathbb{Z}_5$	$\bar y_{24,1}$	$\pm\bar y_{6,2}^4$
$H^{50}\cong\mathbb{Z}_2$	$\bar y_{25,1}$	$\pm\bar y_{10,1}\bar y_{15,4}$
$H^{52}\cong\mathbb{Z}_3$	$\bar y_{26,1}$	$\pm\bar y_{6,2}\bar y_{10,1}^2$
$H^{54}\cong\mathbb{Z}_2$	$\bar y_{27,1}$	$\pm\bar y_{6,2}^2\bar y_{15,4}$
$H^{56}\cong\mathbb{Z}$	$\bar y_{28,1}$	$\pm\bar y_{6,2}^3\bar y_{10,1}$
$H^{62}\cong\mathbb{Z}_2$	$\bar y_{31,2}$	$\pm\bar y_{6,2}\bar y_{10,1}\bar y_{15,4}$
$H^{64}\cong\mathbb{Z}_3$	$\bar y_{32,1}$	$\pm\bar y_{6,2}^2\bar y_{10,1}^2$
$H^{66}\cong\mathbb{Z}_2$	$\bar y_{33,3}$	$\pm\bar y_{6,2}^3\bar y_{15,4}$
$H^{68}\cong\mathbb{Z}_5$	$\bar y_{34,1}$	$\pm\bar y_{6,2}^4\bar y_{10,1}$
$H^{74}\cong\mathbb{Z}_2$	$\bar y_{37,2}$	$\pm\bar y_{6,2}^2\bar y_{10,1}\bar y_{15,4}$
$H^{76}\cong\mathbb{Z}_3$	$\bar y_{38,1}$	$\pm\bar y_{6,2}^3\bar y_{10,1}^2$
$H^{86}\cong\mathbb{Z}_2$	$\bar y_{43,1}$	$\pm\bar y_{6,2}^3\bar y_{10,1}^2\bar y_{15,4}$
$H^{59}\cong\mathbb{Z}$	$d_{59}=\beta^{-1}(s_{29,1}-s_{29,2}-s_{29,3}+s_{29,4}-s_{29,5}+s_{29,6}-s_{29,7}+s_{29,8})$	
$H^{71}\cong\mathbb{Z}$	$\beta^{-1}(2s_{35,1}-3s_{35,2}-s_{35,3}+s_{35,4}+s_{35,5}-s_{35,6}+s_{35,7})$	$\pm\bar y_{6,2}d_{59}$
$H^{79}\cong\mathbb{Z}$	$\beta^{-1}(2s_{39,1}-s_{39,2}-s_{39,3}-s_{39,4}+s_{39,5}-2s_{39,6})$	$\pm\bar y_{10,1}d_{59}$
$H^{83}\cong\mathbb{Z}$	$\beta^{-1}(2s_{41,1}-s_{41,2}+s_{41,3}-s_{41,4}+s_{41,5})$	$\pm\bar y_{6,2}d_{59}$
$H^{91}\cong\mathbb{Z}$	$\beta^{-1}(s_{45,1}-s_{45,2}-s_{45,3}+s_{45,4})$	$\pm\bar y_{6,2}\bar y_{10,1}d_{59}$
$H^{95}\cong\mathbb{Z}$	$\beta^{-1}(s_{47,1}-s_{47,2}+s_{47,3})$	$\pm\bar y_{6,2}^3d_{59}$
$H^{103}\cong\mathbb{Z}$	$\beta^{-1}(-s_{51,1}+s_{51,2})$	$\pm\bar y_{6,2}^2\bar y_{10,1}d_{59}$

References

In problem 15 Hilbert asked for a rigorous foundation of Schubert calculus.

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Baker (1933) and Manin (1969) commented that, to secure "the foundation of a calculus" it suffices to decide:

- The objects to be calculated;
- The rule of the calculation.

1. Baker, H. F. Principles of geometry, Cambridge: Univ. Press, 1933.
2. Manin, Ju. I. On Hilbert's fifteenth problem, Izdat. "Nauka", Moscow, 1969.

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As for the case of Schubert calculus, the foundation consists of two results:

- **The basis theorem of Schubert calculus** tells that the objects to be calculated are the "Schubert symbols", or "Schubert varieties";
- **Our characteristic formula (or intersection formula)** provides an effective rule performing the calculation.

References

This talk is based on the following survey papers on Problem 15:

- Duan, Zhao, Schubert calculus and Intersection theory of Flag manifolds, Russian Math. Surveys, 77 (in Russia, Uspekhi Mat. Nauk, 77), 2022;
- Duan, Zhao, Make Schubert calculus rigorous (in Chinese). Sci Sin Math, 2022.
- Duan, Zhao, On Schubert's Problem of Characteristics, In: Schubert Calculus and Its Applications in Combinatorics and Representation Theory, Springer proceedings in Mathematics and Statistics, 332, 2020.

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where we have concluded that "Problem 15 has been solved satisfactorily".

In particular, the following tasks have been accomplished:

- **Schubert (1870), Severi (1916)**: "The problem of characteristics is the fundamental one of the enumerative geometry";
- **Weil (1963)**: "The classical Schubert calculus amounts to the determination of intersection theory of flag manifolds."

Thanks you so much for your attention!

Computational aspects:

- Data mining: Cartan matrix $\Rightarrow$ Characteristic numbers
- Data Processing: Characteristic numbers $\Rightarrow$ the Chow rings $A^*(G/P)$

Computational aspects:

- Data mining: Cartan matrix $\Rightarrow$ Characteristic numbers
- Data Processing: Characteristic numbers $\Rightarrow$ the Chow rings $A^*(G/P)$

Question

Can AI be helpful to speed up the computation?

Van Der Waerden's paper

Topologische Begründung des Kalküls der abzählenden Geometrie.

Von

Bartel L. van der Waerden in Groningen (Niederlande).

§ 1.

Einleitung.

Eines der Pariser Probleme Hilberts[1]) lautet: „Eine strenge Begründung des Schubertschen Abzählungskalküls".

In früheren Arbeiten[2]) habe ich gesucht darzutun, daß das Kernproblem der abzählenden Geometrie besteht in der Aufstellung einer brauchbaren Definition der „Multiplizitäten" oder der Vielfachheiten, mit denen die Lösungen eines algebraisch-geometrischen Problems gezählt werden müssen, damit das „Prinzip der Erhaltung der Anzahl" für diese Lösungen bei jeder Spezialisierung der Daten des Problems gelte. In der Arbeit W_1 habe ich gezeigt, daß man bei jedem Problem, dessen Gleichungen homogen in den Unbekannten und rational in einigen Parametern sind, die Lösungen für jede spezielle Parameterzahl in einer und nur einer Weise mit solchen Vielfachheiten versehen kann, daß Anzahl und algebraische Eigenschaften der Lösungen bei diesen Parameterspezialisierungen erhalten bleiben, und daß bei allgemeiner Parameterwahl die Multiplizitäten gleich 1 sind. Damit war eine implizite Definition der Multiplizitäten gegeben, aber noch kein brauchbares Mittel, diese in vorliegenden Fällen (außer den allereinfachsten) wirklich zu bestimmen. Eine besondere Schwierigkeit bei der Anwendung war noch, daß mit der Möglichkeit von „Lösungen mit der

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[1]) D. Hilbert, Mathematische Probleme, Gött. Nachr. 1900, S. 253.

[2]) B. L. v. d. Waerden, Diss. Amsterdam 1926. Der Multiplizitätsbegriff der alge-

Serre's elegant formula for the intersection multiplicites

subvarieties in X that intersect properly (i.e. the codimension of $Y \cap Z$ is equal to the sum of the codimensions of Y and Z). Each component W of the intersection $Y \cap Z$ is ascribed some positive integer $i(Y, Z; W)$, which is the local multiplicity of the intersection. There are several definitions of $i(Y, Z; W)$, for example, Serre's Tor-formula:

$$i(Y, Z; W) = \sum_{k\geq} 0(-1)^k l(\operatorname{Tor}_k^A(A/\mathfrak{a}, A/\mathfrak{b})),$$

where A is the local ring $\mathcal{O}_{X,W}$, $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of Y and Z, and l is the length of the A-module. After this, one puts

$$Y \cdot Z = \sum_W i(Y, Z; W) \cdot W,$$

where W runs through the irreducible components of $Y \cap Z$.