Square-tiled surfaces and interval exchanges: geometry, dynamics, combinatorics and applications

Lecture 5. Teichmüller geodesic flow and an idea of renormalization.

Anton Zorich
University Paris Cité

YMSC, Tsinghua University, November 1, 2022

Diffeomorphisms of surfaces

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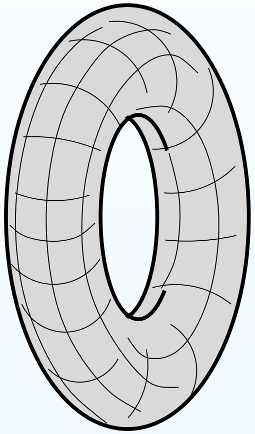
Observation 1. *Surfaces can wrap around themselves.*

Cut a torus along a horizontal circle.



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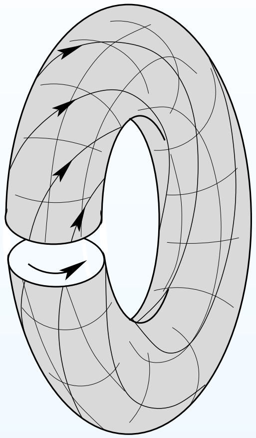
Dehn twist twists progressively horizontal circles up to a complete turn on the opposite boundary component of the cylinder and then identifies the components.



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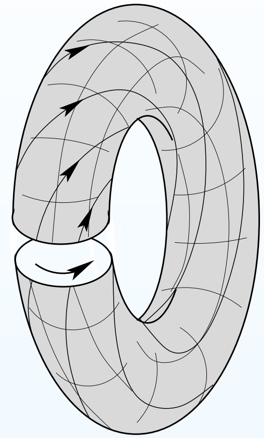

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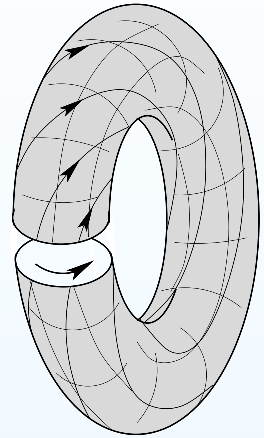

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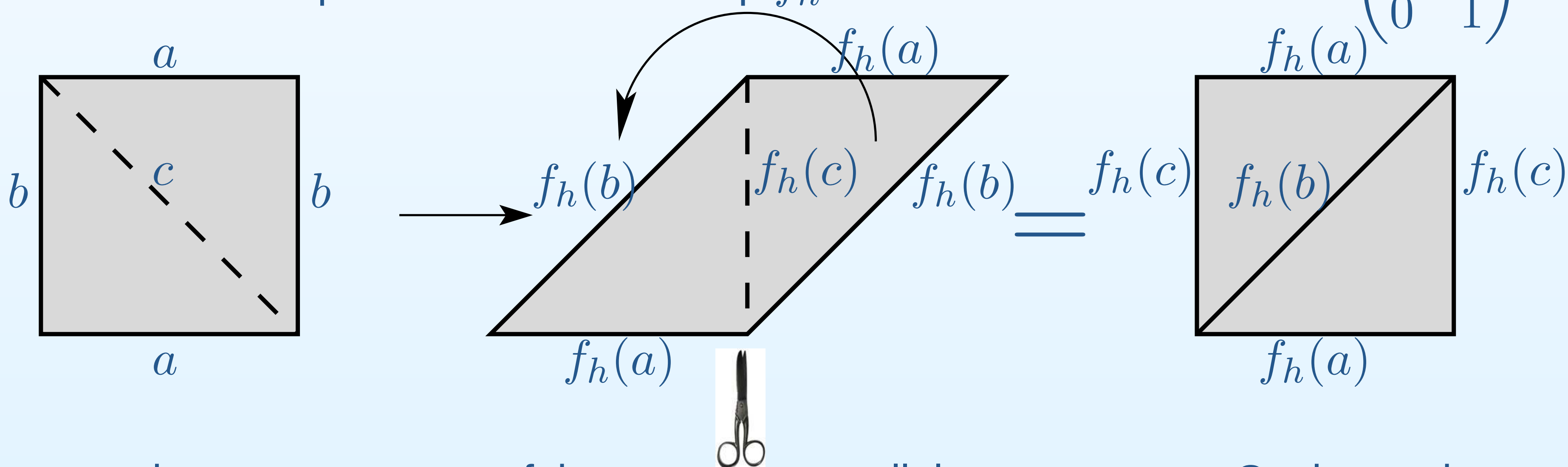

It maps the square pattern of the torus to a parallelogram pattern. Cutting and pasting appropriately we can transform the new pattern to the initial square.

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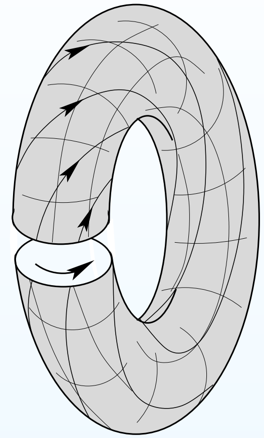

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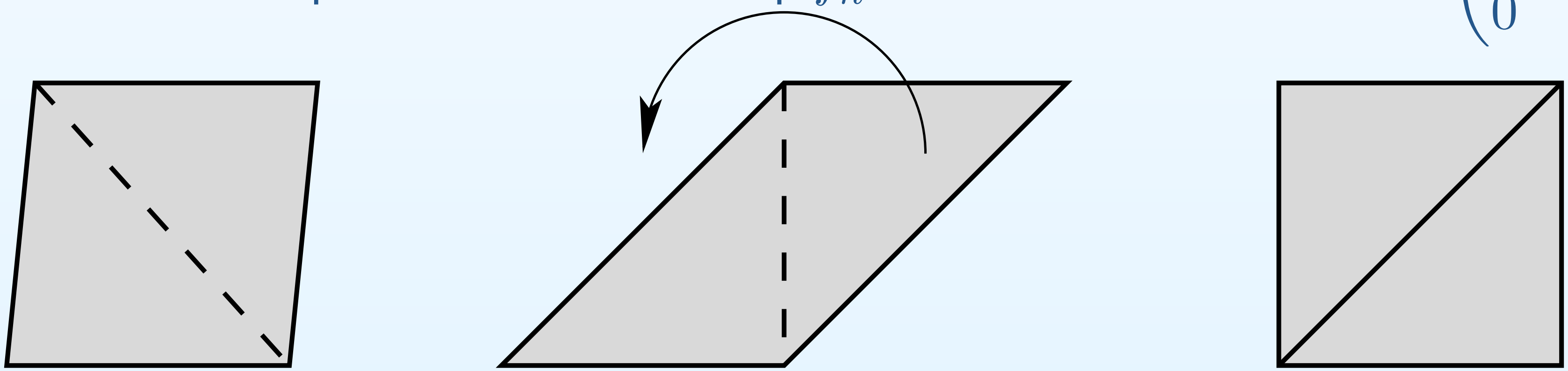

Changing the slope of the parallelogram pattern progressively we get a *closed path* in the space of flat tori.

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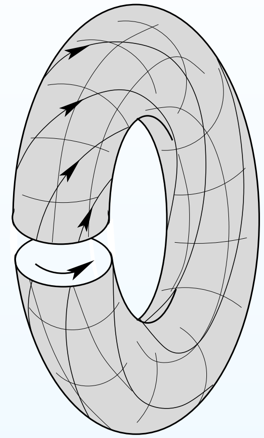

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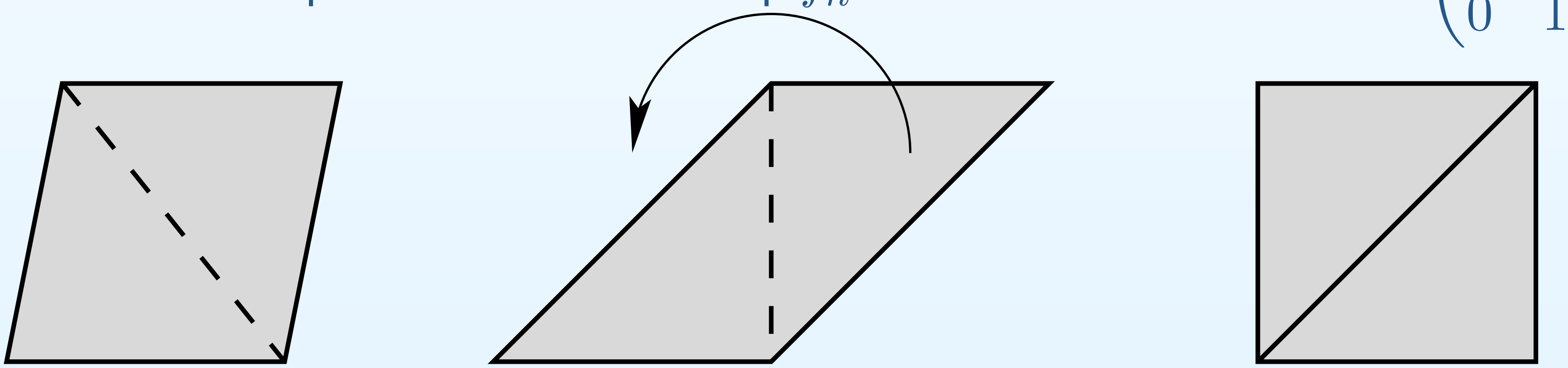

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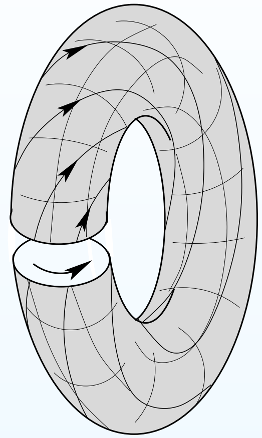

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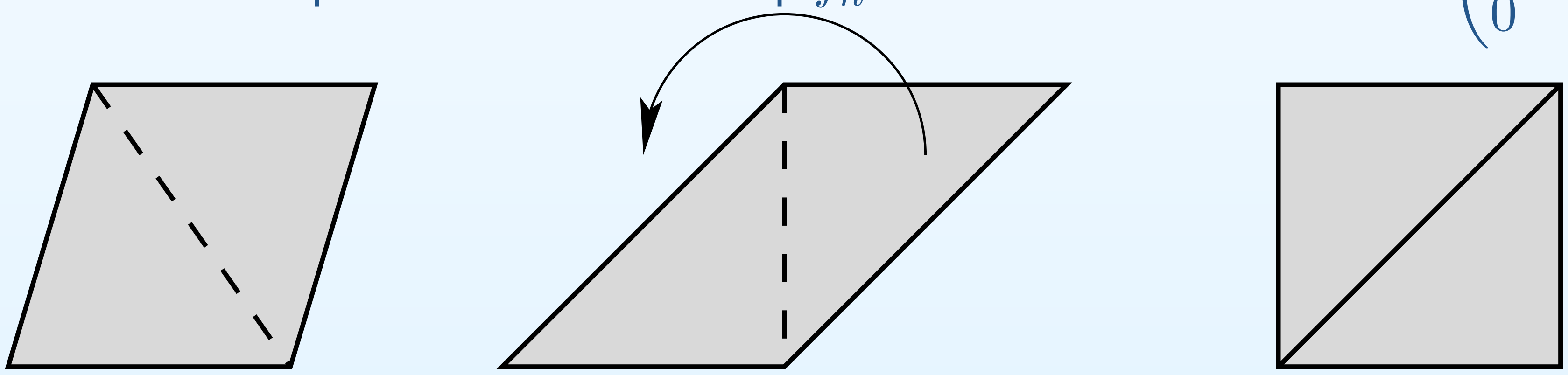

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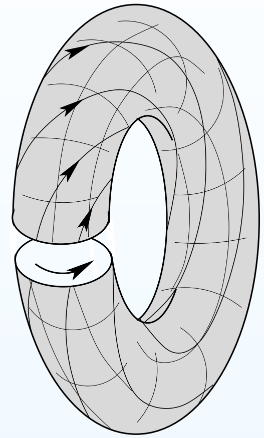

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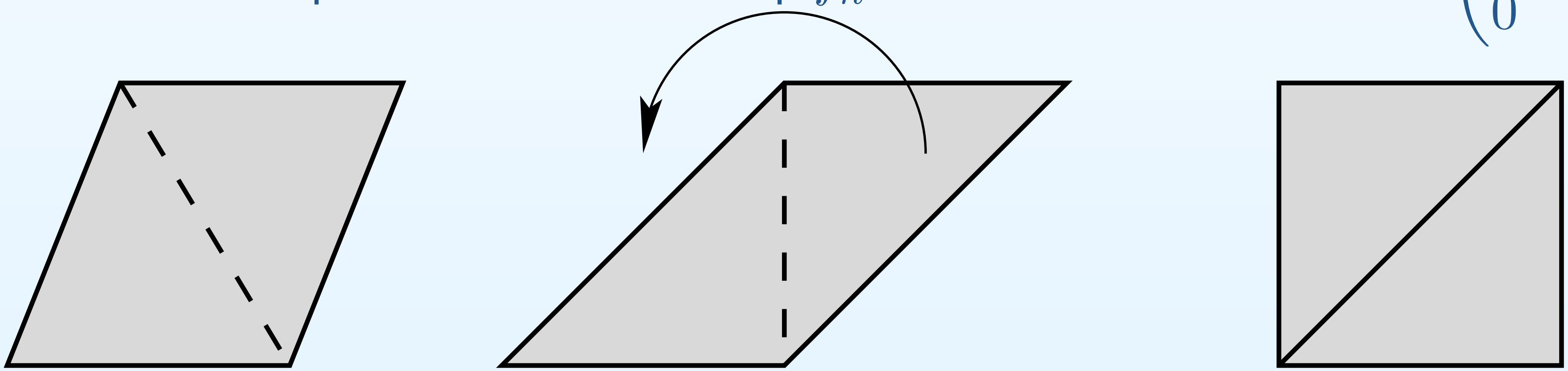

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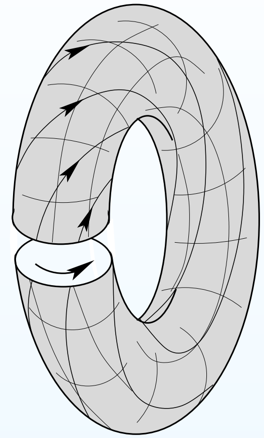

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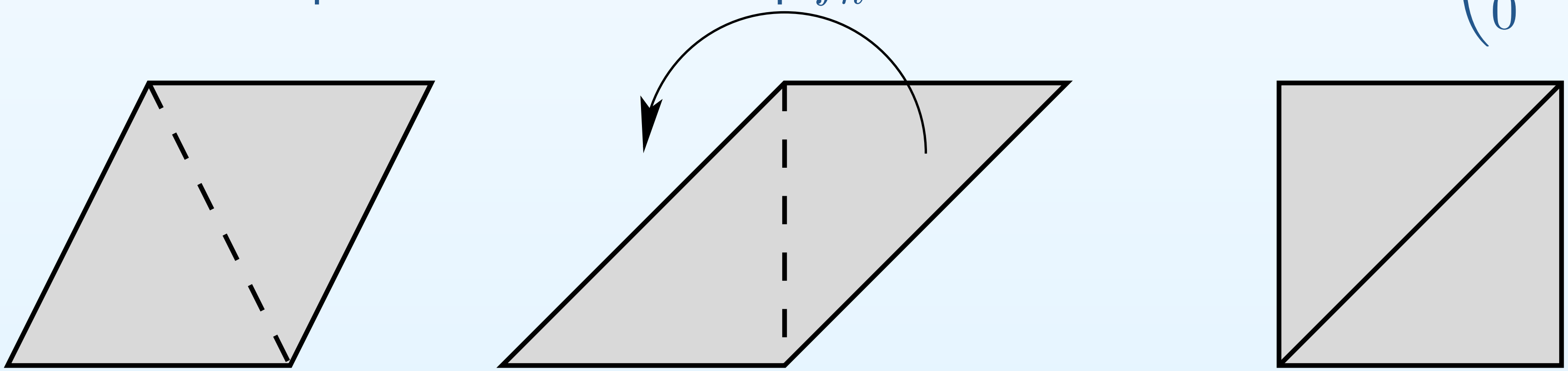

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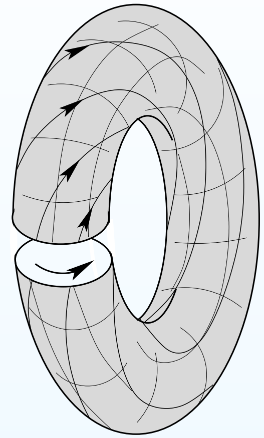

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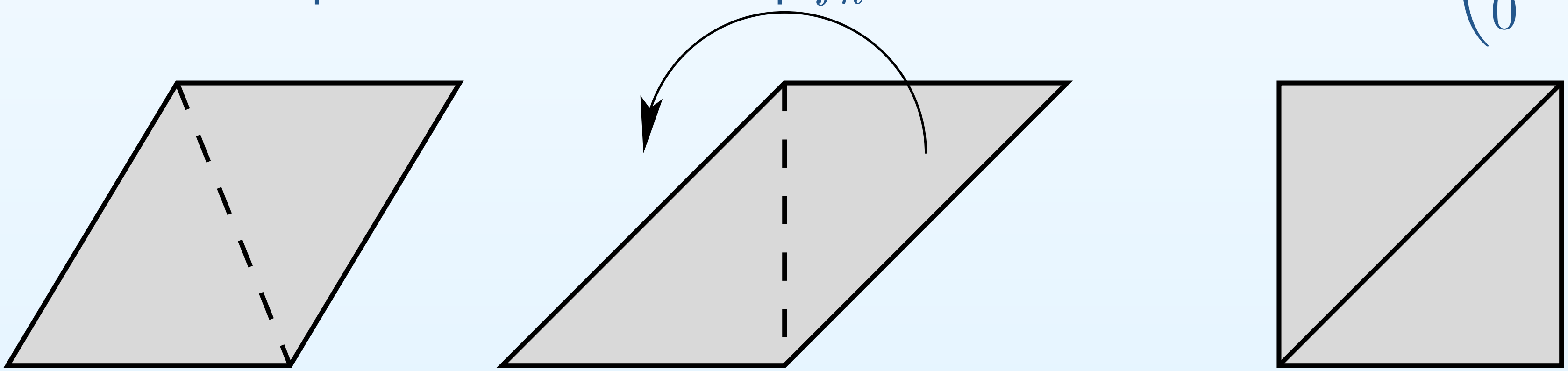

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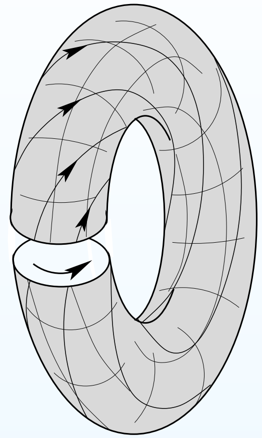

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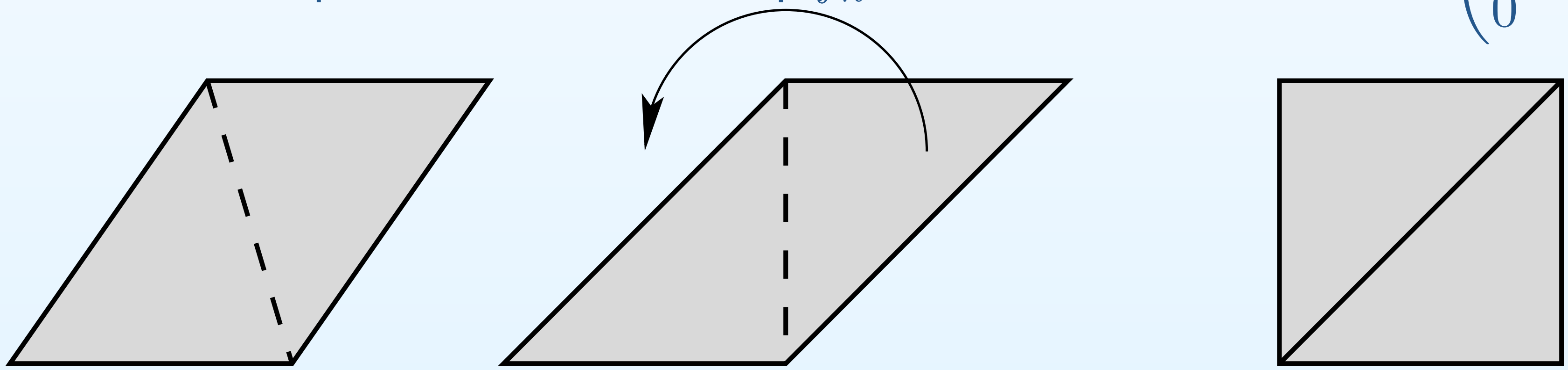

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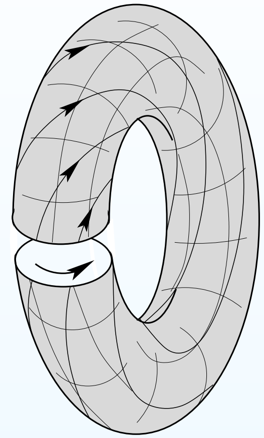

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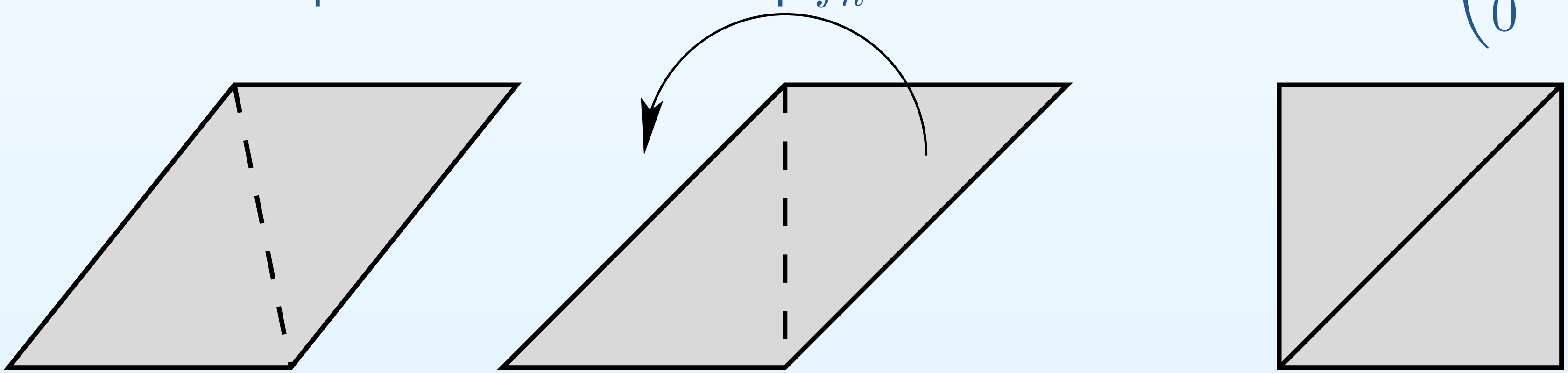

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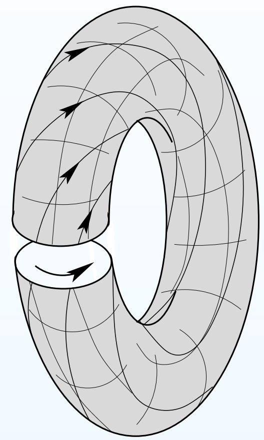

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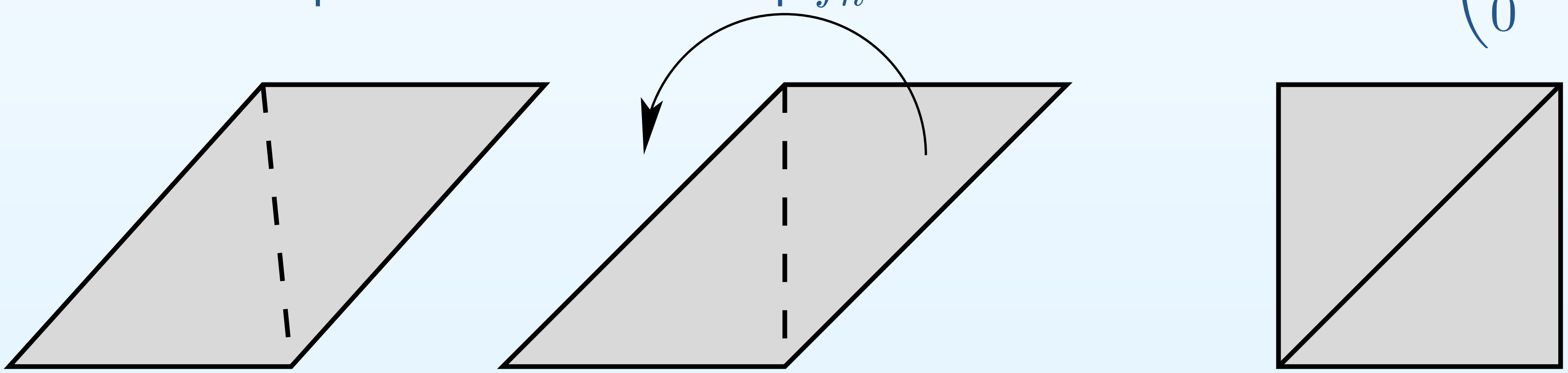

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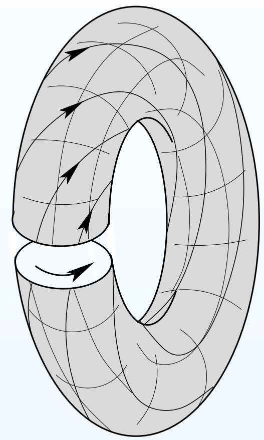

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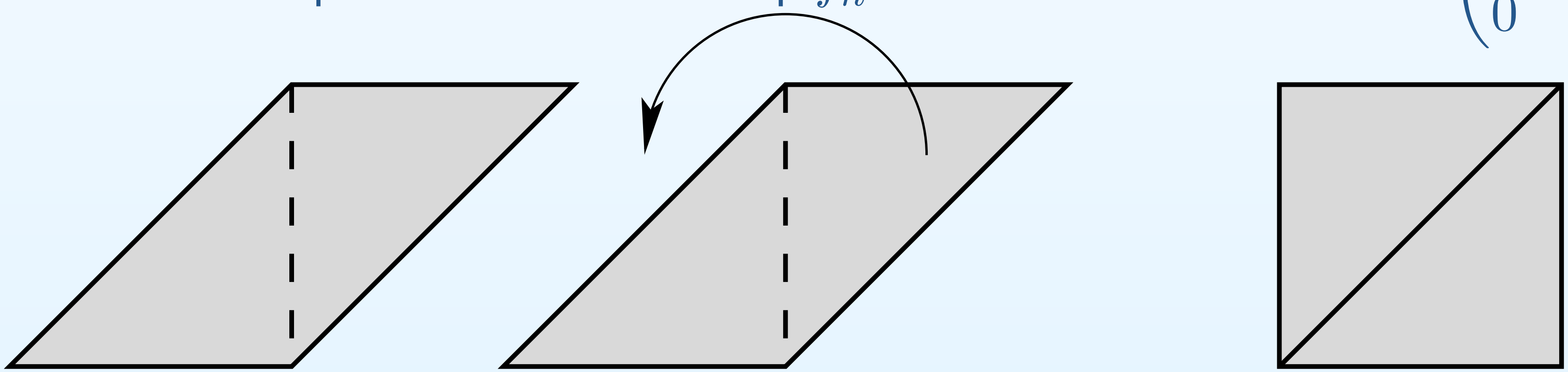

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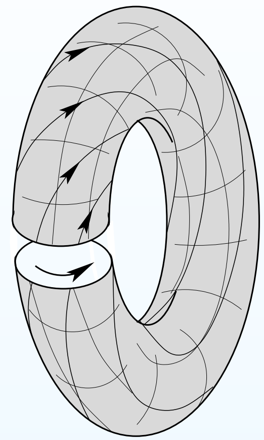

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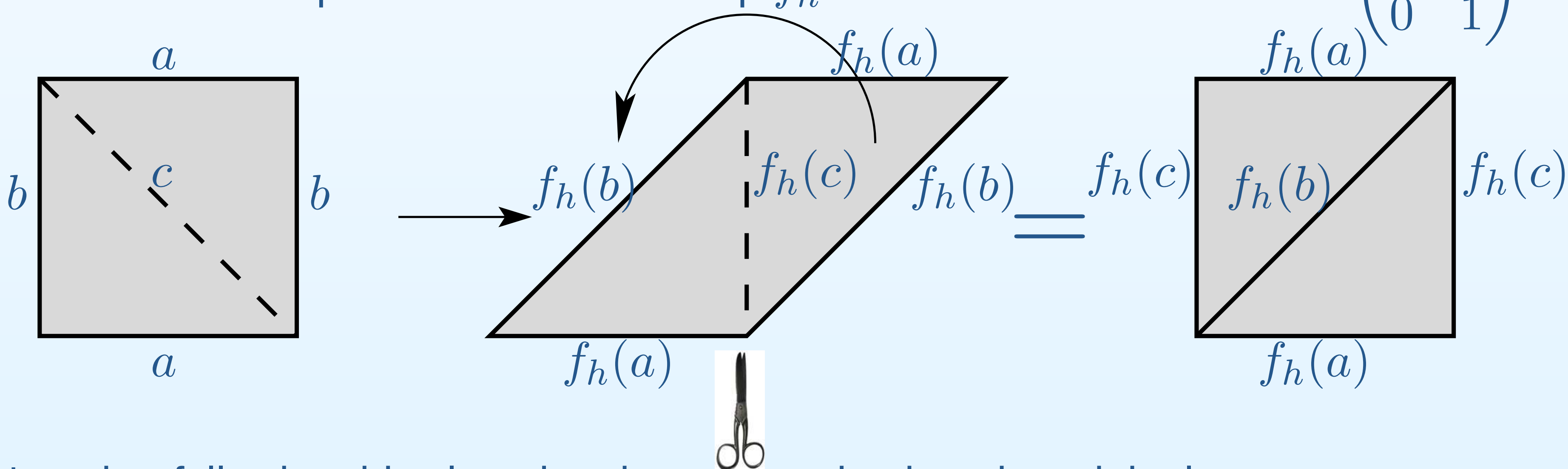
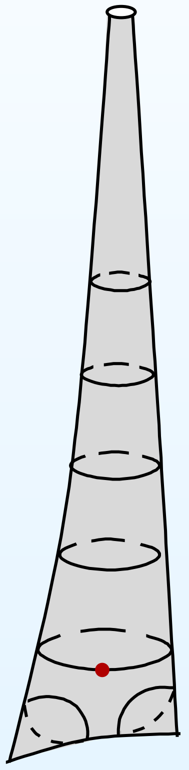
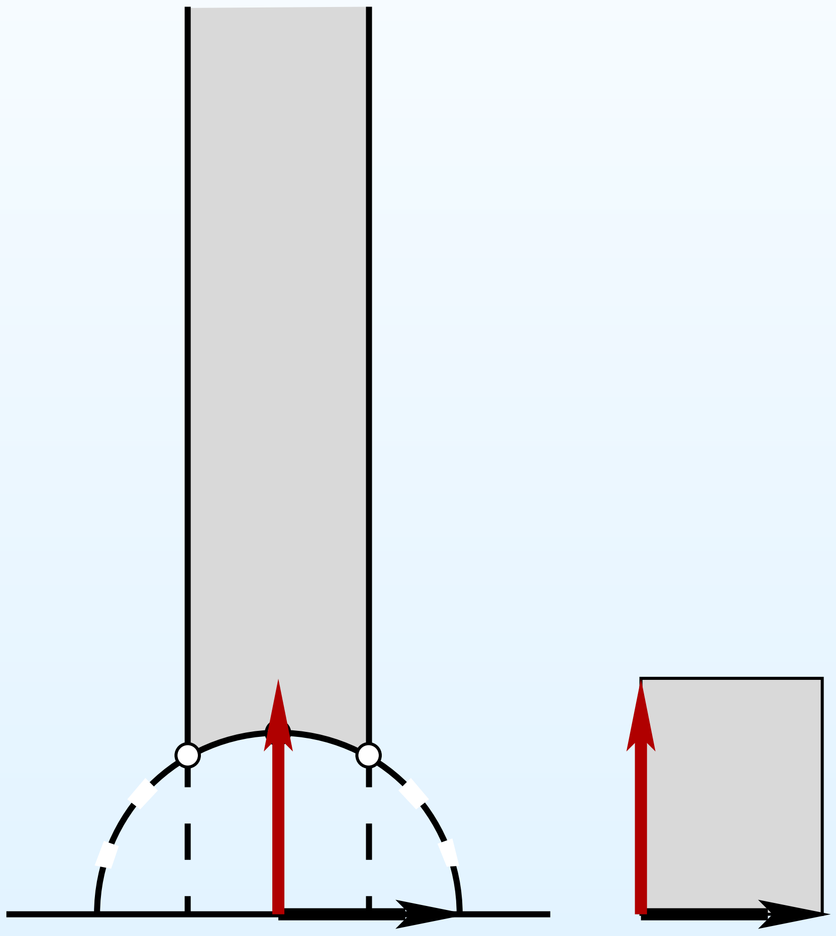


Note that following this closed path we come back to the original square torus having twisted the homology!

Closed horocycle in the moduli space of tori

Projection of a similar closed orbit of the *horocyclic flow* $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ to the moduli space of flat tori.





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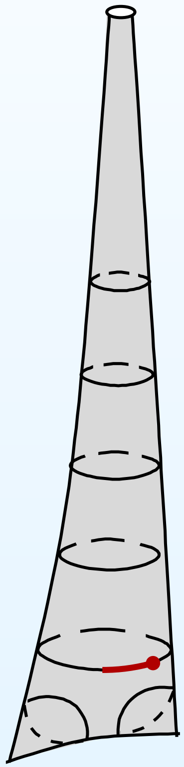

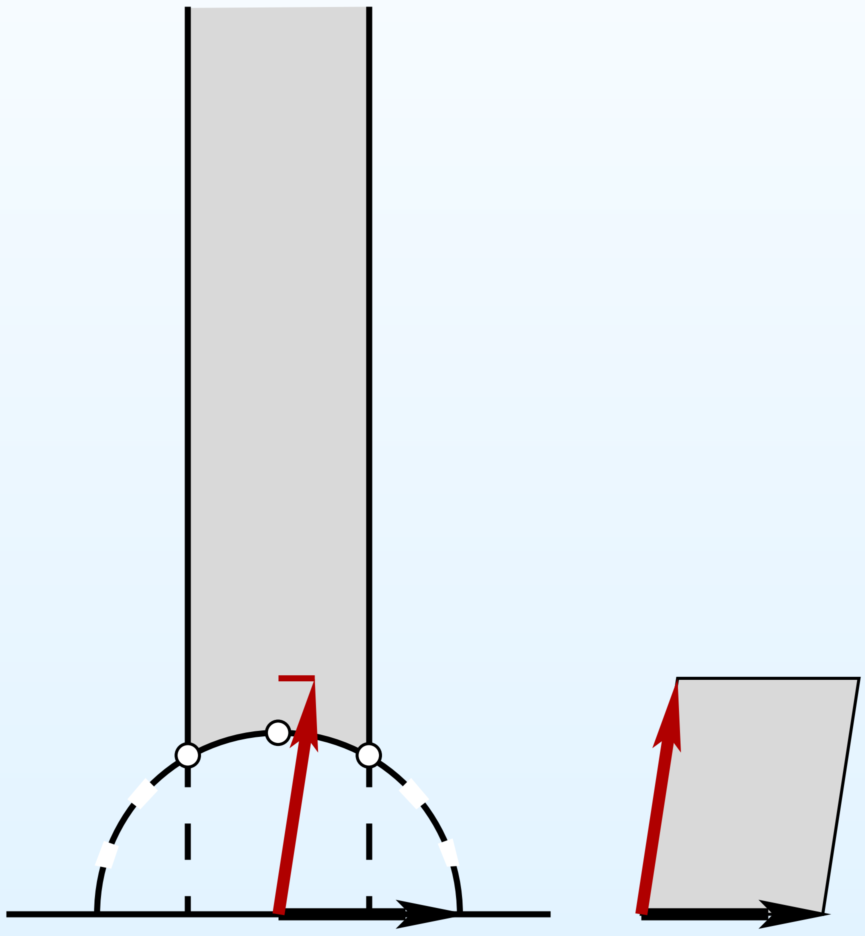

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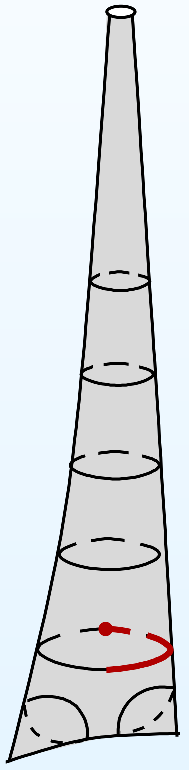

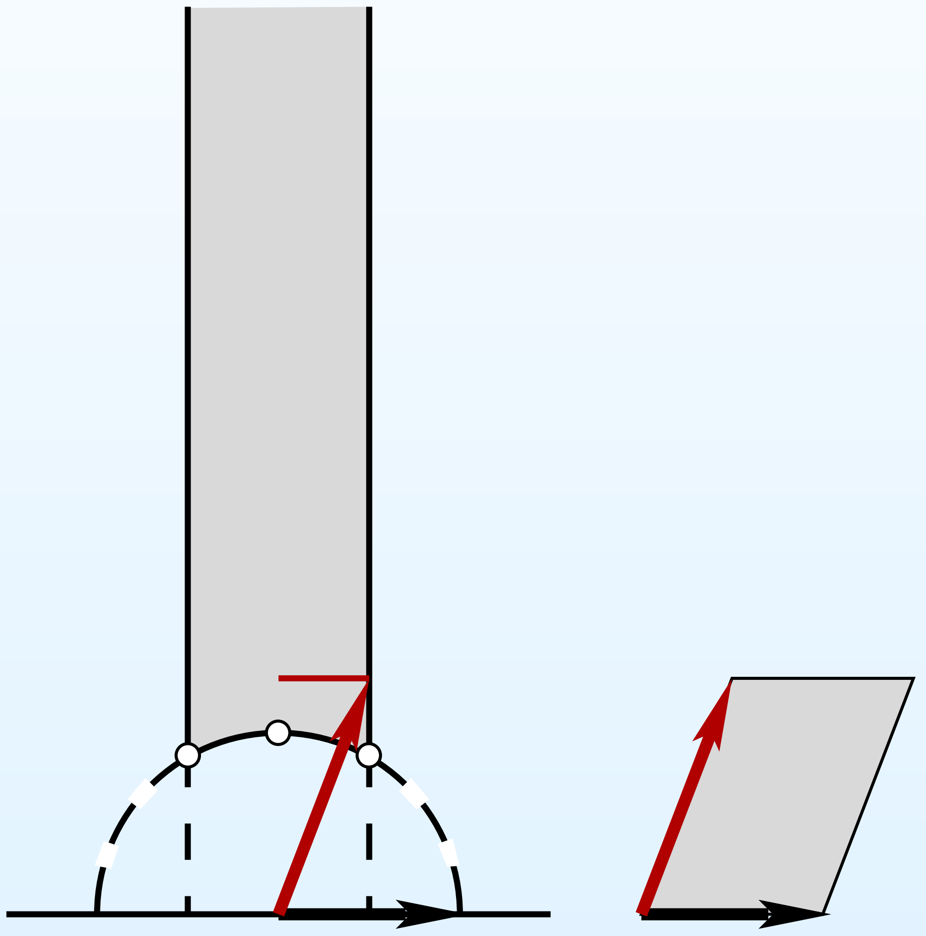

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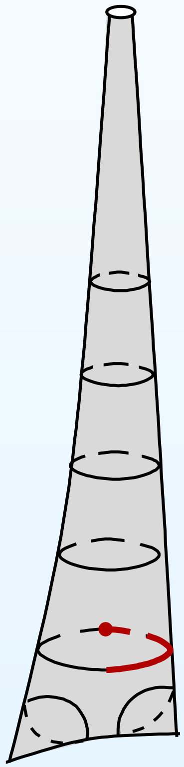

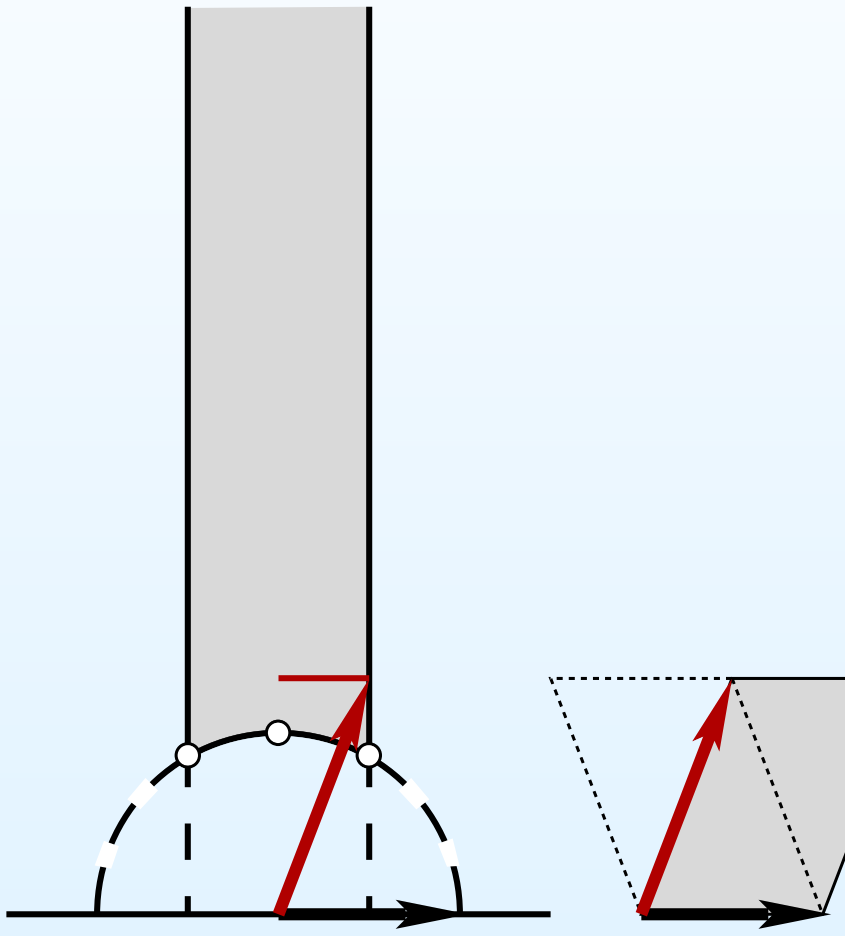

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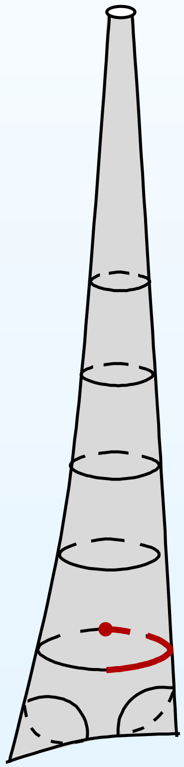

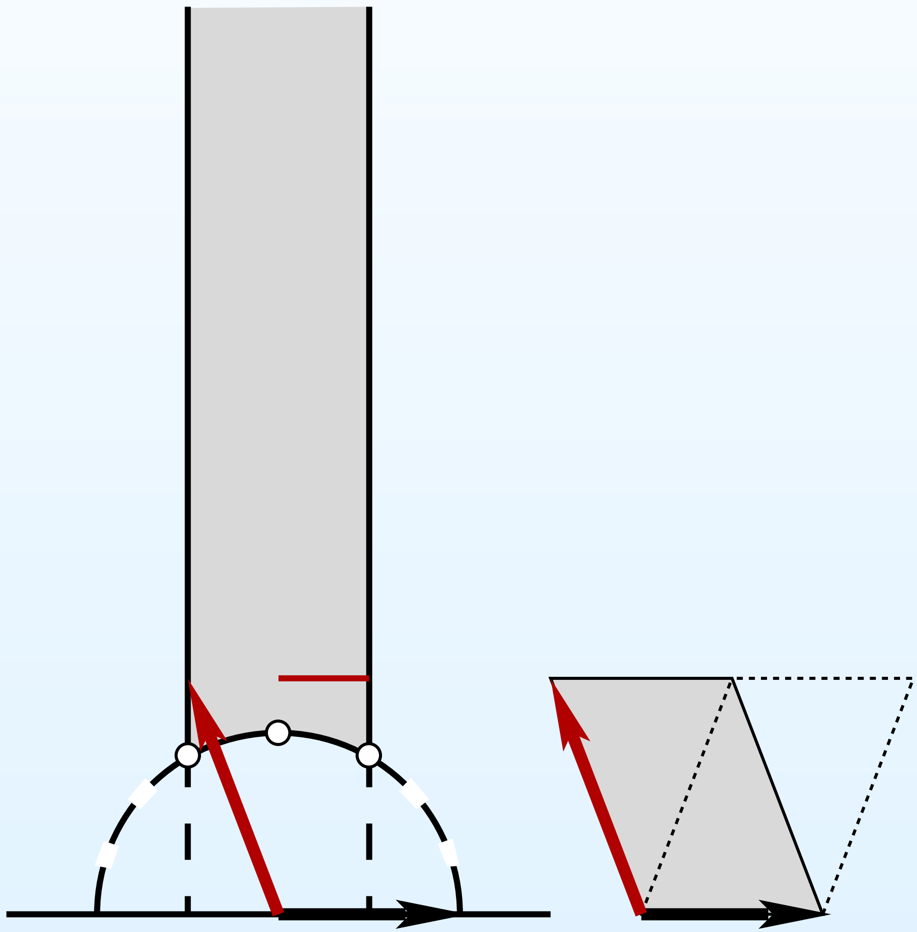

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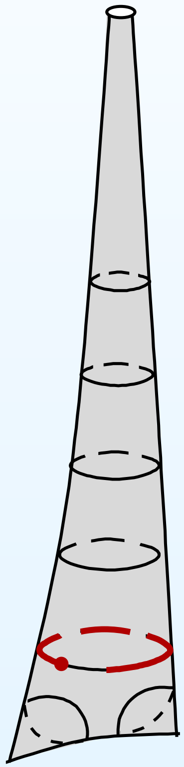

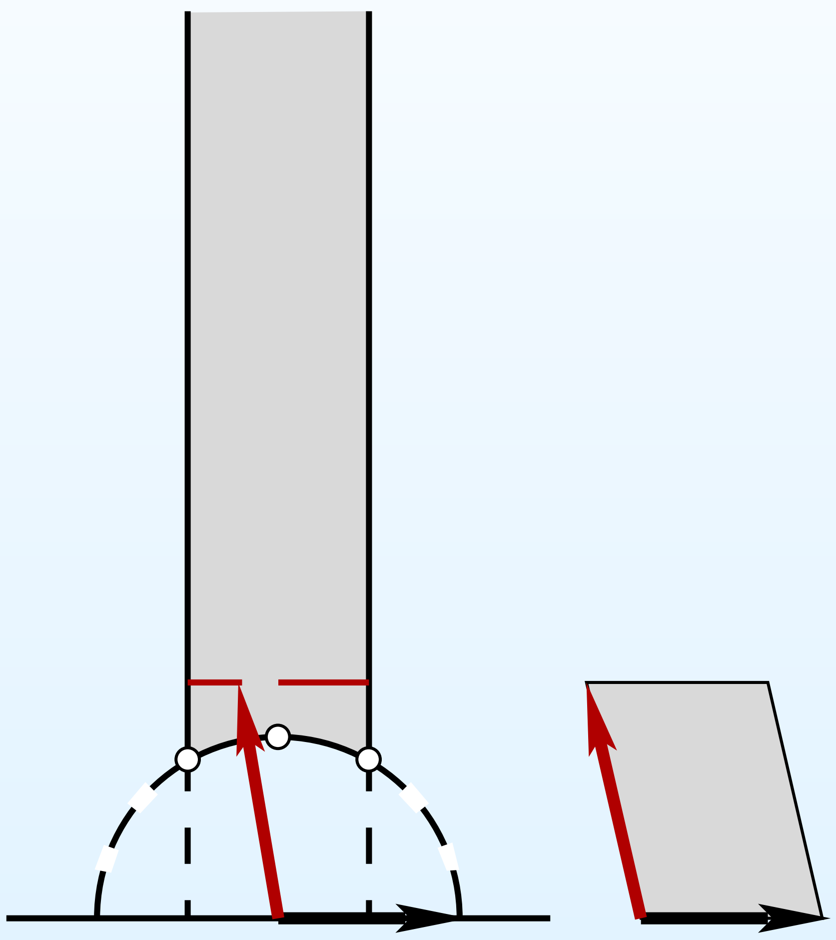

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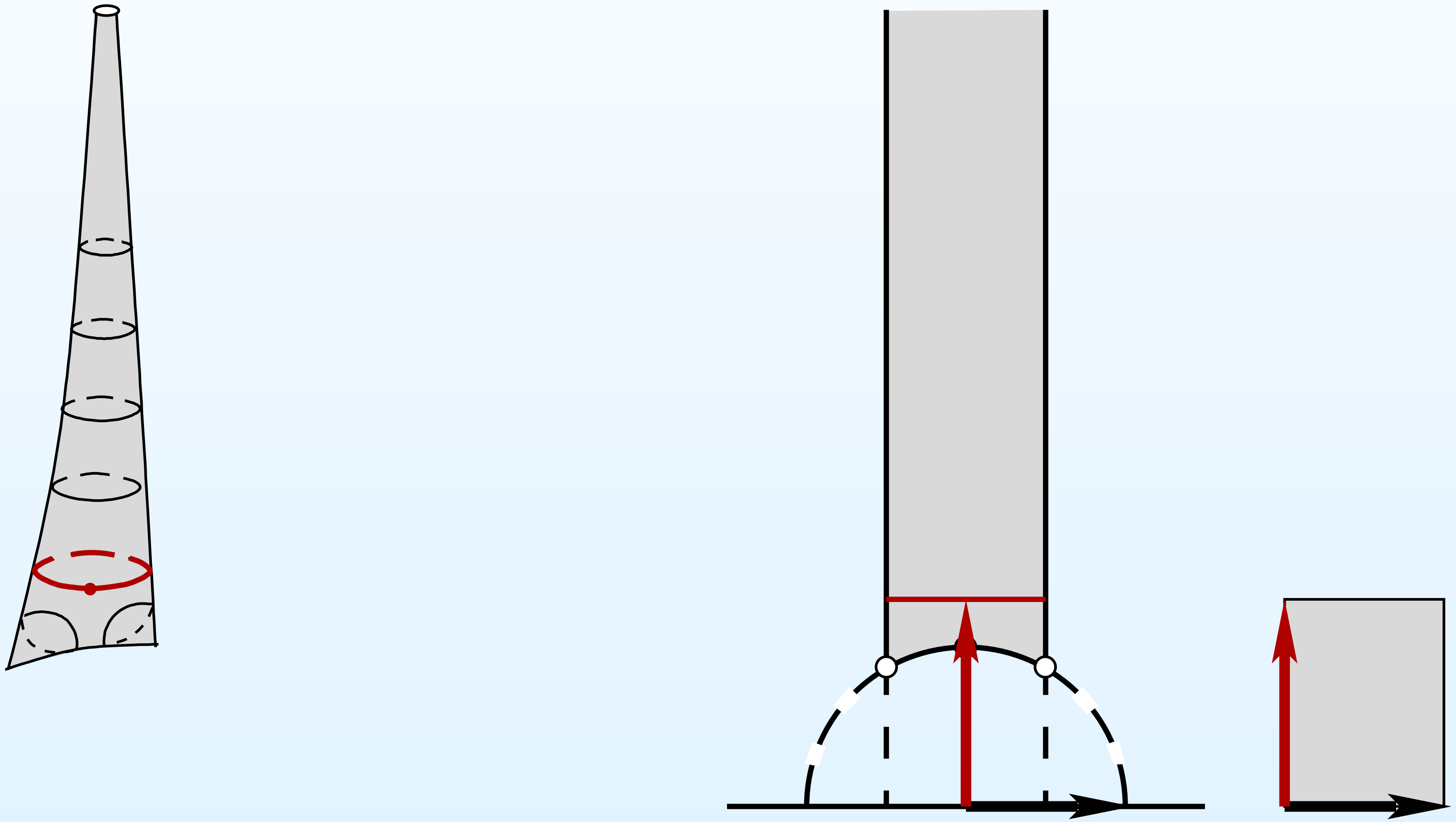
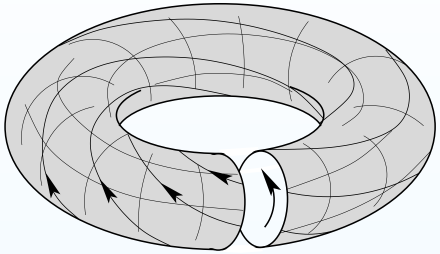
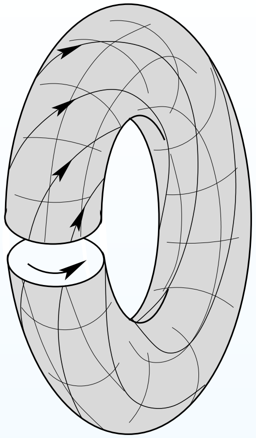


Pseudo-Anosov diffeomorphisms

Consider a composition of two Dehn twists

$g = f_v \circ f_h =$  $\circ$ 

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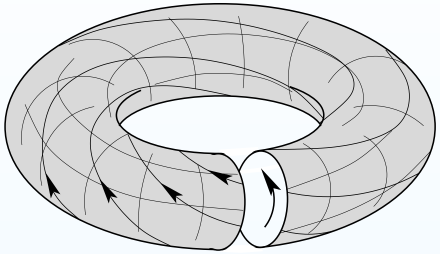
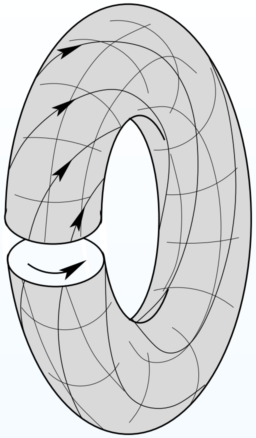
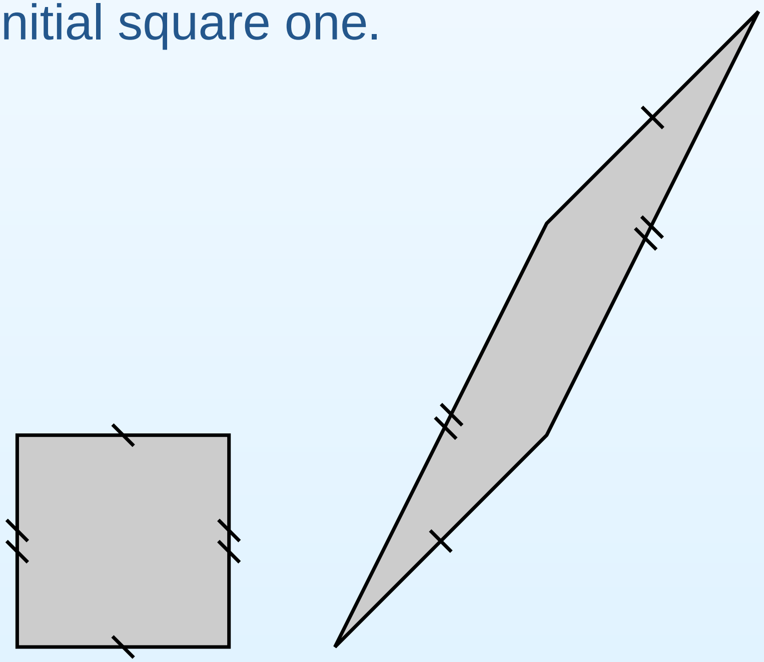
Consider a composition of two Dehn twists $g = f_v \circ f_h =$  $\circ$ 

It corresponds to the integer linear map $\hat{g} : \mathbb{R}^2 \to \mathbb{R}^2$ with matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Cutting and pasting appropriately the image parallelogram pattern we can check by hands that we can transform the new pattern to the initial square one.



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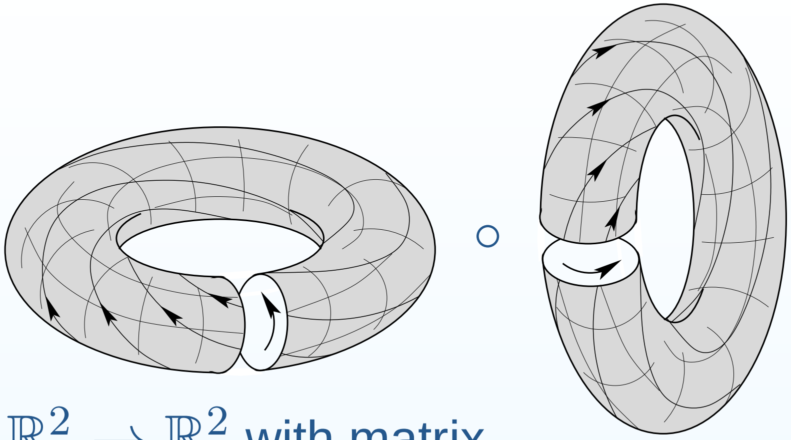
Consider a composition of two Dehn twists $g = f_v \circ f_h =$ 

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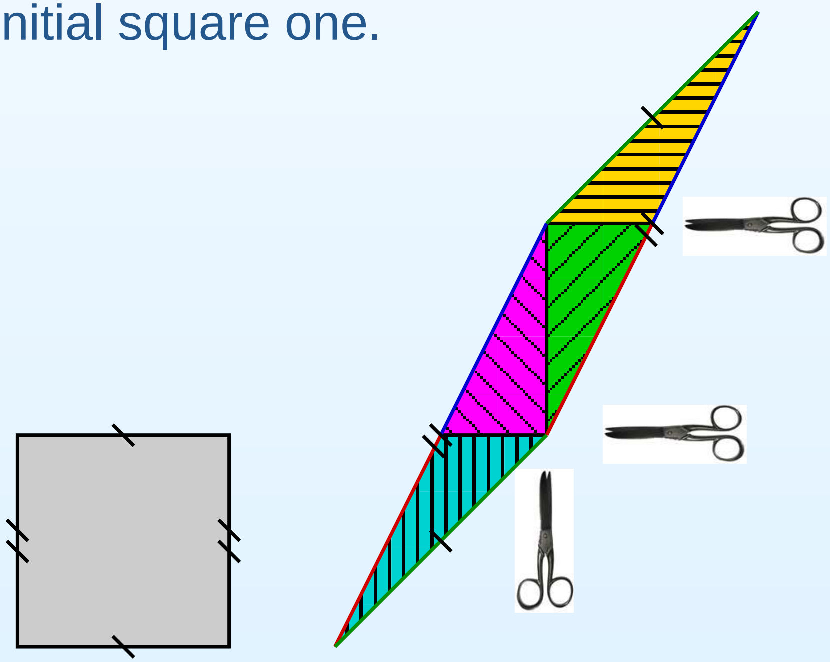

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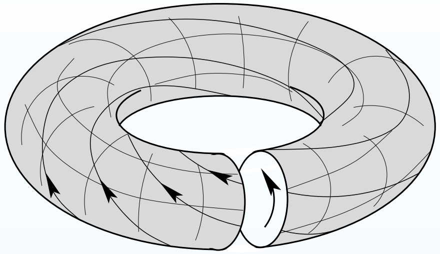
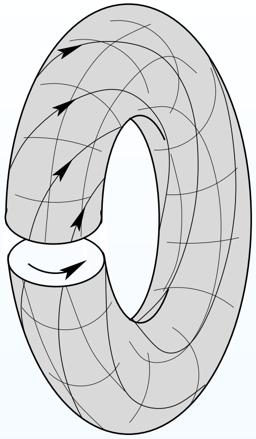
Consider a composition of two Dehn twists $g = f_v \circ f_h =$  $\circ$ 

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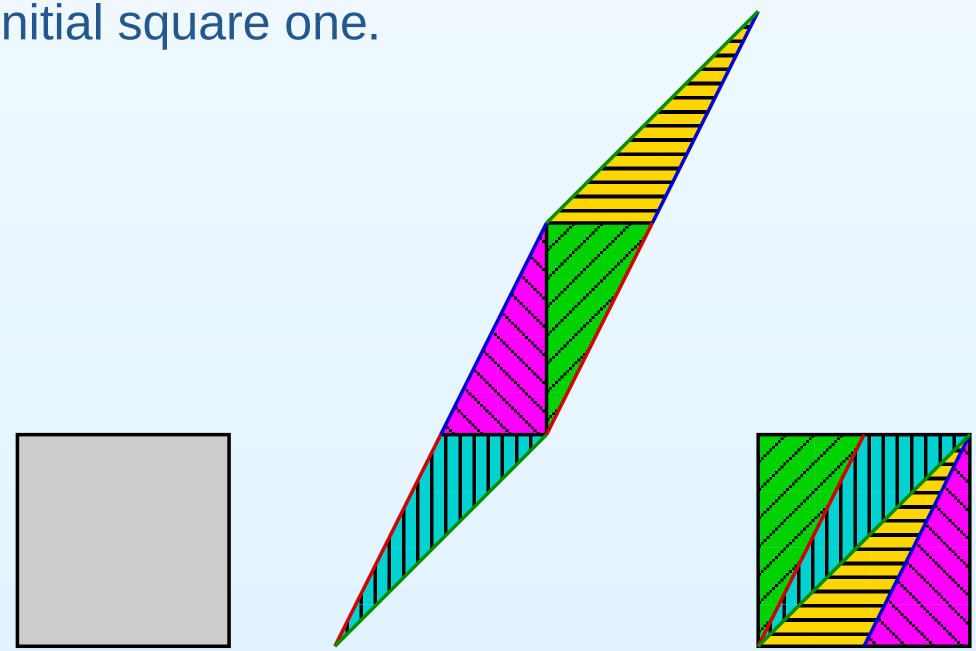

Closed geodesics in the space of tori

Consider eigenvectors $\vec{v}_{exp}$ and $\vec{v}_{contr}$ of the linear transformation $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ corresponding to the eigenvalues $\lambda > 1$ and to $1/\lambda < 1$ respectively. Consider two transversal foliations on the original torus in directions of $\vec{v}_{exp}$ and of $\vec{v}_{contr}$. We have just proved that expanding our torus $\mathbb{T}^2$ by factor λ in direction $\vec{v}_{exp}$ and contracting it by the factor λ in direction $\vec{v}_{contr}$ we get the original torus.

Consider a one-parameter family of flat tori obtained from the initial square torus by a continuous deformation expanding with a factor e^t in directions $\vec{v}_{exp}$ and contracting with a factor e^{-t} in direction $\vec{v}_{contr}$. By construction such one-parameter family defines a closed curve in the space of flat tori: after the time $t_0 = \log \lambda$ it closes up and follows itself.

One can check that this closed curve is, actually, a closed geodesics in the moduli spaces of tori.

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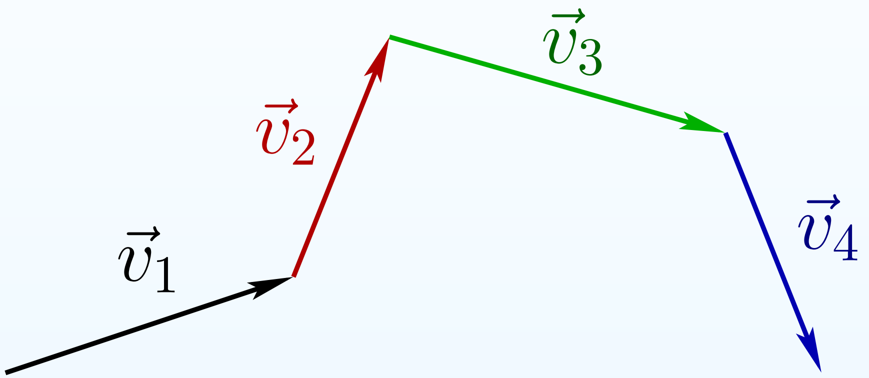
Dynamics in the moduli spaces

Very flat surfaces: construction from a polygon

Consider a broken line constructed from vectors $\vec{v}_1, \ldots, \vec{v}_k$.



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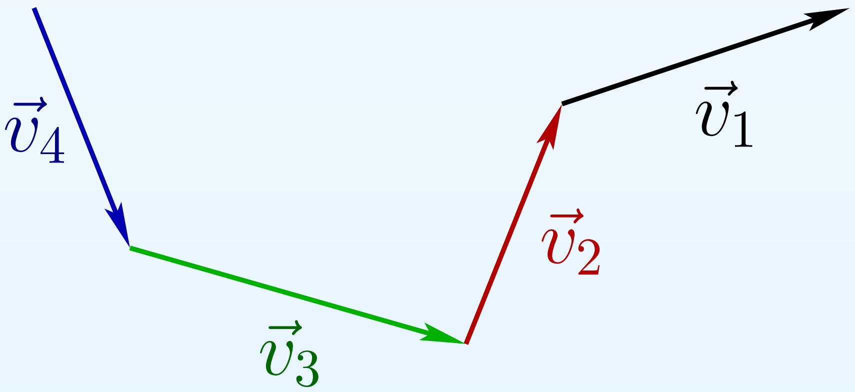

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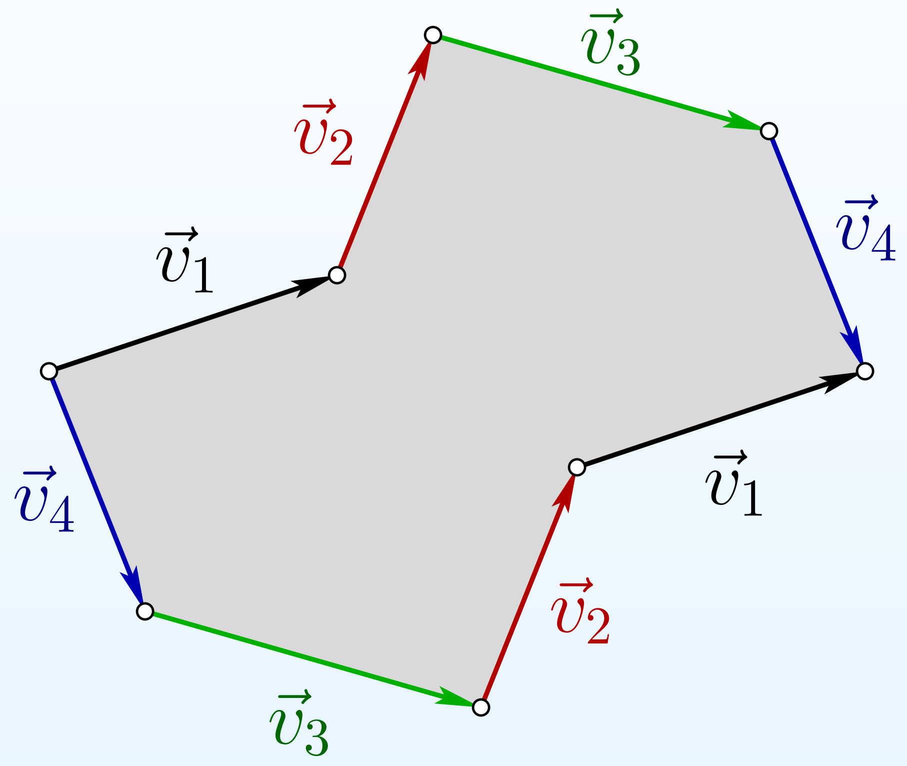
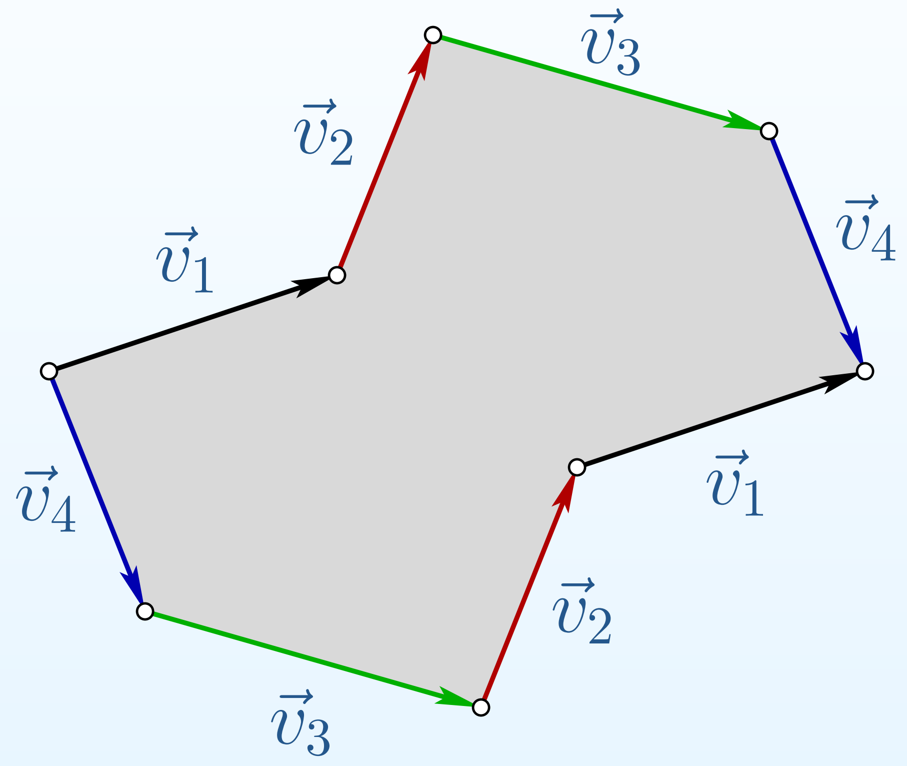


and another one constructed from the same vectors taken in another order. If we are lucky enough the two broken lines do not intersect and form a polygon.

Very flat surfaces: construction from a polygon



Identifying the corresponding pairs of sides by parallel translations we get a closed surface endowed with a flat metric.

Holomorphic 1-form associated to a flat structure

Consider the natural coordinate z in the complex plane, where lives the polygon. In this coordinate the parallel translations which we use to identify the sides of the polygon are represented as $z' = z + const$.

Since this correspondence is holomorphic, our flat surface S with punctured conical points inherits the complex structure. This complex structure extends to the punctured points.

Consider now a holomorphic 1-form dz in the complex plane. The coordinate z is not globally defined on the surface S. However, since the changes of local coordinates are defined as $z' = z + const$, we see that $dz = dz'$. Thus, the holomorphic 1-form dz on $\mathbb{C}$ defines a holomorphic 1-form ω on S which in local coordinates has the form $\omega = dz$.

The form ω has zeroes exactly at those points of S where the flat structure has conical singularities.

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Flat structure defined by a holomorphic 1-form

- Reciprocally a pair (Riemann surface, holomorphic 1-form) uniquely defines a flat structure: $z = \int \omega$.
- In a neighborhood of zero a holomorphic 1-form can be represented as $w^d\, dw$, where d is the **degree** of zero. The form ω has a zero of degree d at a conical point with cone angle $2\pi(d+1)$. Moreover, $d_1 + \cdots + d_n = 2g - 2$.
- The moduli space $\mathcal{H}_g$ of pairs (complex structure, holomorphic 1-form) is a $\mathbb{C}^g$-vector bundle over the moduli space $\mathcal{M}_g$ of complex structures.
- The space $\mathcal{H}_g$ is naturally stratified by the strata $\mathcal{H}(d_1, \dots, d_n)$ enumerated by unordered partitions $d_1 + \cdots + d_n = 2g - 2$.
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- The vectors defining the polygon from the previous picture considered as complex numbers are the relative periods $\int_{P_i}^{P_j} \omega$ of ω, so each stratum $\mathcal{H}(d_1, \dots, d_n)$ is modelled on the relative cohomology $H^1(S, \{P_1, \dots, P_n\}\,;\, \mathbb{C})$ serving as *period coordinates*.

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Volume element

Note that the vector space $H^1(S,\{P_1,\dots,P_n\}\,;\,\mathbb{C})$ contains a natural integer lattice $H^1(S,\{P_1,\dots,P_n\}\,;\,\mathbb{Z}\oplus\sqrt{-1}\,\mathbb{Z})$. Consider a linear volume element $d\nu$ normalized in such a way that the volume of the fundamental domain in this lattice equals one. Consider now the real hypersurface $\mathcal{H}_1(d_1,\dots,d_n)\subset\mathcal{H}(d_1,\dots,d_n)$ defined by the equation $area(S)=1$. The volume element $d\nu$ can be naturally restricted to the hypersurface defining the volume element $d\nu_1$ on $\mathcal{H}_1(d_1,\dots,d_n)$.

Theorem (H. Masur; W. A. Veech) *The total volume* $\operatorname{Vol}(\mathcal{H}_1(d_1,\dots,d_n))$ *of every stratum is finite.*

The Masur–Veech volumes of the first several low-dimensional strata were computed by M. Kontsevich and myself about 2000. The first efficient algorithm for evaluation of the Masur–Veech volume was found by A. Eskin and A. Okounkov. In particular, they proved that the Masur–Veech volume of any stratum always has the form $(p/q)\pi^{2g}$ where p/q is a rational number. By 2003 A. Eskin computed these rational numbers up for all strata to genus 10. By now we have very good knowledge of Masur–Veech volumes for strata of Abelian differentials and for the principal strata of quadratic differentials.

Ergodic transformations

Let μ be a finite measure on a topological space M (for example, a volume element on a manifold M, with a finite total volume). A map $T : M \to M$ *preserves measure* μ (corresp. is *volume preserving*) if for any measurable subset $A \subset M$ one has $\mu(T^{-1}(A)) = \mu(A)$.

A subset $A \subset M$ is called T-*invariant* if $T^{-1}(A) = A$.

The map T is called *ergodic* with respect to the measure μ if any invariant set has measure 0, or the full measure $\mu(M)$.

Examples.

● Rotations of a circle are measure preserving. Irrational rotations are ergodic; rational ones are not.

● The map $z \to z^2$ where $z \in \mathbb{C}$, $|z| = 1$ is not invertible but it preserves the measure and is ergodic.

● A (pseudo)Anosov diffeomorphism preserves the area form and and is ergodic.

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Ergodic Theorem

Consider the orbit $x, T(x), T(T(x)), \dots, T^{(n-1)}(x)$ of a point $x \in M$. By *time average* of a μ-measurable function f on M we call the average

$$\frac{1}{n}\sum_{i=0}^{n-1} f(T^{(i)}(x)),$$

i.e. the mean value of f along first n points of the orbit of x. By *space average* we call

$$\frac{1}{\mu(M)}\int_M f(x)\,d\mu.$$

Ergodic Theorem. *Let T be an ergodic map preserving finite measure μ. For μ-almost any point x of M the time averages converge to space averages:*

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{i=0}^{n-1} f(T^{(i)}(x)) = \frac{1}{\mu(M)}\int_M f(x)\,d\mu.$$

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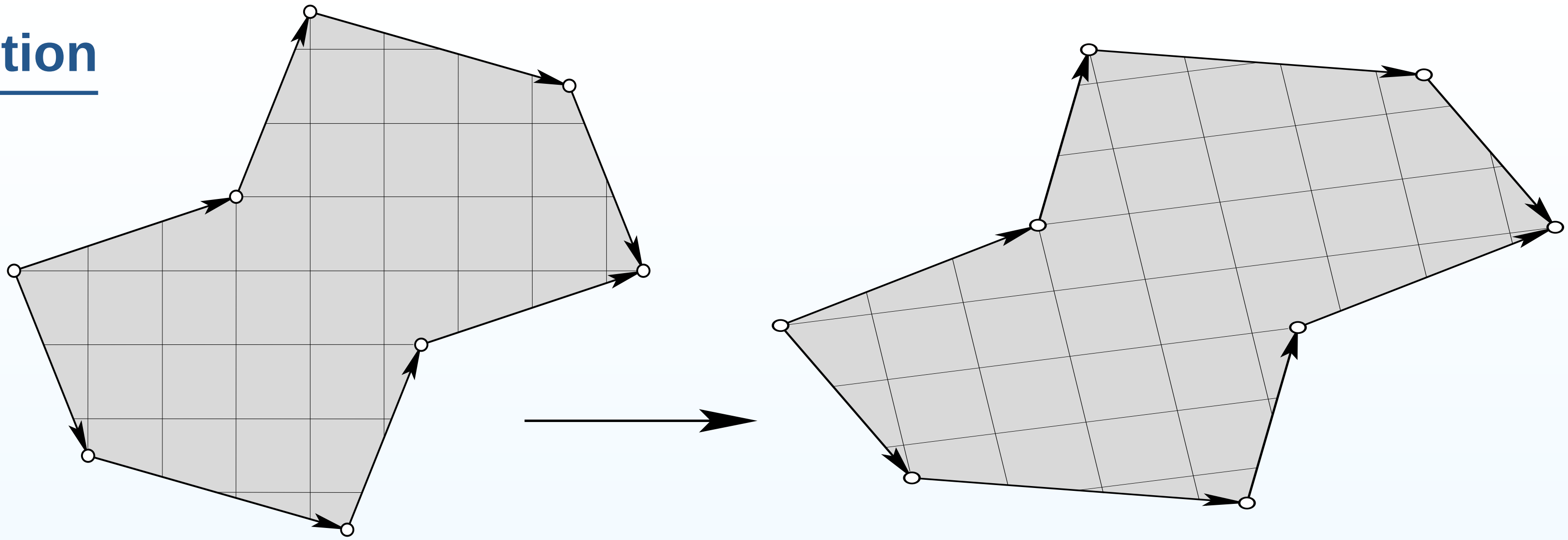
Group action

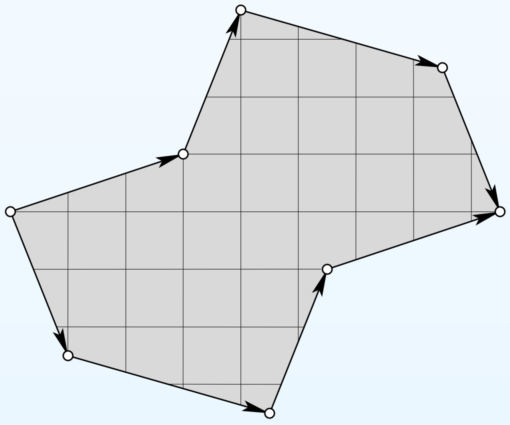


The subgroup $SL(2,\mathbb{R})$ of area preserving linear transformations acts on the "unit hyperboloid" $\mathcal{H}_1(d_1,\dots,d_n)$. The diagonal subgroup $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \subset SL(2,\mathbb{R})$ induces a natural flow on the stratum, which is called the *Teichmüller geodesic flow*.

Key Theorem (H. Masur; W. A. Veech) *The action of the groups* $SL(2,\mathbb{R})$ *and* $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ *preserves the measure* $d\nu_1$*. Both actions are ergodic with respect to this measure on each connected component of every stratum* $\mathcal{H}_1(d_1,\dots,d_n)$.

Masur—Veech Theorem

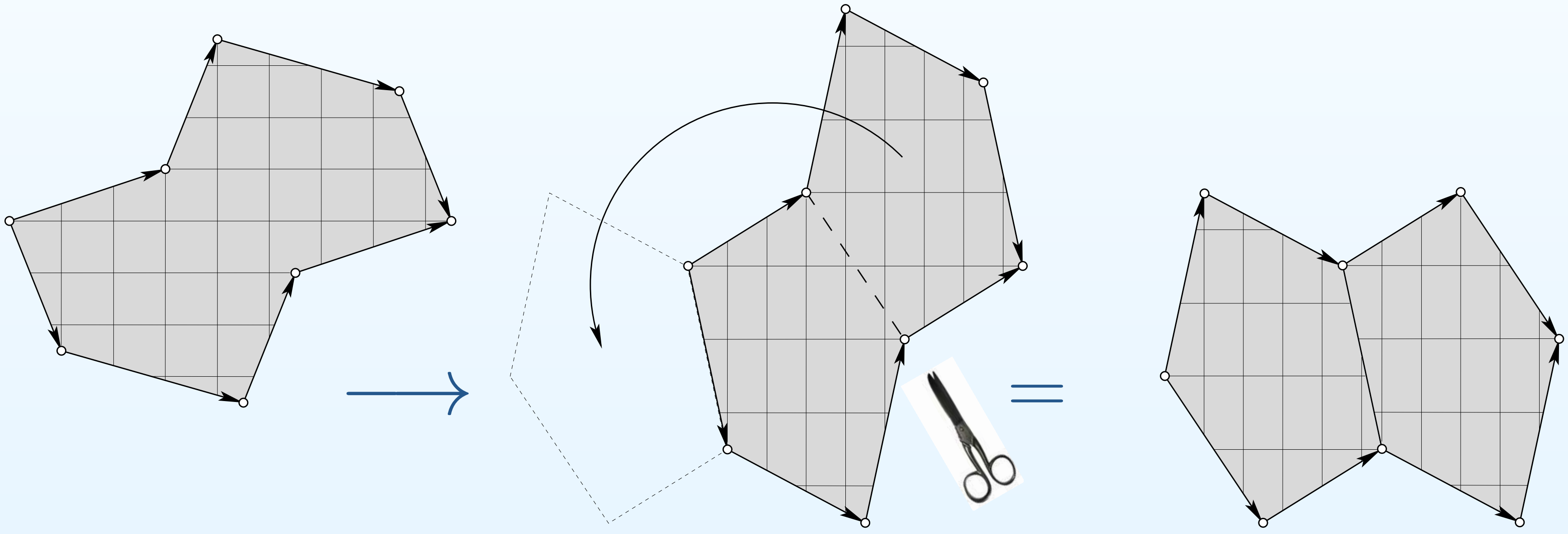
Theorem of Masur and Veech claims that taking an arbitrary octagon as below we can contract it horizontally and expand vertically by the same factor e^t to get arbitrary close to, say, regular octagon.



Masur—Veech Theorem

Theorem of Masur and Veech claims that taking an arbitrary octagon as below we can contract it horizontally and expand vertically by the same factor e^t to get arbitrary close to, say, regular octagon.

There is no paradox since we are allowed to cut-and-paste!



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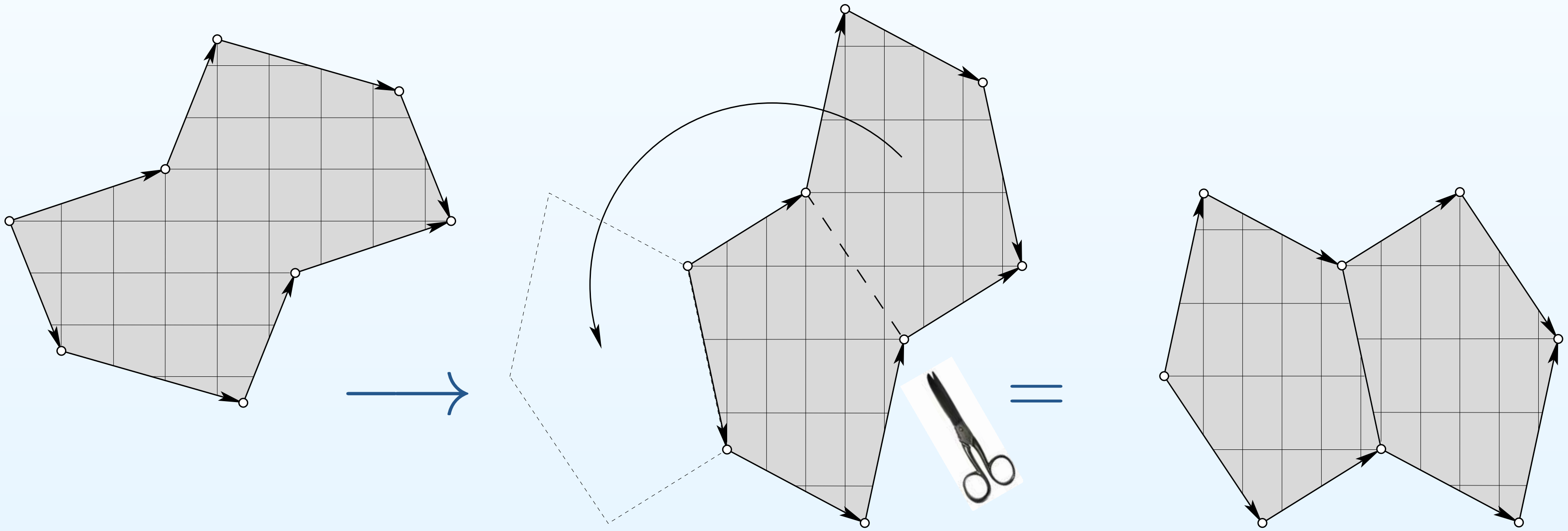

The first modification of the polygon changes the flat structure while the second one just changes the way in which we unwrap the flat surface

Moduli spaces of Abelian differentials

We have seen that any stratum $\mathcal{H}(m_1, \ldots, m_n)$ of all pairs *(Riemann surface S, holomorphic 1-form with n zeroes of degrees $m_1, \ldots, m_n$)* is locally modeled on $H^1(S, \{n \text{ points}\}; \mathbb{C})$ and, thus, is endowed with a canonical volume element $d\nu$ (the one normalized by the integer lattice).

The group $\mathrm{SL}(2,\mathbb{R})$ acts on the second term in the tensor product

$$H^1(S, \{n \text{ points}\}; \mathbb{R} \oplus i\mathbb{R}) \simeq H^1(S, \{n \text{ points}\}; \mathbb{R}) \otimes \mathbb{R}^2 .$$

The projectivized stratum $\mathrm{P}\mathcal{H}(m_1, \ldots, m_n) \simeq \mathcal{H}_1(m_1, \ldots, m_n)/\,\mathrm{SO}(2,\mathbb{R}) \simeq \mathcal{H}(m_1, \ldots, m_n)/\mathbb{C}^*$ is foliated by hyperbolic planes $\mathbb{H}^2 = \mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2,\mathbb{R})$ called *Teichmüller discs*. The natural projection of such disc to $\mathcal{M}_g$ is an isometric immersion, so Teichmüller discs are *complex geodesics* in the Teichmüller metric on $\mathcal{M}_g$.

Similarly, any stratum of meromorphic quadratic differentials with at most simple poles is locally modeled on the anti-invariant subspace of $H^1(\hat{S}, \{n \text{ points}\}; \mathbb{C})$, where $p : \hat{S} \to S$ is the canonical double cover such that $p^*q = \omega^2$ becomes a global square of a holomorhic form ω.

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Invariant measures and orbit closures

Magic Wand Theorem (A. Eskin–M. Mirzakhani–A. Mohammadi, 2014). *The closure of any* $\mathrm{SL}(2,\mathbb{R})$*-orbit is a suborbifold. In period coordinates any* $\mathrm{GL}(2,\mathbb{R})$*-orbit closure is represented by a complexification of an* $\mathbb{R}$*-linear subspace.*

Any ergodic $\mathrm{SL}(2,\mathbb{R})$*-invariant measure is supported on a suborbifold. In period coordinates this suborbifold is represented by an affine subspace, and the invariant measure is just a usual affine measure on this affine subspace.*

Theorem (S. Filip, 2014) *Any* $\mathrm{SL}(2,\mathbb{R})$*-invariant orbifold is, actually, a complex orbifold.*

“But still, my homeward way has proved too long.
While we were wasting time there, old Poseidon,
it almost seems, stretched and extended space.”

J. Brodsky

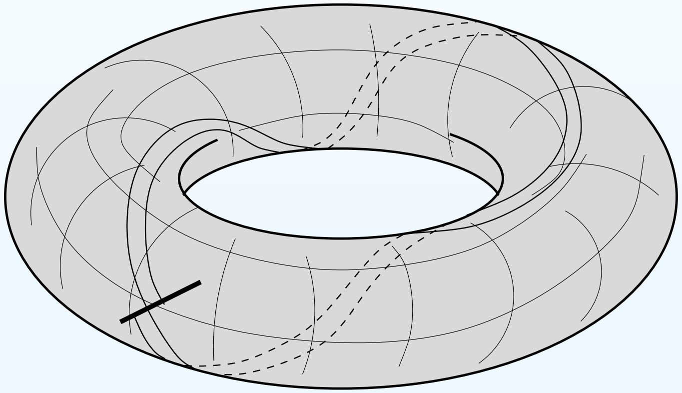
И все-таки ведущая домой
дорога оказалась слишком длинной,
как будто Посейдон, пока мы там
теряли время, растянул пространство.

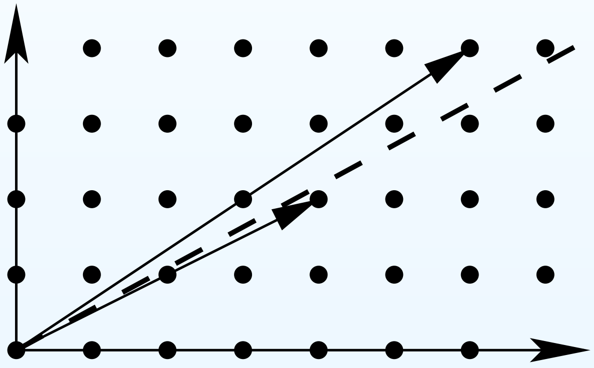
И. Бродский

Idea of Renormalization

Asymptotic cycle for a torus

Consider a leaf of a measured foliation on a surface. Choose a short transversal segment X. Each time when the leaf crosses X we join the crossing point with the point x_0 along X obtaining a closed loop. Consecutive return points $x_1, x_2, \ldots$ define a sequence of cycles $c_1, c_2, \ldots$.





The *asymptotic cycle* is defined as $\lim_{n\to\infty} \dfrac{c_n}{n} = c \in H_1(\mathbb{T}^2; \mathbb{R})$.

Theorem (S. Kerckhoff, H. Masur, J. Smillie, 1986.) *For any flat surface directional flow in almost any direction is uniquely ergodic.*

This implies that for almost any direction the asymptotic cycle exists and is the same for all points of the surface.

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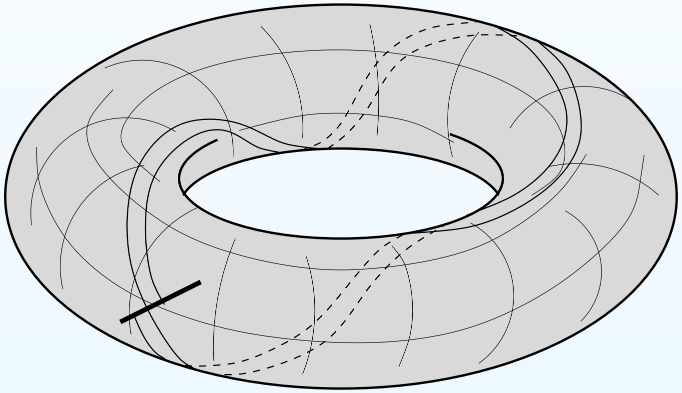

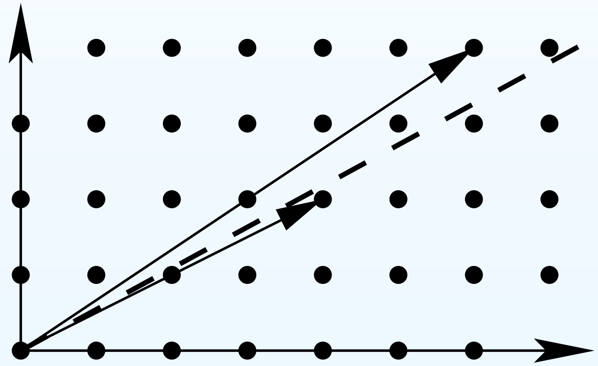

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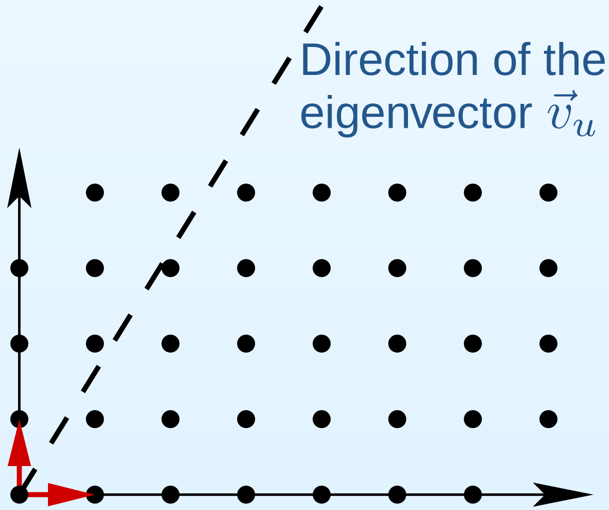
Asymptotic cycle in the pseudo-Anosov case

Consider a model case of the foliation in direction of the expanding eigenvector $\vec{v}_u$ of the Anosov map $g : \mathbb{T}^2 \to \mathbb{T}^2$ with $Dg = A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Take a closed curve γ and apply to it k iterations of g. The images $g_*^{(k)}(c)$ of the corresponding cycle $c = [\gamma]$ get almost collinear to the expanding eigenvector $\vec{v}_u$ of A, and the corresponding curve $g^{(k)}(\gamma)$ closely follows our foliation.

The first return cycles to a short subinterval exhibit exactly the same behavior by a simple reason that they are images of the first return cycles to a longer subinterval under a high iteration of g.



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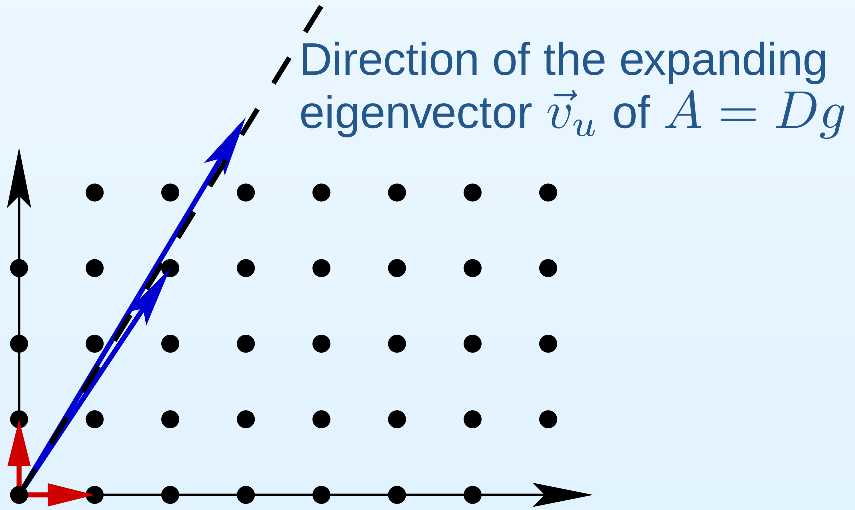

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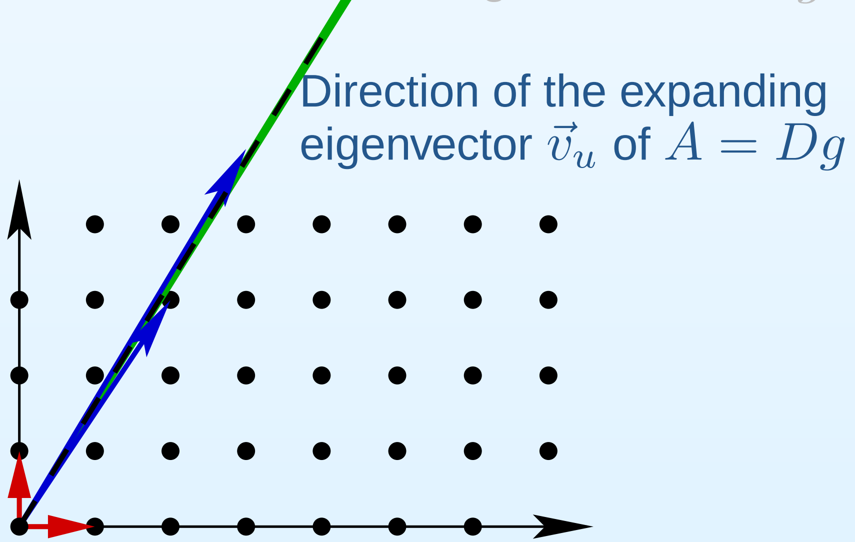

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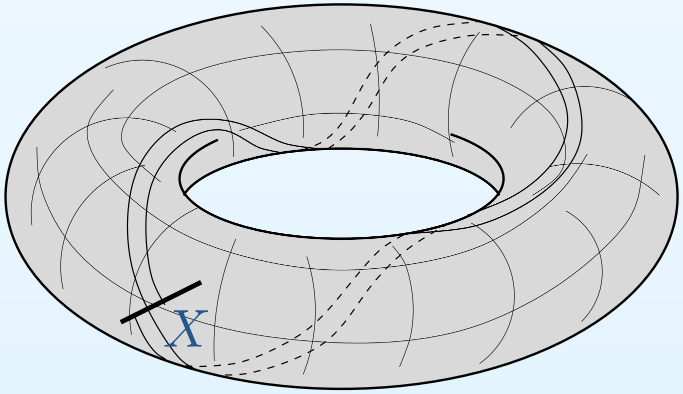
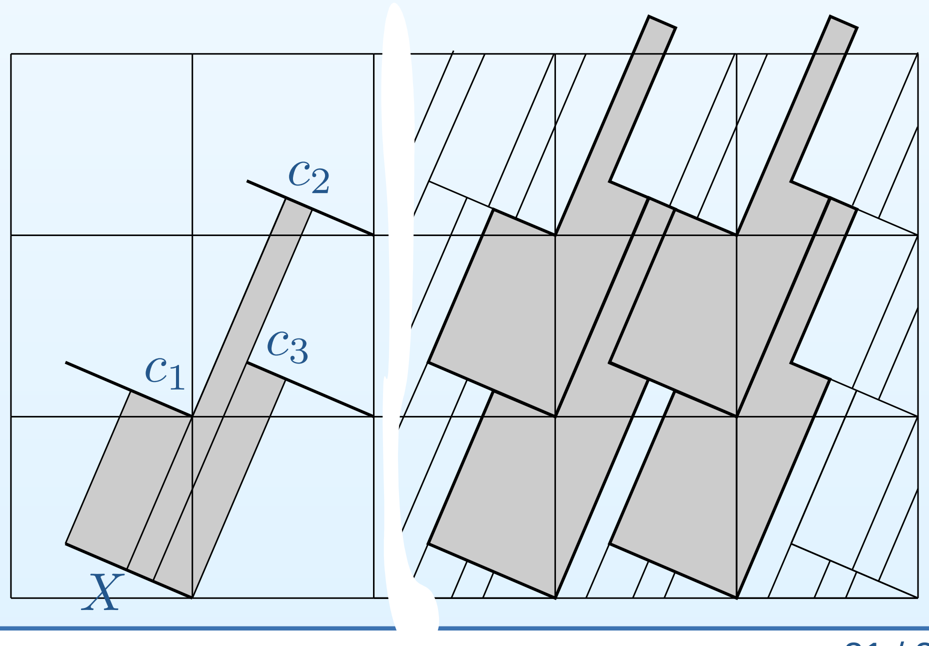
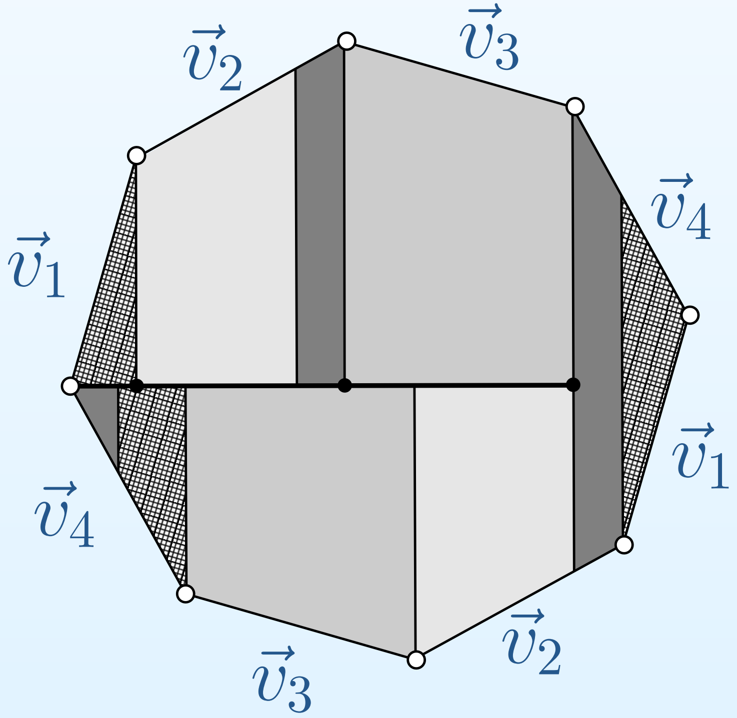




First return cycle $c_i(g(X))$ to $g(X)$ is $g_*(c_i(X))$

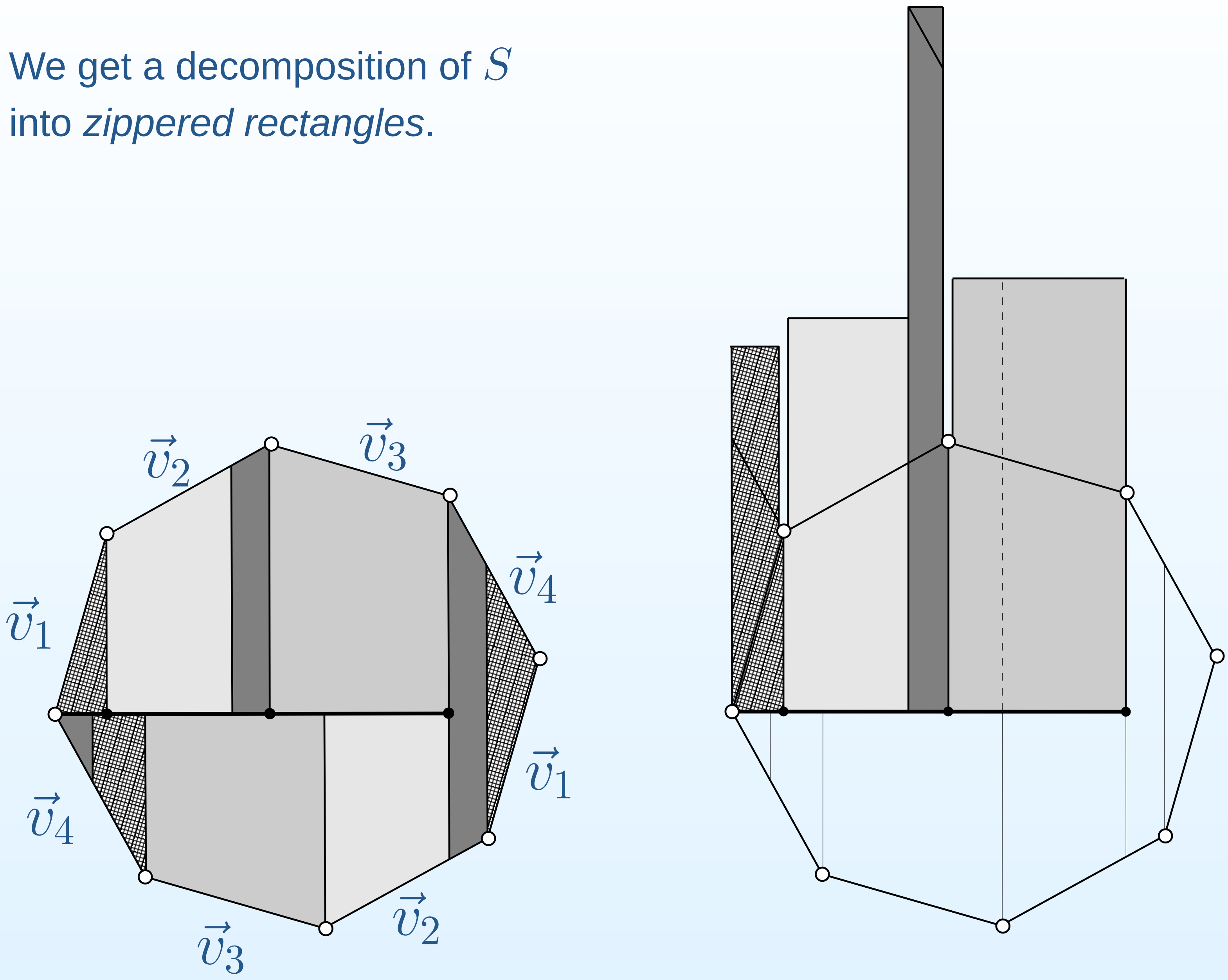
Zippered rectangles

For a general flat surface S the first return map of the vertical flow to a horizontal segment X also induces an interval exchange transformation $T : X \to X$.



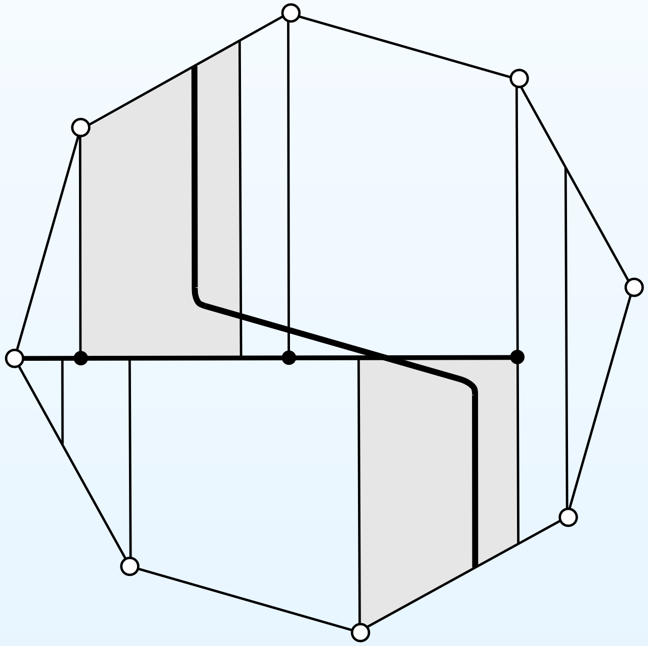
Zippered rectangles

We get a decomposition of S into *zippered rectangles*.



First return cycles

Launch the vertical trajectory from a point $x \in X$. When the trajectory intersects X for the first time join the corresponding point $T(x)$ to the original point x along X to obtain a closed loop $c(x)$. (In the picture this "first return cycle" is smoothed.)



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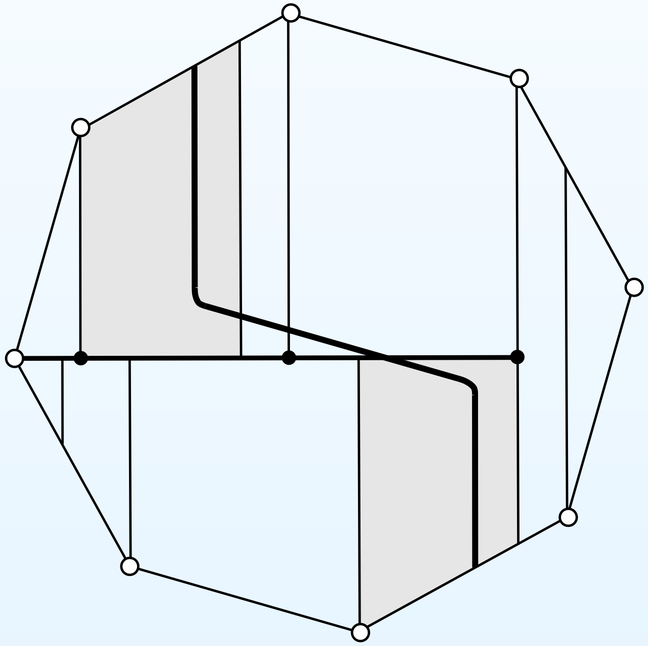
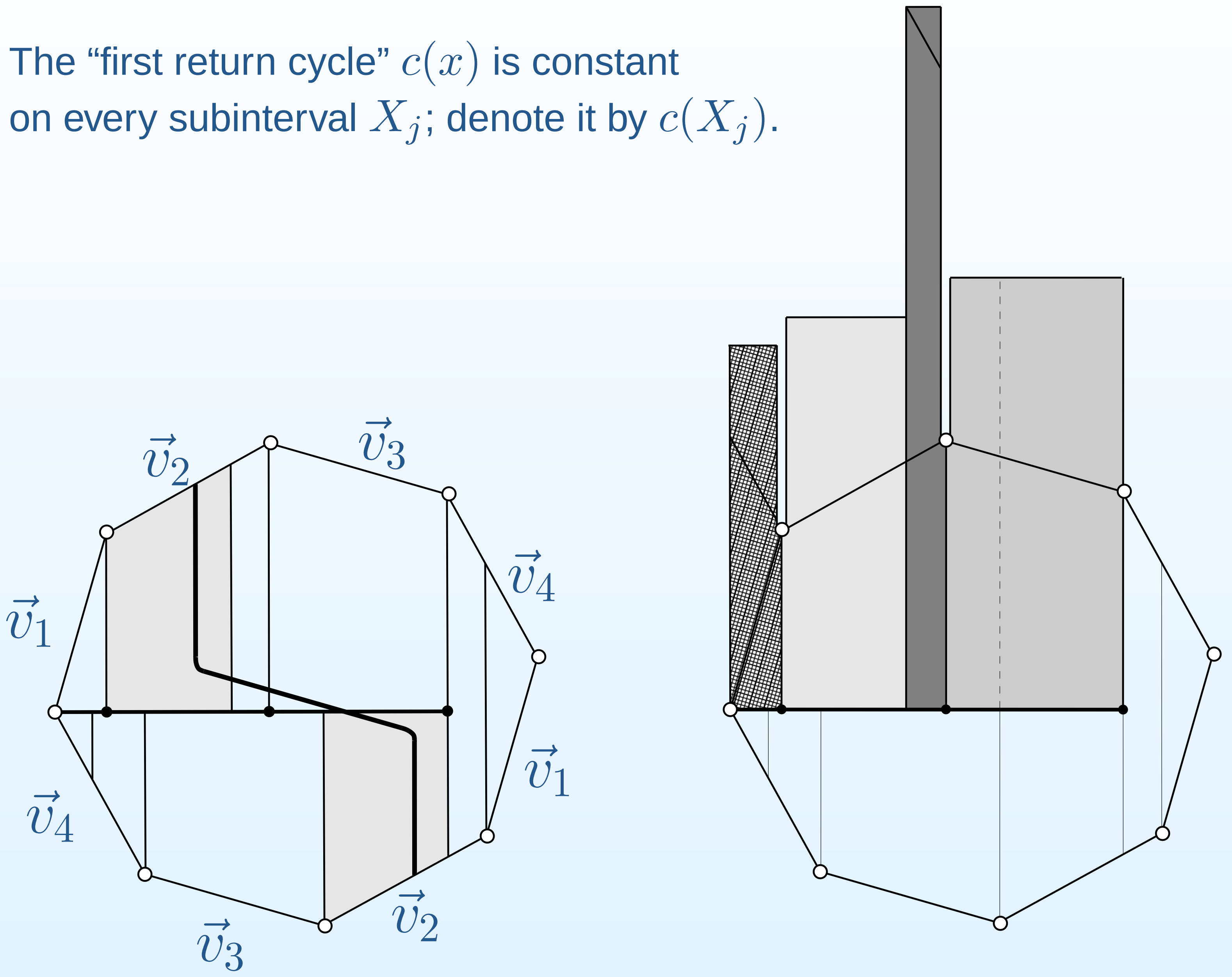


The cycle $c_N(x)$ obtained after N returns of the vertical trajectory to X can be computed as:

$$c_N(x) = c(x) + c(T(x)) + \cdots + c(T^{N-1}(x))$$

First return cycles

The “first return cycle” $c(x)$ is constant on every subinterval X_j; denote it by $c(X_j)$.



One step of renormalization

Consider a subinterval $X' \subset X$. Choose it in such way that that the first return map to X' induces an interval exchange transformation $T' : X' \to X'$ of the same number n of subintervals.

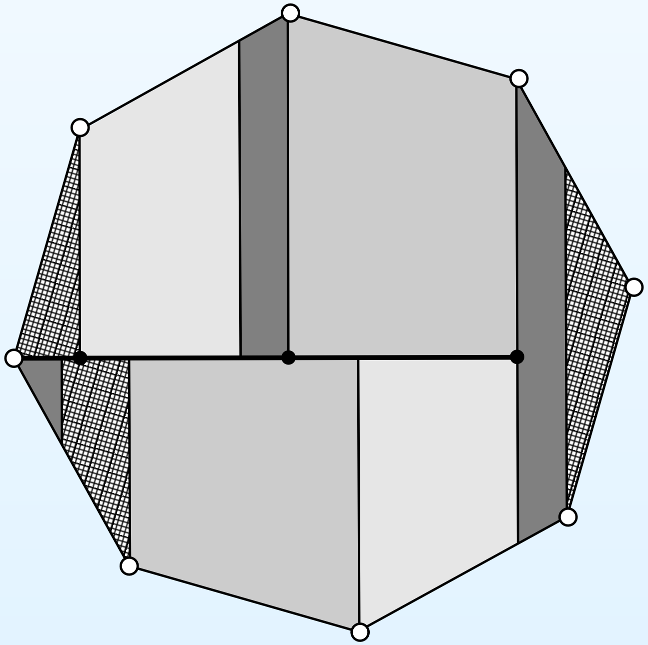
New first return cycles $c'(X'_k)$ to the interval X' are expressed in terms of the initial first return cycles $c(X_j)$ by linear relations; the lengths $|X'_k|$ of subintervals of the new partition $X' = X'_1 \sqcup \cdots \sqcup X'_n$ are expressed in terms of the lengths $|X_j|$ of subintervals of the initial partition by dual linear relations:

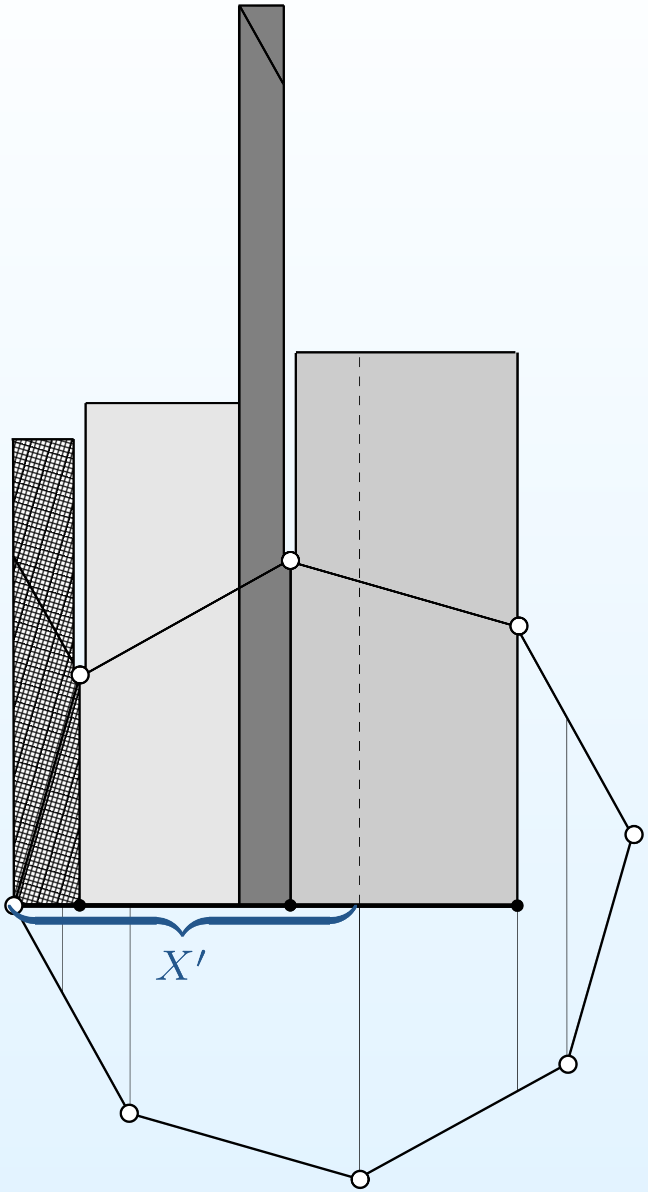
$$c'(X'_k) = \sum_{j=1}^{n} A_{jk} \cdot c(X_j) \qquad\qquad |X_j| = \sum_{k=1}^{n} A_{jk} \cdot |X'_k|\,,$$

Here a nonnegative integer matrix A_{jk} is completely determined by the initial interval exchange transformation $T : X \to X$ and by the choice of $X' \subset X$.

Idea of renormalization

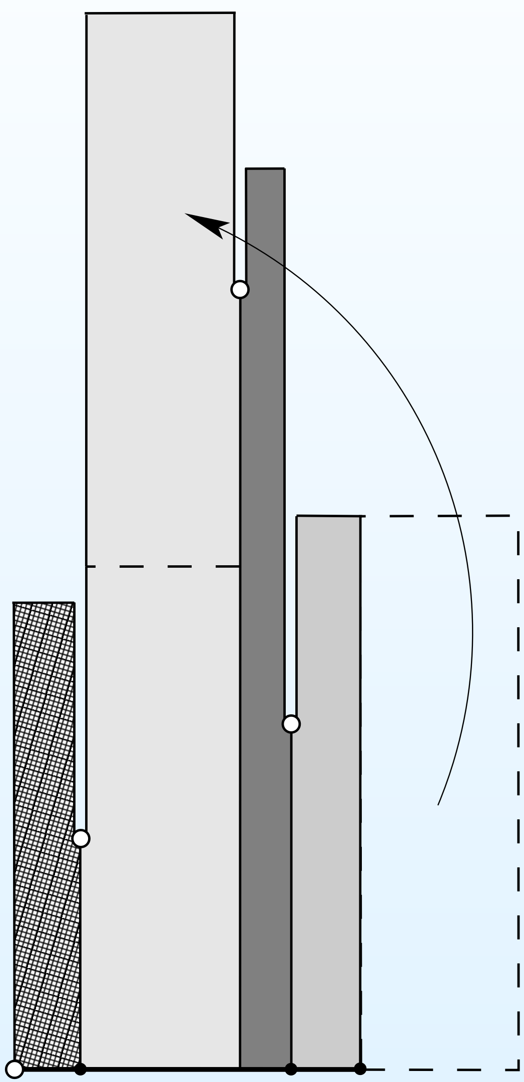
Unwrap the flat surface into “zippered rectangles”. Shorten the base.



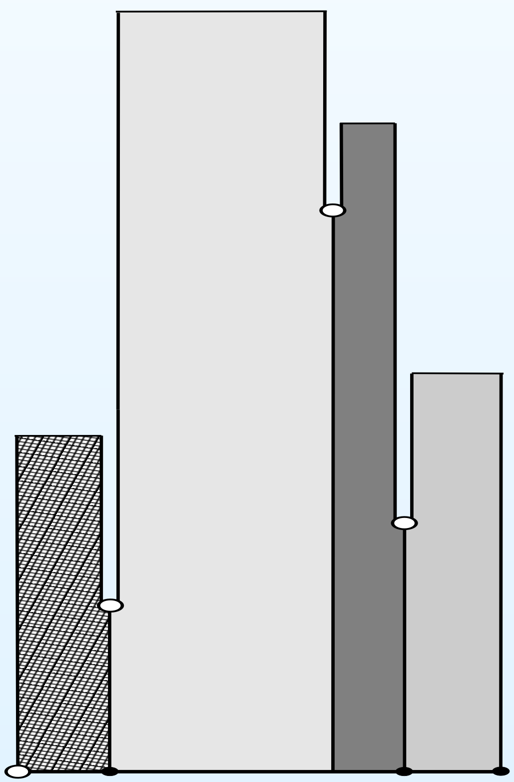


Idea of a renormalization

Expand the resulting tall and narrow zippered rectangles horizontally and contract them vertically by a factor e^{t_0}.



$$\begin{pmatrix} e^{t_0} & 0 \\ 0 & e^{-t_0} \end{pmatrix} \longrightarrow$$



Time acceleration machine

To construct the cycle c_N representing a long piece of trajectory of the vertical flow we follow the trajectory $x, T(x), \dots, T^{N-1}(x)$ of the corresponding interval exchange transformation and compute the corresponding ergodic sum $c_N(x) = c(x) + \dots + c(T^{N-1}(x))$.

Passing to a subinterval $X' \subset X$ we can follow the trajectory $x, T'(x), \dots, (T')^{N'-1}(x)$ of the new interval exchange transformation $T' : X' \to X'$. Since X' is shorter than X we cover the initial piece of trajectory of the vertical flow in a smaller number N' of steps.

Passing from T to T' we accelerate the time: that the trajectory $x, T'(x), \dots, (T')^{N'-1}(x)$ follows the trajectory $x, T(x), \dots, T^{N-1}(x)$ but jumps over several iterations of T at a time.

This renormalization is a "time acceleration machine": instead of getting c_N by following the trajectory $x, ..., T^{N-1}(x)$ of the initial interval exchange transformation for the exponential time $N \sim \exp(const \cdot s)$ we obtain the cycle c_N applying only s steps of the renormalization map.