Recap. X: Fano mfd of cx dim n.

$-K_X$ ample, $\mathrm{Aut}(X)$ discrete.

<u>Main thm of the minicourse</u>

$(X, -K_X)$ admits an ACB metric at level m

iff $\mathrm{Ding}(\mathcal{X}, \mathcal{L}) + \mathrm{Chow}_m(\mathcal{X}, \mathcal{L}) \geq 0$

for all very ample test configurations $(\mathcal{X}, \mathcal{L})$

of exponent m, with equality iff $(\mathcal{X}, \mathcal{L})$

is trivial.

Prop (Keller–H.)

1. For any hermitian $A \in \mathfrak{gl}(H^0(X, -mK_X))$ there exists a sequence $\{A_p\}_p$ of hermitian matrices with rational eigenvalues s.t. the flat limit of $e^{tA}.X \subset \mathbb{P}(H^0(X, -mK_X)^\vee)$ agrees with that of $e^{tA_p}.X$ for $p \gg 1$ and $A_p \to A \quad (p \to \infty)$.

2. For any hermitian A, there exist hermitian matrices A_α and A_β s.t.

(i) $A = A_\alpha + A_\beta$,

(ii) A_α has rational eigenvalues,

(iii) The flat limits of $e^{tA}.X$, $e^{tA_\alpha}.X$, $e^{tA_\beta}.X$ are all equal (as a subscheme in $\mathbb{P}(H^0(X, -mK_X)^\vee)$).

We want to show

$$\lim_{t\to+\infty}\frac{d}{dt}\mathcal{D}_m(H_t)>0 \quad \text{for all hermitian}$$

$$A\in \mathfrak{gl}(H^0(X,-mK_X)),$$

$$H_t := e^{-tA^*}H_{0,m}e^{-tA}\in \mathcal{B}_m.$$

↑ reference herm. form.

by assuming F-stability of Saito-Takahashi,

i.e. $$\lim_{t\to+\infty}\frac{d}{dt}\mathcal{D}_m(H_t')>0, \quad \text{for all hermitian}$$

$$A'\in \mathfrak{gl}(H^0(X,-mK_X))$$

with rational eigenvalues.

(To exclude $A=\text{const.}\,\mathrm{id}$, so assume $\mathrm{tr}\,A=0$).

Step 1. Show that for any hermitian $A \in \mathfrak{gl}(H^0(X, -mK_X))$ we have

$$\lim_{t \to +\infty} \frac{d}{dt} \mathcal{D}_m(H_t) = \sum_{i=1}^{N_m} \lambda_i C_i(\underline{\text{flat limit of } e^{tA} \cdot X})$$

!! $\mathcal{X}_0$.

$\lambda_1, \dots, \lambda_{N_m}$: eigenvalues of A.

This follows from

$$\frac{d}{dt} \mathcal{L}(FS(H_t)) = \sum_{i=1}^{N_m} \lambda_i \int_{e^{tA} \cdot X} \frac{|z_i|^2}{\sum_\ell |z_\ell|^2} \tilde{h}^{1/m}$$

where $[z_1, \dots, z_{N_m}]$ is the hmg coords on $\mathbb{P}(H^0(X, -mK_X)^\vee)$, $\tilde{h}$ is the FS metric on $\mathbb{P}(H^0(X, -mK_X)^\vee)$ w.r.t the reference form $H_{0,m}$.

Note $\tilde{h}^{\frac{1}{m}}\Big|_{e^{tA}\cdot X} = FS(H_t)$ hem. metric on $-K_X$

i.e. volume form when $t \in \mathbb{R}_{\geq 0}$.

Show that $\displaystyle\int_{e^{tA}\cdot X} \frac{|z_i|^2}{\sum_\ell |z_\ell|^2} \tilde{h}^{\frac{1}{m}}$ converges to a number $C_i(\mathcal{X}_0)$ which depends only on the flat limit $\mathcal{X}_0$ of $e^{tA}\cdot X$, by showing that $\tilde{h}^{\frac{1}{m}}\Big|_{\mathcal{X}_{0,red}^{reg}}$ defines a volume form which may be degenerate (but nondeg on at least one point in $\mathcal{X}_{0,red}^{reg}$).

Write down the volume form $\tilde{h}^{\frac{1}{m}}\Big|_{e^{tA}\cdot X}$ in terms of hmg coords and take the limit $t \to +\infty$.

Pick coord system near each $p \in \mathcal{X}_{0,\mathrm{red}}^{\mathrm{reg}}$ which can be perturbed to a coord. system for a pt in $e^{tA}\cdot X$ $(t \gg 0)$ (essentially IFT).

(of cx n by flatness)

Observing that $\tilde{h}^{\frac{1}{m}}$ is smooth globally on $\mathbb{P}(H^0(X, -mK_X)^\vee)$ shows that the limit of $\tilde{h}^{\frac{1}{m}}\Big|_{e^{tA}\cdot X}$ defines a possibly degenerate volume form on $\mathcal{X}_{0,\mathrm{red}}^{\mathrm{reg}}$.

Step 2.

Prop 1 $\Rightarrow$ $$\lim_{t\to+\infty} \frac{d}{dt} \mathcal{D}_m(H_t) = \lim_{p\to\infty} \sum_{i=1}^{N_m} \lambda_{i,p} C_i(x_0)$$

$$= \lim_{p\to\infty} \lim_{t\to+\infty} \frac{d}{dt} \mathcal{D}_m(H_{t,p})$$

$$\geqslant 0 \quad \text{by stability.}$$

Prop 2 $\Rightarrow$ $$\lim_{t\to+\infty} \frac{d}{dt} \mathcal{D}_m(H_t) = \sum_{i=1}^{N_m} (\lambda_{i,\alpha} + \lambda_{i,\beta}) C_i(x_0)$$

$$= \underline{\lim_{t\to+\infty} \frac{d}{dt} \mathcal{D}_m(H_{t,\alpha})} > 0$$

$$+ \underline{\lim_{t\to+\infty} \frac{d}{dt} \mathcal{D}_m(H_{t,\beta})} \geqslant 0$$

by stability. > 0

$\square$

§9. Coupled KE metrics and coupled Ding stability

Suppose that there exist ample $\mathbb{Q}$-line bundles $L_1, \cdots, L_k$ on X s.t. $-K_X = L_1 + \cdots + L_k$.

Pick $m \in \mathbb{N}$ suff. large and divisible so that mL_i is a very ample line bundle for $i = 1, \cdots, k$ and that the natural multiplication map

$$H^0(X, mL_1) \otimes \cdots \otimes H^0(X, mL_k) \to H^0(X, mL_1 + \cdots + mL_k)$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad H^0(X, -mK_X).$$

is surjective.

So we get a sequence of embeddings

$$\begin{tikzcd} \iota_{cpd} : X \arrow[r, hook] & \mathbb{P}(H^0(X, -mK_X)^\vee) \arrow[d, hook] \\ & \mathbb{P}(H^0(X, mL_1)^\vee \otimes \cdots \otimes H^0(X, mL_k)^\vee). \end{tikzcd}$$

We also have

$$\iota_i : X \hookrightarrow \mathbb{P}(H^0(X, mL_i)^\vee) \quad i = 1, \ldots, k.$$

Pick hermitian forms $H_{0,m}^{(i)}$ on $H^0(X, mL_i)$

$i = 1 = 1, \ldots, k$

and define

$$h_0 := \iota_{cpd}^* FS_{cpd}(H_{0,m}^{(1)} \otimes \cdots \otimes H_{0,m}^{(k)}) \text{ herm. metric on } -K_X.$$

$$h_0' := \iota_1^* FS_1(H_{0,m}^{(1)}) \otimes \cdots \otimes \iota_k^* FS_k(H_{0,m}^{(k)})$$

$$\underbrace{\iota_1^* FS_1(H_{0,m}^{(1)})}_{=: h_0^{(1)}} \qquad \underbrace{\iota_k^* FS_k(H_{0,m}^{(k)})}_{=: h_0^{(k)}}$$

herm. metric on $-K_X$.

<u>Def</u>. $\vec{\mathcal{H}} := \mathcal{H}_1 \times \cdots \times \mathcal{H}_k$

$$\mathcal{H}_i := \{ \phi \in C^\infty(X, \mathbb{R}) \mid \theta_i + \sqrt{-1}\partial\bar\partial\phi > 0 \}$$

Kähler metric associated to $h_0^{(i)}$

$\theta_i \in c_1(L_i)$.

The coupled Ding functional

$\mathcal{D}^{cpd} : \vec{\mathcal{H}} \to \mathbb{R}$, def by

$$\mathcal{D}^{cpd}(\phi_1, \cdots, \phi_k) := \mathcal{L}^{cpd}(\phi_1, \cdots, \phi_k) - \sum_{i=1}^{k} \mathcal{E}(\phi_i)$$

where

$$\mathcal{L}^{cpd}(\phi_1, \cdots, \phi_k) := -\log \int_X e^{-\sum_{i=1}^{k} \phi_i} \, d\mu_0'$$

volume form associated to h_0'.

All of these were introduced by Hultgren-Witt Nyström, who also introduced the coupled KE metrics, i.e.

$\omega_1 \in c_1(L_1), \cdots, \omega_k \in c_1(L_k)$, satisfying

$$\mathrm{Ric}(\omega_1) = \mathrm{Ric}(\omega_2) = \cdots = \mathrm{Ric}(\omega_k) = \sum_{i=1}^{k} \omega_i.$$

They showed that the crit. pt. of $\mathcal{D}^{cpd}$ is precisely the coupled KE metrics.

HWN show $\exists$ coupled KE metrics $\Rightarrow$ "coupled K-stability".

We want to find the slope formula for $\mathcal{D}^{\mathrm{cpl}}$.

Let $\left(H_t^{(1)}, \ldots, H_t^{(k)}\right) \in \mathcal{B}_m^{(1)} \times \cdots \mathcal{B}_m^{(k)}$

where $\mathcal{B}_m^{(i)} = GL(N_m^{(i)}, \mathbb{C}) / U(N_m^{(i)})$

$$N_m^{(i)} := \dim H^0(X, mL_i).$$

$$H_t^{(i)} := e^{-tA^{(i)*}} H_{0,m}^{(i)} e^{-tA^{(i)}}$$

for some $A^{(i)} \in \mathfrak{gl}(H^0(X, mL_i))$

$(i = 1, \ldots, k)$.

Lemma. $\exp\left(-\sum_{i=1}^{k} FS_i(H_t^{(i)})\right) d\mu_0'$

$$= \exp\left(-FS_{cpd}(\vec{H}_t)\right) d\mu_0$$

where $\tilde{H}_t := H_t^{(1)} \otimes \cdots \otimes H_t^{(k)}$ is the herm. form on $H^0(X, mL_1) \otimes \cdots \otimes H^0(X, mL_k)$

In particular,

$$\mathcal{L}^{cpd}\left(FS_1(H_t^{(1)}), \ldots, FS_k(H_t^{(k)})\right) = \mathcal{L}\left(FS_{cpd}(\vec{H}_t)\right).$$

Pf. $\displaystyle LHS = \prod_{i=1}^{k} \left(\sum_{j_i=1}^{N_m^{(i)}} \| s_{j_i}^{(i)} \|^2_{h_0^{(i)}} \right)^{-\frac{1}{m}} d\mu_0'$

$\{s_{j_i}^{(i)}\}_{j_i=1}^{N_m^{(i)}}$ is an $H_t^{(i)}$-onb.

$$= \left(\sum_{j_1=1}^{N_m^{(1)}} \cdots \sum_{j_k=1}^{N_m^{(k)}} \| S_{j_1}^{(1)} \otimes \cdots \otimes S_{j_k}^{(k)} \|_{h_0^{(1)} \otimes \cdots \otimes h_0^{(m)}} \right)^{-\frac{1}{m}} d\mu_0'$$

$$= \left(\sum_{i=1}^{k} \sum_{j_i=1}^{N_m^{(i)}} \| \vec{S}_{j_i} \|^2_{h_0'} \right)^{-\frac{1}{m}} d\mu_0'$$

$\{\vec{S}_{j_i}\}$ is an $\vec{H}_t$-onb for $H^0(X, mL_1) \otimes \cdots \otimes H^0(X, mL_k)$.

$$= \left(\sum_{i=1}^{k} \sum_{j_i=1}^{N_m^{(i)}} \| \vec{S}_{j_i} \|_{h_0} \right)^{-\frac{1}{m}} d\mu_0$$

$$= \exp\left(-FS_{cpd}(\vec{H}_t)\right) d\mu_0 \qquad \square$$

So finding the slope formula reduces to finding

$$\lim_{t \to t_0} \frac{d}{dt} \mathcal{L}(FS_{cpd}(\vec{H}_t)).$$

Suppose that each $A^{(i)} \in \mathfrak{gl}(H^0(X, mL_i))$ above has integral eigenvalues. Then we get a k-tuple of test config's

$$(\mathcal{X}_i, \mathcal{L}_i) := (\mathcal{X}_{A^{(i)}}, \mathcal{L}_{A^{(i)}}),$$

each for (X, L_i) of exponent m, as the Zariski closure of

$$\tau^{A^{(i)}} \cdot X \quad \text{in} \quad \mathbb{P}(H^0(X, mL_i)^\vee) \times \mathbb{C}$$

where $\tau : \mathbb{C}^* \longrightarrow GL(H^0(X, mL_i))$

$$\tau \longmapsto \tau^{A^{(i)}} \qquad \tau = e^{-t}$$

We define a test configuration $(\mathcal{Y}, \mathcal{L}_{\mathcal{Y}})$ by the Zariski closure of

$$L_{cpd}(X) \subset \mathbb{P}\left(H^0(X, mL_1)^\vee \otimes \cdots \otimes H^0(X, mL_k)^\vee\right)$$

under the 1-PS $\tau^{A^{(1)}} \otimes \cdots \otimes \tau^{A^{(k)}}$.

$\mathcal{L}_{\mathcal{Y}}$ = restriction of the hyperplane bundle.

<u>Prop</u>.

$$\lim_{t \to +\infty} \frac{d}{dt} \mathcal{L}\left(FS_{cpd}(\vec{H}_t)\right) = -1 + lct\left(\mathcal{Y}^\vee, D_{\mathcal{Y}^\vee}; \mathcal{Y}_0^\vee\right)$$

Thm.

$$\lim_{t\to\infty}\frac{d}{dt}\mathcal{D}^{cpd}\left(FS_1(H_t^{(1)}),\ \dots,\ FS(H_t^{(k)})\right)$$

$$= -\sum_{i=1}^{k}\frac{\left(\overline{\mathcal{L}^{\nu}_{A^{(i)}}}\right)^{n+1}}{(n+1)m^{n+1}\mathrm{Vol}(L_i)}$$

$$-1+\mathrm{lct}\left(\mathcal{Y}^{\nu}, D_{\mathcal{Y}^{\nu}};\ \mathcal{Y}^{\nu}_0\right)$$

$$=: \mathrm{Ding}\left(\left(\mathcal{X}_{A^{(i)}}, \mathcal{L}_{A^{(i)}}\right)_{i=1}^{k}\right).$$

<u>coupled Ding invariant</u>

Def. $(X, -K_X; L_1, \dots, L_k)$ is **coupled Ding stable** if for any $m \in \mathbb{N}$ and for any very ample test config $(\mathcal{X}, \mathcal{L}_1), \dots, (\mathcal{X}_k, \mathcal{X}_k)$ we have $\mathrm{Ding}((\mathcal{X}_i, \mathcal{L}_i)_{i=1}^k) \geq 0$ with equality iff all $(\mathcal{X}_i, \mathcal{L}_i)$'s are trivial.

Remarks

1. Original def of HWN further assumed $\mathcal{Y} = \mathcal{X}_1 = \cdots = \mathcal{X}_k$ in their def of stability. (e.g. when all t.c's are product, given by the same holo. v.f.)

Product case considered by Futaki – Zhang, Delcroix – Hultgren, Nakamura, ...

2. $\mathcal{Y}$ is determined by the generators of the $\mathbb{C}^*$-actions defining $(\mathcal{X}_1, \mathcal{L}_1), \dots, (\mathcal{X}_k, \mathcal{L}_k)$,

But different exponents may lead to different $\mathcal{Y}$.

e.g. If $\mathcal{Y}_m$ is def by $(\mathcal{X}_1, \mathcal{L}_1^{\otimes m}), \dots, (\mathcal{X}_k, \mathcal{L}_k^{\otimes m})$,

there's no reason to believe $\mathcal{Y}_m \cong \mathcal{Y}_1$.

Cor. Suppose that $(X, -K_X; L_1, \dots, L_k)$ admits a coupled KE metric. Then it is coupled Ding semistable, i.e.

$$\mathrm{Ding}\left((\mathcal{X}_i, \mathcal{L}_i)_{i=1}^k \right) \geq 0$$

$\forall m \in \mathbb{N}$, $\forall$ very ample t.c.'s of exponent m.

Open problems

1. Can the above be improved to uniform stability?

Should we use $J^{NA}(\mathcal{Y}, \mathcal{L}_{\mathcal{Y}})$ or $\sum_{i=1}^k J^{NA}(\mathcal{X}_i, \mathcal{L}_i)$?

2. Can we prove uniform coupled Ding stability $\Rightarrow \exists$ coupled KE metric?

3. Is it possible that the non-Archimedean metric on $-K_X$ rep by $(\mathcal{Y}, \mathcal{L}_{\mathcal{Y}})$ can be defined by the NA metrics rep. by $(\mathcal{X}_i, \mathcal{L}_i)$?

$\delta_m^{\text{coupled}} > 1$

Takahashi defined a coupled ACB metrics.

<u>Thm</u>. $\exists$ coupled ACB metric at level m

$\Leftrightarrow$ $\mathrm{Ding}\left(\left(\mathcal{X}_i, \mathcal{L}_i\right)_{i=1}^{k}\right) + \sum_{i=1}^{k} \mathrm{Chow}_m(\mathcal{X}_i, \mathcal{L}_i) \geq 0$ for all very ample t.c's of exponent m, with equality iff all $\mathcal{X}_i$'s are trivial.

$\exists$ coupled KE metrics

$\iff$ coercivity of $\mathcal{D}^{cpd}$

HWN, Takahashi

$\iff$ Coupled analytic δ-invariant > 1

$\parallel$ K. Zhang.

coupled (algebraic) δ-invariant