Lecture No 17   May 6, 2022 (Fri)

Recall that we are discussing the SDE:

$$dX_t = \alpha(t, X_t)\, dB_t + b(t, X_t)\, dt, \qquad (1)$$

where $B = (B_t)$ is an N-dimensional $(\mathcal{F}_t)$-Brownian motion, and related $\mathcal{L}_t$-martingale problem.

[Fact] (We state without proof)

- Equivalence between the martingale problem and the weak solution: We showed that the distribution of the weak solution is a solution of the $\mathcal{L}_t$-martingale problem.
- Conversely, if the $\mathcal{L}_t$-martingale problem has a solution P, one can construct (via martingale representation theorem) a Brownian motion and a weak solution of (1), whose distribution coincides with P.
- Accordingly, the existence and uniqueness of weak solution is equivalent to those of the solution of the $\mathcal{L}_t$-martingale problem.

- Result corresponding to Perron's theorem for ODEs:
  If the coefficients α, b are continuous in (t, x) (and satisfy the linear growth condition for non-explosion of solution, but no Lipschitz condition), the SDE (1) has a weak solution (a solution of martingale problem).
- It is constructed by Euler-Maruyama approximation.
  Instead of successive approximation, we consider division of time interval $[0, T]$ and construct approximating sequence step by step in time. This approximation converges only in law sense (by showing the tightness).

Uniqueness of weak solutions

- Yamada-Watanabe's theorem: If the strong solution is unique, then the weak solution is also unique.
- Stroock-Varadhan's theorem: If $a(t,x) = (a^{ij}(t,x))$ is bounded continuous and uniformly positive definite (in this case, we call $\mathcal{L}_t$ uniformly elliptic):
$$^\exists c > 0 \text{ s.t. } \sum_{ij} a^{ij}(t,x)\xi_i\xi_j \geq c|\xi|^2, \ ^\forall x, \xi \in \mathbb{R}^d$$
and $b(t,x)$ is a bounded Borel measurable function, then the weak solution is unique.
- This result is very different from ODE. For example, it is well-known that the solution of ODE is not unique, if we take $b = b(x) = (\operatorname{sgn} x)|x|^\alpha, 0 < \alpha < 1, x \in \mathbb{R}$.
- However, if we add non-degenerate Brownian motion, roughly, X_t immediately escapes the "bad points" of b and we obtain the uniqueness of the weak solution. (b is treated by Cameron-Martin-Girsanov-Maruyama's formula discussed later.)

Tanaka's example for $^{\exists}$weak solution but $^{\nexists}$strong solution:
Consider the SDE:

$$dX_t = \text{sgn}\,(X_t) dB_t, \tag{2}$$

where $\text{sgn}(x) := 1_{\{x>0\}} - 1_{\{x \le 0\}}$.

1. Existence of weak solution

⛶ If a solution X_t exists, it is given by a stochastic integral so that it is a martingale with quadratic variation

$$\langle X \rangle_t = \int_0^t \left(\text{sgn}\,(X_s)\right)^2 ds = t.$$

Therefore, by Lévy's theorem, X_t is a Brownian motion. By this consideration, let X_t be a Brownian motion defined on a certain probability space and set

$$B_t := \int_0^t \text{sgn}\,(X_s) dX_s.$$

Then, as above, we see that B_t is a Brownian motion. From $dB_t = \text{sgn}(X_t) dX_t$, noting $\frac{1}{\text{sgn}(x)} = \text{sgn}(x)$, we obtain

$$\mathrm{sgn}(X_t)dB_t = dX_t.$$

Thus, we see that X_t is a solution of the SDE (2) with B_t constructed by us, so that X_t is a weak solution of (2). □

2. Non-existence of strong solution

☺ If a strong solution X exists, it is $\sigma\{B_\cdot\}$-measurable, i.e. $\mathcal{F}_t^X \subset \mathcal{F}_t^B$ holds. However, by $\mathrm{sgn}(X_t)dB_t = dX_t$, we have

$$B_t = \int_0^t \mathrm{sgn}\,(X_s)dX_s \underset{(\star)}{=} |X_t| - \int_0^t \delta_0(X_s)ds$$

$(\star)$ is shown if one can apply Itô's formula for $f(X_t)$ taking $f(x) = |x|$, since $f'(x) = \mathrm{sgn}(x), f''(x) = 2\delta_0(x)$. Indeed, the last term has only a formal meaning, but if we interpret this term as

$$\lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t 1_{\{|X_s| \le \varepsilon\}} ds,$$

then it is known that the above formula holds. This is called Tanaka's formula and we will discuss later.

The RHS of $(\star)$ depends only on $\{|X_s|; 0 \le s \le t\}$ so that we obtain $\mathcal{F}_t^B \subset \mathcal{F}_t^{|X|}$. Therefore, $\mathcal{F}_t^X \subset \mathcal{F}_t^{|X|}$ holds. However, since X is a Brownian motion, this never happen (if so, $X_t =^{\exists} f(|X_t|)$ i.e. $x = f(|x|)$ on $\mathbb{R}$, which is not true). This proves the non-existence of strong solution. □

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§17 Relation of SDEs to PDEs and Feynman-Kac's formula

Here we consider the temporary homogeneous case, that is, we assume the coefficients α, b of the SDE do not depend on t and denote the operator $\mathcal{L}_t$ by $\mathcal{L}$. The SDE is written as

$$dX_t = \alpha(X_t)\, dB_t + b(X_t)\, dt \tag{3}$$

and for $u = u(x) \in C^2(\mathbb{R}^d)$

$$\mathcal{L}u(x) = \frac{1}{2} a^{ij}(x) \frac{\partial^2 u}{\partial x^i \partial x^j}(x) + b^i(x) \frac{\partial u}{\partial x^i}(x),$$

where $a = \alpha\, \alpha^*$, that is, $a^{ij}(x) = \sum_{k=1}^{N} \alpha_k^i(x) \alpha_k^j(x)$.

17.1 Dirichlet problem

We generalize Kakutani's theorem (Theorem 13.2) for Brownian motion to the solution of the SDE (3).

[Theorem 17.1] Let $D \subset \mathbb{R}^d$ be a bounded open set and assume $u \in C^2(D) \cap C(\bar{D})$ solves the Dirichlet problem:

$$\begin{aligned} \mathcal{L}u(x) &= 0, \quad x \in D \\ u(x) &= f(x), \quad x \in \partial D \end{aligned}$$

Moreover, we assume that u can be extended to D^c such that $u \in C_b^2(\mathbb{R}^d)$. Then, u has a probabilistic representation:

$$u(x) = E_x\left[f(X_\sigma)\right], \quad x \in D,$$

where $X = (X_t)$ is a solution of the SDE (3) starting at x and $\sigma := \inf\{t > 0; X_t \in \partial D\}$. We assume that $\sigma < \infty$ a.s. In particular, the solution of the Dirichlet problem is unique. □

[Remark] If $(a^{ij}(x))$ is uniformly elliptic (and D is bounded), $\sigma < \infty$ (a.s.) holds. □

[Proof] By Proposition 16.1,

$$u(X_t) - \int_0^t \mathcal{L}u(X_s)\,ds$$

is a martingale. Therefore, noting that $\mathcal{L}u(X_s) = 0$ for $s < \sigma$, by Doob's optional sampling theorem, we see that $u(X_{t\wedge\sigma})$ is a bounded martingale. Thus, we obtain

$$u(x) \underset{\text{(i)}}{=} E_x[u(X_{t\wedge\sigma})] \underset{t\to\infty}{\longrightarrow} E_x[u(X_\sigma)] \underset{\text{(ii)}}{=} E_x[f(X_\sigma)]$$

and this shows the conclusion. Here, (i) follows by the martingale property for ${}^\forall t \geq 0$. For the next limit procedure, we apply Lebesgue's convergence theorem by noting that u is bounded and $\sigma < \infty$ a.s. by our assumption. Finally, (ii) is shown similarly to the proof of Kakutani's theorem by noting that $X_\sigma \in \partial D$ and $u = f$ on ∂D.

The uniqueness of the solution of the Dirichlet problem is shown similarly to Kakutani's theorem. □

17.2 Cauchy problem and Feynman-Kac formula

- Theorem 17.1 was for elliptic equations. Now, we consider parabolic equations on $\mathbb{R}^d$.
- Let an initial value $f \in C_b(\mathbb{R}^d)$ and a potential function $V \in C_b(\mathbb{R}^d)$ be given.
- We assume the coefficients α, b of the SDE (3) satisfy the linear growth condition.
- Recall the differential operator $\mathcal{L}$ associated with the SDE:

$$\mathcal{L}u(x) := \tfrac{1}{2}a^{ij}(x)\frac{\partial^2 u}{\partial x^i \partial x^j}(x) + b^i(x)\frac{\partial u}{\partial x^i}(x)$$

$$a^{ij}(x) := \sum_{k=1}^{N} \alpha^i{}_k(x)\alpha^j{}_k(x)$$

[Theorem 17.2]
Let $u = u(t,x) \in C^{1,2}((0,\infty) \times \mathbb{R}^d) \cap C([0,\infty) \times \mathbb{R}^d)$ be a solution of the initial value problem (Cauchy problem) of the following parabolic equation:

$$\frac{\partial u}{\partial t} = \mathcal{L}u + Vu, \quad t > 0,\ x \in \mathbb{R}^d$$
$$u(0,x) = f(x), \quad x \in \mathbb{R}^d,$$

and assume that it grows at most polynomially in x, that is, for $^\forall T > 0$, $^\exists C = C_T$, $^\exists p = p_T > 0$ such that

$$|u(t,x)| \leq C(1 + |x|^p), \quad t \in [0,T],\ x \in \mathbb{R}^d. \tag{4}$$

Then, u is represented by means of the solution $X = (X_t)$ of the SDE (3) starting at x as

$$u(t,x) = E_x\left[f(X_t)e^{\int_0^t V(X_s)\,ds}\right].$$

□

[Proof] Fix $T > 0$ and set

$$M_t := u(T-t, X_t) e^{\int_0^t V(X_s)\,ds}, \quad t \in [0, T].$$

Then by Itô's formula (note that $d\{u(\cdots)\}d\{e^{\cdots}\} = 0$)

$$\begin{aligned} dM_t =& e^{\int_0^t V(X_s)\,ds} d\{u(T-t, X_t)\} + u(T-t, X_t) d\{e^{\int_0^t V(X_s)\,ds}\} \\ =& \sum_{i=1}^{d} \sum_{k=1}^{N} \frac{\partial u}{\partial x^i}(T-t, X_t) {\alpha^i}_k(X_t) e^{\int_0^t V(X_s)\,ds}\, dB_t^k \\ &+ \left[-\tfrac{\partial u}{\partial t}(T-t, X_t) + \mathcal{L}u(T-t, X_t) + V(X_t)u(T-t, X_t)\right] \\ &\qquad \times e^{\int_0^t V(X_s)\,ds}\, dt. \end{aligned}$$

However, by the equation satisfied by u, the term of dt cancels. Accordingly, M_t is a martingale. Thus, we have $E_x[M_T] = E_x[M_0]$. However, since

$$\text{(LHS)} = E_x\left[f(X_T) e^{\int_0^T V(X_s)\,ds}\right], \ \text{(RHS)} = u(T, x),$$

we obtain the conclusion by taking $t = T$. □

(The last argument is rough→More precisely, we need the cut-off argument in the next page)

- It requires some work to show that M_t is a martingale (for example, to show its integrability $\leftarrow$ (4), (5)).
- By the linear growth property of the coefficients, one can show that the solution X_t satisfies

$$E\left[\sup_{0\le t\le T}|X_t|^p\right] < \infty \tag{5}$$

  for ${}^\forall p > 1$, $T > 0$. P: Show this estimate. (Use Burkholder, Hölder, Gronwall and cut-off by σ_n below.)
- We introduce a cut-off for X_t by n by setting $\sigma_n := \inf\{t > 0; |X_t| > n\}$. Then, one can show $E_x[M_0] = E_x[M_{T\wedge\sigma_n}]$. Here, $E_x[M_0] = u(T, x)$ holds. On the other hand, $E_x[M_{T\wedge\sigma_n}]$ converges as $n \to \infty$ to $E[M_T]$ by showing that $M_{T\wedge\sigma_n}$ is uniformly integrable in n from at most polynomial growth property of u and (5). Thus, we obtain the conclusion.

§18 Cameron-Martin-Girsanov-Maruyama's formula

[Setting]

- $(\Omega, \mathcal{F}, \mathcal{P})$, $(\mathcal{F}_t)_{t\geq 0}$: Probability space and its filtration
- $B = (B_t(\omega))_{t\in[0,1]}$: 1-dimensional $(\mathcal{F}_t)$-Brownian motion starting at 0: $B_0 = 0$.
- $\eta_t = \eta(t,\omega)$, $t \in [0,1]$: $(\mathcal{F}_t)$-adapted stochastic process and absolutely continuous in $t^{\star)}$ s.t. $\eta_0 = 0$ and

$$\sup_{\omega\in\Omega} \int_0^1 \dot{\eta}_t(\omega)^2 \, dt < \infty.$$

- See the next page for $\star$).

[Remark for the absolute continuity]

- $\eta(t)$, $t \in [0,1]$ is absolutely continuous in $t^{\star)}$
  $\underset{\text{def}}{\Longleftrightarrow}$ For $^{\forall}\varepsilon > 0$, $^{\exists}\delta > 0$ s.t. $\sum_{i=1}^{n} |\eta(b_i) - \eta(a_i)| < \varepsilon$
  holds for any disjoint $\cup_{i=1}^{n}(a_i, b_i] \subset [0,1]$ s.t.
  $\sum_{i=1}^{n}(b_i - a_i) < \delta$.
- If η is absolutely continuous, then its derivative $\dot{\eta}_t$ exists for a.e. t, and $\eta_b - \eta_a = \int_a^b \dot{\eta}_t dt$ holds for $0 \leq a < b \leq 1$.
- In fact, by definition, $\eta(t)$ is of bounded variation in t. Therefore, it defines Lebesgue-Stieltjes (signed) measure $d\eta$ on $[0,1]$, which is absolutely continuous with respect to the Lebesgue measure dt.
- $\dot{\eta}(t)$ is the Radon-Nikodým derivative: $\dot{\eta}(t) = \frac{d\eta}{dt}$.

[Theorem 18.1] (CMGM's formula) One can define*) a (new) probability measure Q on $(\Omega, \mathcal{F})$ by

$$\frac{dQ}{dP}(\omega) = \exp\left\{\int_0^1 \dot{\eta}_t dB_t - \frac{1}{2}\int_0^1 \dot{\eta}_t^2\, dt\right\},$$

and under Q,

$$\hat{B}_t := B_t - \eta_t, \quad t \in [0,1]$$

is a Brownian motion. □

[Remark] Above formula $\exp\{\cdots\}$ in RHS is considered under the probability measure P. Under the original measure P, B_t is a Brownian motion and $\int_0^1 \dot{\eta}_t dB_t$ is defined as a stochastic integral. □

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*) If we define a measure Q by $dQ := \frac{dQ}{dP} dP$ i.e. $Q(A) = \int_A \exp\{\cdots\} dP, A \in \mathcal{F}$, then Q is a probability measure, i.e. $Q(\Omega) = 1$ i.e. $E^P[\exp\{\cdots\}] = 1$ holds.

In a sense, the converse holds.

[Corollary 18.2] (Cameron-Martin's formula)

- $\bar{P}$: Wiener measure on $W^1 \equiv C([0,1],\mathbb{R})$
  (i.e., the distribution of B)
- $\eta = (\eta_t) \in C^1([0,1])$: non-random (i.e. independent of ω)
- $\bar{Q}$: The distribution of $(B_t + \eta_t)_{t\in[0,1]}$ on W^1

$\Longrightarrow$ $\bar{Q} \prec \bar{P}$ (i.e. $\bar{Q}$ is absolutely continuous with respect to $\bar{P}$) and its Radon-Nikodým density function is given by

$$\frac{d\bar{Q}}{d\bar{P}}(w) = \exp\left\{\int_0^1 \dot{\eta}_t dw_t - \frac{1}{2}\int_0^1 \dot{\eta}_t^2\, dt\right\},$$

where $w = (w_t)_{t\in[0,1]} \in W^1$. Note that w_t in the RHS is a Brownian motion under $\bar{P}$. □

- We give the proof of Corollary 18.2 assuming that Theorem 18.1 is shown.
- We will give the proof of Theorem 18.1 later.

[Proof of Corollary 18.2] • Defining a probability measure Q on W^1 by

$$\frac{dQ}{d\bar{P}}(w) = \exp\left\{\int_0^1 \dot{\eta}_t dw_t - \frac{1}{2}\int_0^1 \dot{\eta}_t^2\, dt\right\},$$

it is enough to prove $Q = \bar{Q}$.

• By applying Theorem 18.1 with $\Omega = W^1, P = \bar{P}$, $\hat{w}_t(w) := w_t - \eta_t$ is a Brownian motion under Q. Thus, for any bounded measurable function $\Phi = \Phi(w)$ on W^1, we have

$$E^Q[\Phi(w)] \underset{\text{Definition of } \hat{w}}{=} E^Q[\Phi(\hat{w} + \eta)] \underset{\hat{w} \text{ is BM}}{=} E^{\bar{P}}[\Phi(w+\eta)] \underset{\text{Definition of } \bar{Q}}{=} E^{\bar{Q}}[\Phi(w)]$$

This shows $Q = \bar{Q}$, since the test function Φ is arbitrary. □

- Higher dimensional versions of Theorem 18.1 and Corollary 18.2 are known.
- Corollary 18.2 can be shown easily by a heuristic (physical) argument based on a Feynman measure.

Wiener measure $\bar{P}$ is heuristically written as

$$\bar{P}(dw) = \frac{1}{Z} e^{-\frac{1}{2}\int_0^1 \dot{w}_t^2 dt} \prod_{t \in [0,1]} dw_t$$

The part of the measure is a formal infinite product of Lebesgue measure called Feynman measure. Z is a normalization constant to make RHS a probability measure.

⚇ Dividing the interval $[0,1]$ as $0 = t_0 < t_1 < \cdots < t_n = 1$,

$$\begin{aligned} \bar{P}(w_{t_1} \in dx_1, \ldots, w_{t_n} \in dx_n) &= \prod_{i=1}^{n} p(t_i - t_{i-1}, x_{i-1}, x_i) dx_i \\ &= \frac{1}{Z_n} \exp\Big\{ -\sum_{i=1}^{n} \frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})} \Big\} \prod_{i=1}^{n} dx_i \end{aligned}$$

Z_n is a certain constant. However, the equation in the exponential would converge as

$$= -\frac{1}{2} \sum_{i=1}^{n} \Big(\frac{x_i - x_{i-1}}{t_i - t_{i-1}} \Big)^2 (t_i - t_{i-1}) \longrightarrow -\frac{1}{2} \int_0^1 \dot{w}_t^2 dt$$

□

Based on the above heuristic form of $\bar{P}$, we would have

$$\begin{aligned}\bar{Q}(d\omega) &= \bar{P}(w+\eta \in d\omega) \qquad (\because\ w \text{ is a BM under } \bar{P}) \\ &= \bar{P}(w \in d\omega - \eta) \\ &\underset{w=\omega-\eta}{=} \frac{1}{Z} e^{-\frac{1}{2}\int_0^1 (\dot{\omega}_t - \dot{\eta}_t)^2 dt} \prod_{t\in[0,1]} d(\omega_t - \eta_t) \\ &= e^{\int_0^1 \dot{\omega}_t \dot{\eta}_t dt - \frac{1}{2}\int_0^1 \dot{\eta}_t^2 dt} \bar{P}(d\omega)\end{aligned}$$

For the last equality, since η_t is non-random and Lebesgue measure is shift-invariant, one would have $d(\omega_t - \eta_t) = d\omega_t$ and then use the above heuristic form of $\bar{P}$ again. Thus, regarding

$$\int_0^1 \dot{\omega}_t \dot{\eta}_t dt = \int_0^1 \dot{\eta}_t d\omega_t,$$

Corollary 18.2 is shown (at least heuristically). □