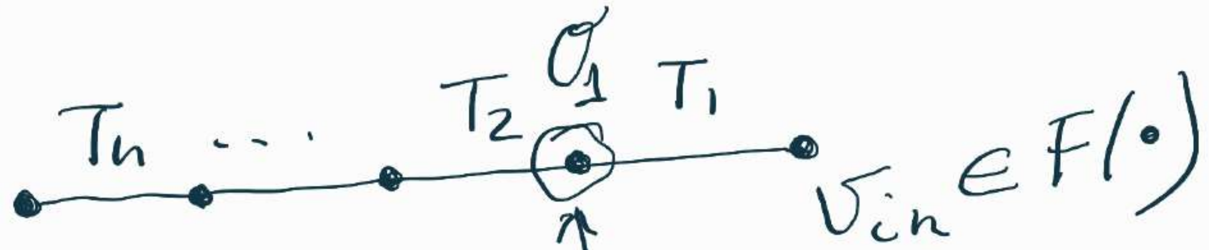
Higher topological theories and problems in pure math, or how to get a Fields medal.

In HTQM

$$F\Big(\underset{T}{\bullet\!\!\rule[0.5ex]{3em}{1pt}}\Big) = \exp(tH + dt\,G)$$

where $H = \{Q, G\}$

In HTQM correlators



$v_{out} \in F(\bullet)$ $\quad O \in End(V)$

$$F(\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet) =$$

$$v_{out} \ldots O_2\, e^{T_2 H + dT_2 G}\, O_1\, e^{T_1 H + dT_1 G}\, v_{in} \quad \bullet$$

(*) $\quad \Omega^\bullet$(configuration space)

space of positions of points

coord. on configuration space are $T_1, \ldots, T_n$

Object (*) has several meanings

It is instructive to integrate it over configuration space

1. This integral turns out an interesting math. number in math applications

2. This integral is a deformation of HTQM

3. This integral satisfies interesting quadratic relations relations between math. problems in point 1.

Key to mirror formulas

math. numbers of 1. correspond to def. of. T. Theory

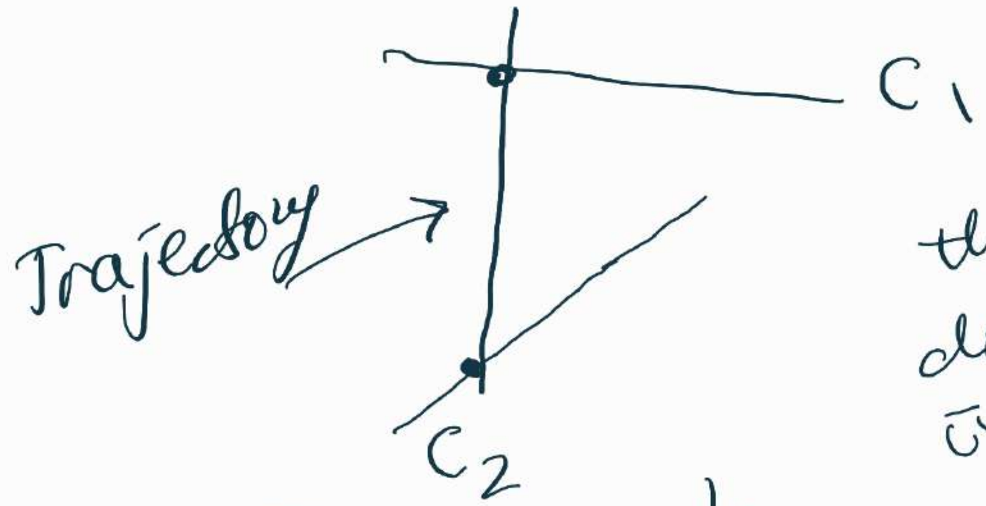
Up to now we study it in dim 1, However, we will generalize it to $\dim_{\mathbb{R}}=2$ and even to real dim $X = 4$ by conjectural equivalence between Functorial and Feynman approaches to HTQFT.

Simplest possible example

$$v_1 = w^{\epsilon}_{c_1}$$
$$v_2 = w^{\epsilon}_{c_2}$$

$$\int_{R_+} \int_X w^{\epsilon}_{c_2}\, e^{tH + dTG}\, w^{\epsilon}_{c_1}$$

$H = Lv \qquad G = 2v$



they do not intersect.

and I do not care about time trajectory goes from c_1 to c_2

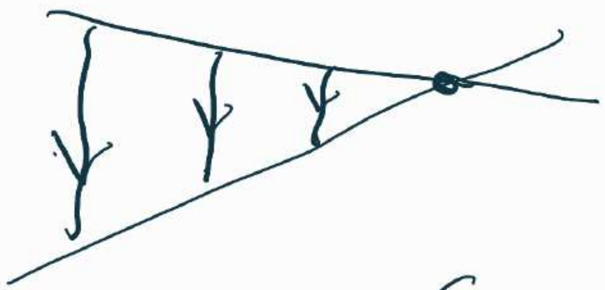
Compare with the easy problem

$$\int_X \omega^{\epsilon}_{C_2}\, e^{TH}\, \omega^{\epsilon'}_{C_1} = f(T)$$

— a function of T

One can show that $f(T)$ is actually a constant.

$Q = d$



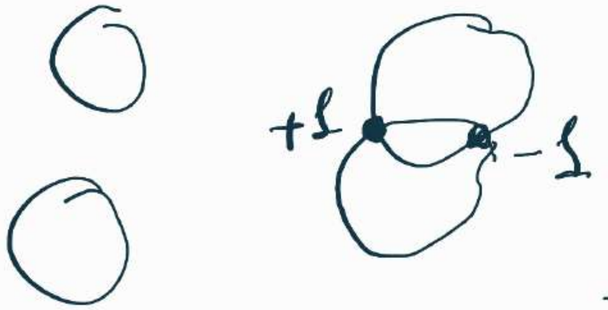
$$\frac{\partial f(T)}{\partial T} = \int_X \omega^{\epsilon}_{C_2}\, H e^{TH}\, \omega^{\epsilon}_{C_1}$$

now $H = \{d, i_v\}$ and integrate by parts $\rightsquigarrow 0$ $\boxed{\text{not very interesting}}$

$C_1 \quad \tilde{C}_1$

it is a homotopy

$$\omega^{\epsilon}_{C_1} - \omega^{\epsilon}_{\tilde{C}_1} = d(\dots)$$



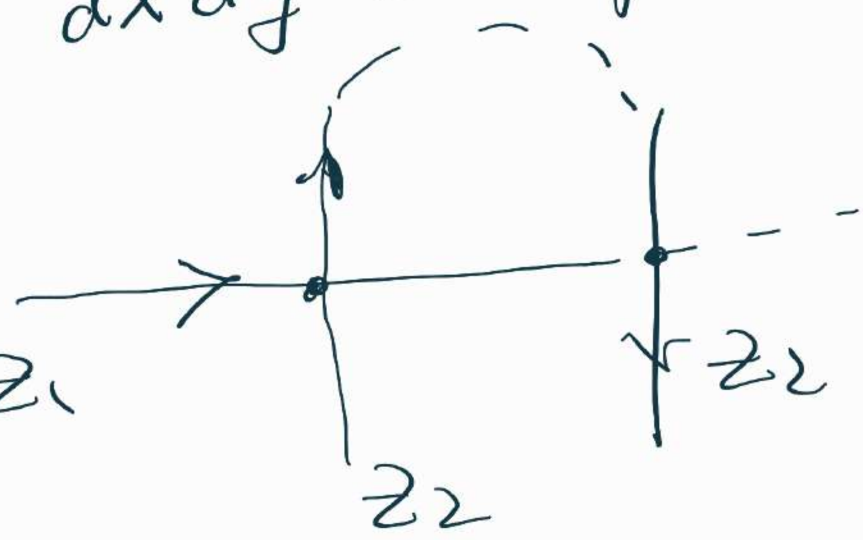
intersection should be taken with signs

$$\omega^{\epsilon}_{Z_1}\cdot\omega^{\epsilon}_{Z_2} = \omega_{Z_1\cap Z_2}$$

Z_2 was considered as orienta

$$\int_X \omega^{\epsilon}_{Z_1}\cdot\omega^{\epsilon}_{Z_2} = \text{number of points taken with signs}$$

$$dx\,dy = -dy\,dx$$



However

$$\int_{\overline{R_+}} \omega_{c_2}^{\epsilon}\, e^{\underline{\underline{\underline{I L_{v}\, i_v dT}}}}\, \omega_{c_1}^{\epsilon}$$

is a new enumerative problem.

In particular, if $\dim X = 3$ and $\dim C_i = 1$ we tipically have a nonzero answer.

Relation between new numbers and def. theory

In functonial approach to QFT we observed that deformation of QM is

$$H \to H + \Phi$$

$$e^{t(H+\varphi)} = e^{tH} + \int\limits_0^t e^{(t-\tilde t)H}\,\varphi\, e^{\tilde t H} + \ldots\ldots$$

Integrals over config. space correspond to deformation of $H \to H + \underbrace{\{G, \mathcal{O}\}}_{\varphi}$

in particular in the deformed theory $H = \{Q, G\} \to \{G, Q\} + \{G, \mathcal{O}\}$

Deformed theory again is topological with a new!

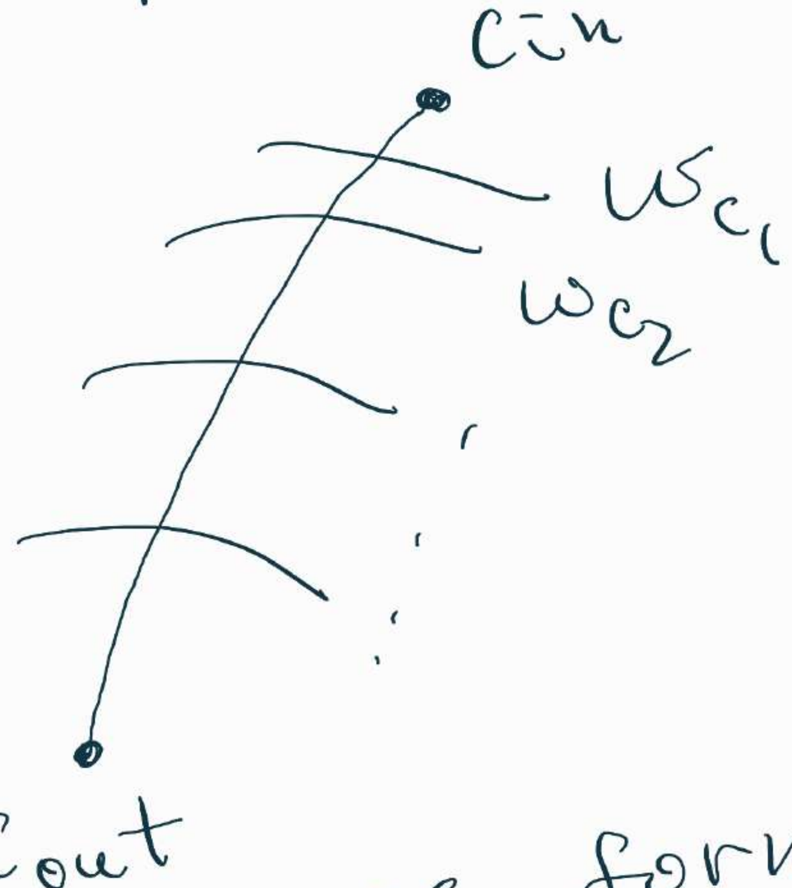
$$Q \longrightarrow Q + \mathcal{O}$$

Example: $Q = d$

$\mathcal{O} = \omega_C^\epsilon : d + \omega_C^\epsilon$

(Witten-Novikov deformation)

Witten – Novikov alg. coor. to enumerative problem:



A bit more formal

Let me introduce gene. parameters τ_i for a set of cycles c_i, such that $p(\tau_i) + \operatorname{codim} c_i = 1 \mod (2)$

$\qquad\qquad \| \quad \epsilon$

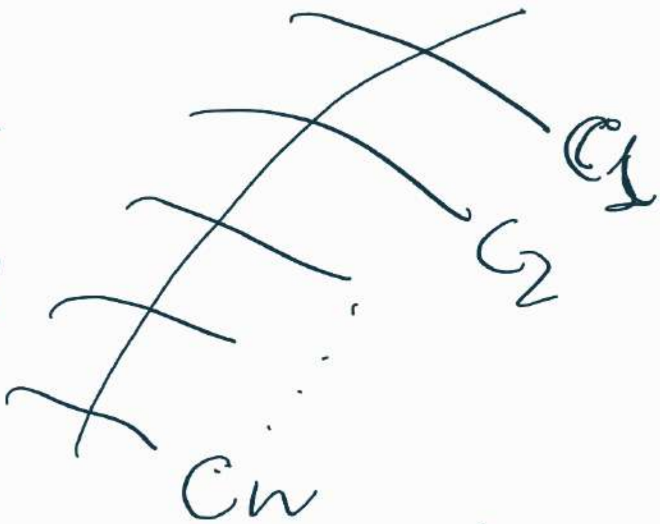
$\qquad\qquad \dim \omega_{c_i}$

$$\mathcal{O}(\tau) = \sum_i \tau_i \, \omega^t_{c_i}$$

Then deformation

$d \to d + \mathcal{O}(\tau)$ correspond to generating function for enumerative problems

number of trajectories



in any order.

$$F(\tau) = \sum n(c_{i_1}, \ldots, c_{i_n}) \, \tau_{i_1} \ldots \tau_{i_n}$$

corresponds to a deformed top. theory

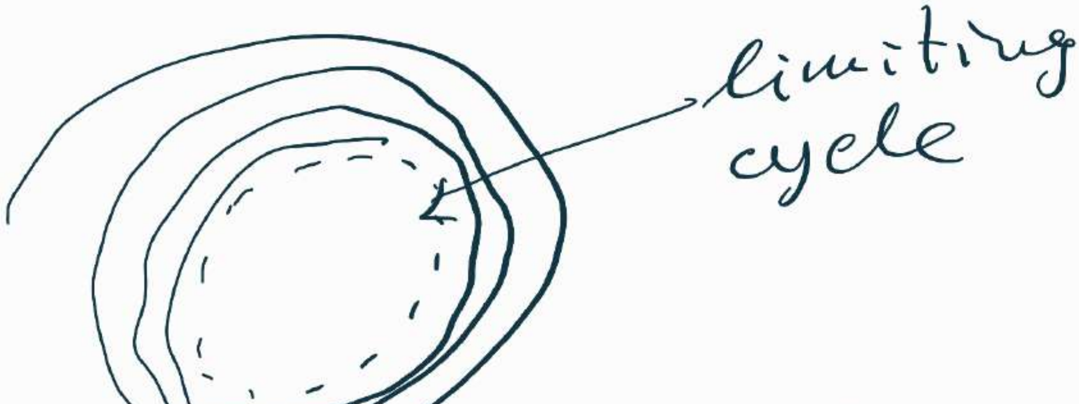
1. Consider vector field of the Morse type:

$$v^i = g^{ij} \frac{\partial W}{\partial x^j}, \quad \text{here}$$

W is a function on X and g^{ij} is a metric on X.

Such vector fields do not have limiting cycles.

Trajectory of a vector field can go like this



Morse vector fields have no limiting cycles, only limiting points

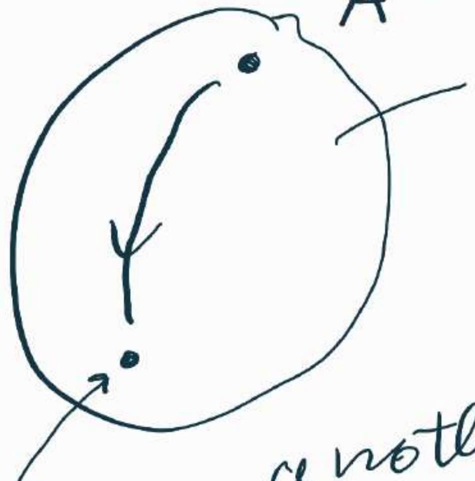
Reason: Consider the derivative of W along the trajectory.

$$\frac{dW}{dt} = \frac{\partial W}{\partial x^i} v^i(x) =$$
$$= \frac{\partial W}{\partial x^i} \frac{\partial W}{\partial x^j} g^{ij} > 0$$

so trajectory cannot make rounds! Limiting cycle is impossible.

Look for enumerative problem

A – fixed point of X v_{Morse}



B – another fixed point.

we may study (Morse did) number of trajectories going from $A \to B$: n^A_B

In general, it is not a number – they form families

Morse proposed to study only isolated trajectories.

Morse proved:

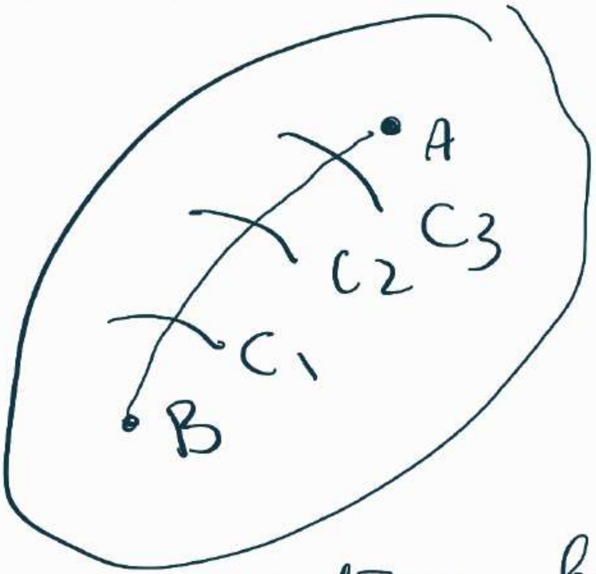
1) $\sum_B n^A_B \cdot n^B_C = 0$, so there is a complex, spanned by critical points with n^A_B being a differential

2) This complex is quasiisomorphic to $H^\bullet X$

Morse theory may be refined.

Namely:

$n^A_B(C_1, \ldots, C_n)$ — number of trajectories going from A to B and passing through cycles $C_1 \ldots C_n$



New enumerative problem

There are relations between

these numbers

$$n^A{}_B(\emptyset)\, n^B{}_C(\emptyset) = 0$$

$$n^A{}_B(C_1)\cdot n^B{}_C(\emptyset) + n^A{}_B(\emptyset)\, n^B{}_C(C_1] = 0$$

$$n^A{}_B(C_1, C_2)\cdot n^B{}_C(\emptyset) + n^A{}_B(C_1)\, n^B{}_C(C_2) + (1 \leftrightarrow 2) + \ldots\ n^A{}_B(\emptyset)\cdot n^B{}_C(C_1, C_2) = 0$$

There are new numbers and quadratic relations.
All these relations may be written unifomely as

$$n^A{}_B(\tau) = \sum n^A{}_B(c_{i_1}, \ldots, c_{i_n})\, \tau_{i_1} \ldots \tau_{i_n}$$

$$\boxed{n^A{}_B(\tau)\cdot n^B{}_C(\tau] = 0}$$

New numbers correspond to deformation of the morse complese.
(one may say something about these new numbers)

$n^A_B(C)$ turns out to be just action of W^ϵ_C on cohomology of X associated to points A and B.
(old operation)

However $n^A_B(C_1, C_2)$ is a new operation
(Refined Massey operation)

Rational homotopy type of the manifold is given by its Massey operations!
Fields medal to Sullivan.

Consider variation of the topic:

Replace one dimensional QFT by two-dimensional

Replace equations

$$\frac{dX^i}{dt} = v^i(X(t))$$ by

Holomorphicity equations

$$\bar\partial X^i = 0,$$

then we will get

2-dim QFT

$$\int \mathcal{D}p\,\mathcal{D}\pi\,\mathcal{D}X\,\mathcal{D}\psi\,\mathcal{D}\bar p\,\mathcal{D}\bar\pi\,\mathcal{D}\bar X\,\mathcal{D}\bar\psi$$

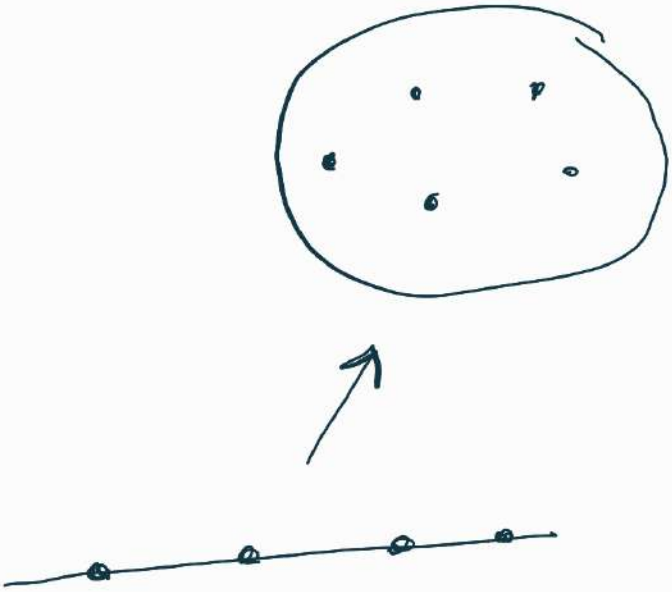
$$\exp\left[p_i\bar\partial X^i + \bar p_i\partial\bar X^i + \pi_i\bar\partial\psi^i + \bar\pi_i\partial\bar\psi^i\right].$$

$$\mathcal{O}_{\omega_1}(z_1)\ldots\mathcal{O}_{\omega_n}(z_n)$$

$$G(w_1)\ldots G(w_K)$$

$$\mathcal{O}_{\omega_1}(z_1) = \omega_1(X(z_1), \Psi(z_1))$$

dif. forms



conf space is the space of points on $\mathbb{CP}^1$

And again we study dif. forms on configuration space.

$$\omega_i \to \omega_{c_i}^{\epsilon}$$

$$N(c_1, \dots, c_k) -$$

number of holomorphic maps passing through cycles $c_1 \dots c_k$.

Kontsevich and Manin wrote equations for these numbers generalizing →

→ and in many cases all numbers were computed.

---

what if we go to dim 4:

$$\frac{dX}{dt} = V(X)$$ dim of QFT = 1

$$\bar{\partial} X = 0$$ dim of QFT = 2

Gauge theory and equations are ← dim of QFT = 4

$$F = *F$$ ↔ Donaldson theory

Corresponding functional integral is

$$S = \int P_+ F + \text{fermions}$$ →

→ $N=2$ SYM theory!

May be studied by tools of QFT and also by tools of pure mathematics.

In particular, conjectures on M-theory may be checked in the study of this theory!

However, analogues of equations $h^A_B(\tau)\, h^B_C(\tau) = 0$ was <u>not</u> find yet

---

Idea: how to get quadratic relations

From the property of being HTQFT we may show that

$$Q_2\, e^{T, H + dT, G}\, \mathcal{O}, v_{in}$$

$v_{out} \ldots$

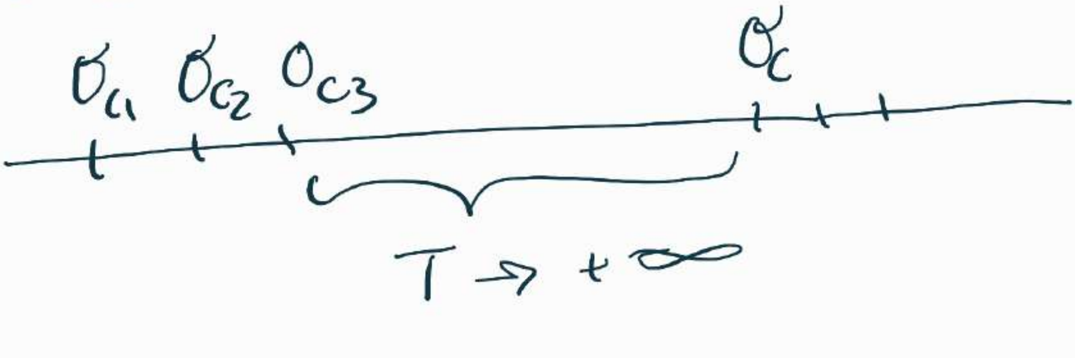
BEFORE being integrated over configuration space is a <u>closed</u> dif-form.

<u>Idea:</u> $\int$ closed dif. form $=$
$= 0$

boundary

integrate over a boundary of configuration space

Boundary corresponds to some $T \to +\infty$



contribution of the component of the boundary is a product of interals over configuration spaces with smaller number of points $\longrightarrow$.

combining all components, we will get equations like

$$n^{A}{}_{B}(\tau)\cdot n^{B}_{C}(\tau) = 0$$