Numbers and Geometry

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Numbers:

Define

$$\mathcal{M}_2 = \{ (m_1, m_2) \mid m_1, m_2 \ \mathbb{Z}^{\geq 0},\ \gcd(m_1, m_2) = 1 \}.$$

$(0,0) \notin \mathcal{M}_2$

Fix $(n_1, n_2) \in \mathcal{M}_2$.

Define

$$\alpha : \mathcal{M}_2 \longrightarrow \mathbb{Z}$$

as follows. For each (m_1, m_2), can write

$$(m_1, m_2) = a(n_1, n_2) + b\, e \quad , \qquad a, b \geq 0 \quad \text{and} \quad e = (1,0) \ \text{or} \ (0,1).$$

Put $\alpha(m_1, m_2) = a + b$.

clearly, $\alpha > 0$.

Example:

- $(n_1, n_2) = (1,1) \Rightarrow \alpha \geq 1.$
- $(n_1, n_2) = (1,2) \Rightarrow \alpha \geq 1.$
- $(n_1, n_2) = (1,n) \Rightarrow \alpha \geq 1.$
- $(n_1, n_2) = (2,3) \Rightarrow \alpha \geq \frac{2}{3}$
- $(n_1, n_2) = (n, n-1) \Rightarrow \alpha \geq \frac{2}{n}$
- $(n_1, n_2) = (n, 2n-1) \Rightarrow \alpha \geq \frac{2}{n}.$

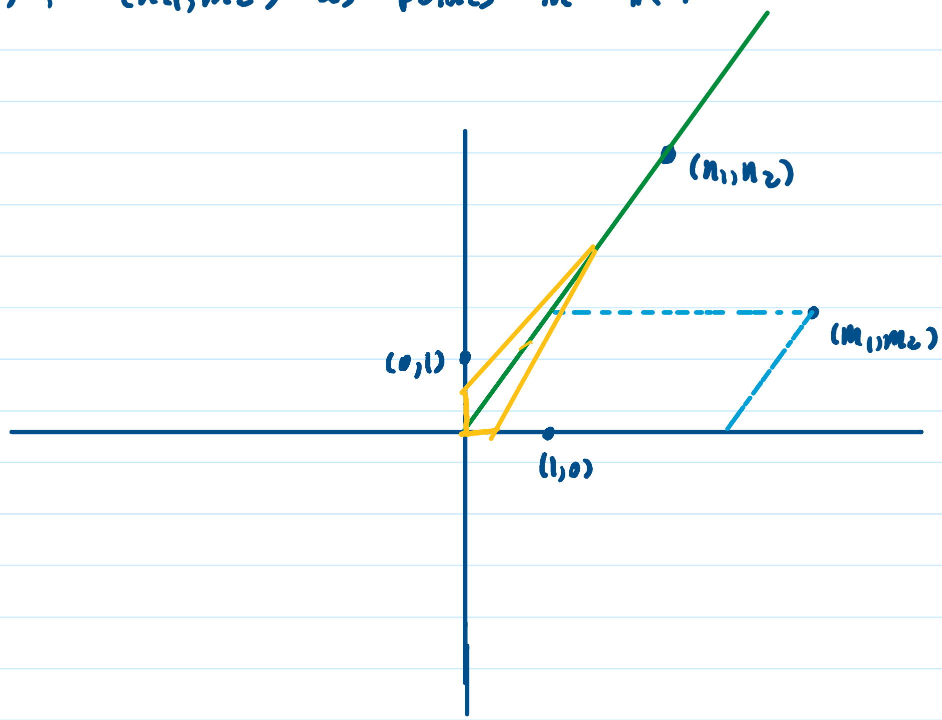
Fact: For $\varepsilon > 0$, $\exists \ell$ s.t.

$$\alpha \geq \varepsilon \Rightarrow \min\{n_1, n_2\} \leq \ell.$$

Exercise: Prove the fact.

Geometric interpretation:

Consider (n_1, n_2), (m_1, m_2) as points in $\mathbb{R}^2$.



$\alpha \geq \varepsilon \iff \nexists\ (m_1, m_2) \in$ Interior of Polytope given by

$(0,0)$, $(\varepsilon, 0)$, $(0, \varepsilon)$, $(\varepsilon n_1, \varepsilon n_2)$

Higher rank:

Define
$$\mathcal{M}_d = \{ (m_1, \dots, m_d) \mid m_i \in \mathbb{Z}^{\geq 0}, \ \gcd(m_1, \dots, m_d) = 1 \}.$$

Fix $(n_1, \dots, n_d) \in \mathcal{M}_d$.

Define
$$\alpha : \mathcal{M}_d \longrightarrow \mathbb{Q}$$

as follows. For each $(m_1, \dots, m_d)$, can write
$$(m_1, \dots, m_d) = a(n_1, \dots, n_d) + \sum_{i \neq j} b_i e_i \qquad \text{for some } 1 \leq j \leq d$$

where
$$e_1 = (1, 0, 0, \cdots), \quad e_2 = (0, 1, 0, \cdots), \quad \cdots \qquad a, b_i \geq 0.$$

Put
$$\alpha(m_1, \dots, m_d) = a + \sum_{i \neq j} b_i .$$

Example:

- $(n_1, n_2, n_3) = (1,1,1) \Rightarrow \alpha \geq 1.$
- $(n_1, n_2, n_3) = (1,1,2) \Rightarrow \alpha \geq 1.$
- $(n_1, n_2, n_3) = (1, n_2, n_3) \Rightarrow \alpha \geq 1.$
- $(n_1, n_2, n_3) = (10, 11, 19) \Rightarrow \alpha(1,1,2) = \frac{5}{11}.$
- In general, $\alpha \geq \frac{1}{\min\{n_1, \ldots, n_d\}}$

- $(n_1, \ldots, n_4) = (20, 57, 133, 210) \Rightarrow \alpha \geq 1$
- $(n_1, \ldots, n_4) = (32, 41, 71, 101) \Rightarrow \alpha \geq 1$

$\Big\}$ (Sankaran–Santos)

Conj (Birkar) = Thm (Sankaran – Santos) ($d=3$ Y. Chen)

For $\varepsilon > 0$, $\exists\, \ell$ s.t.

$$\alpha \geq \varepsilon \implies \min\{n_1, \dots, n_d\} \leq \ell.$$

The proof uses topological properties of subgroups of $\mathbb{R}^d$.

Problem: Give a direct proof.

Another geometric interpretation:

The vectors

$$e_1, e_2, \dots, e_d$$

generate a cone in $\mathbb{R}^d$.

This defines the toric variety $\mathbb{A}^d$ with coordinates $t_1, \dots, t_d$.

Adding the vector

$$(n_1, \dots, n_d)$$

corresponds to a weighted blowup

$$\varphi : X \longrightarrow \mathbb{A}^d.$$

The condition

$$\alpha \geqslant \varepsilon$$

is equivalent to

$$X \text{ has } \varepsilon\text{-lc singularities.}$$

A much harder problems

Fix $\varepsilon > 0$.

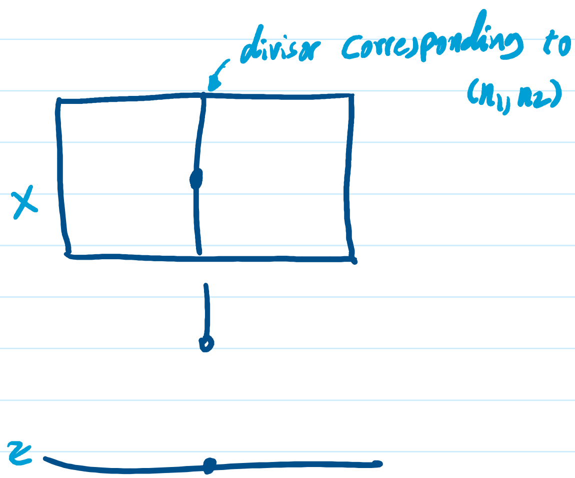
pick $(n_1, \dots, n_d) \in \mathcal{N}_d$.

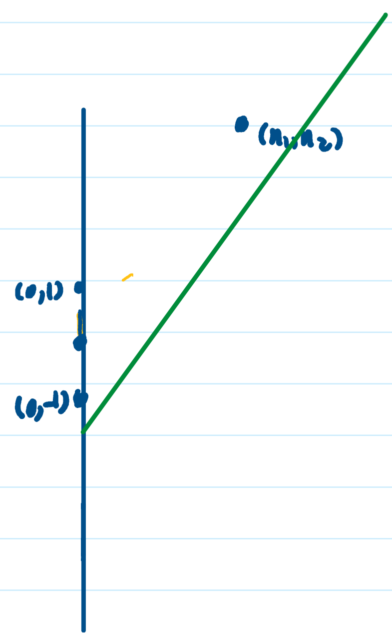
Consider the toric variety X given by

$$e_2, \dots, e_d, \quad -\sum_2^d e_i, \quad (n_1, \dots, n_d).$$

we have a projection

$$X \xrightarrow{f} Z := \mathbb{A}^1.$$





Thm (Birkar-Chen) $\exists\, \ell$ s.t. X ϵ-lc $\Rightarrow$ $n_1 \leq \ell$.

But we don't want to assume X ϵ-lc.

Let

$$\psi : W \longrightarrow X$$

be a resolution of singularities with exceptional divisors $E_1, \dots, E_p$.

Assume $T \subseteq W$ corresponds to $(n_1, \dots, n_d)$.

$$\boxed{\begin{array}{l} \text{Thm (Birkar, 2023).} \\ \text{For } \epsilon, r > 0, \ \exists\, \ell \text{ s.t.} \\ T \not\subseteq \mathbf{B}\left(K_W + \sum_1^r (1-\epsilon) E_j - \psi^*(rK_X)\right) / Z \quad \Rightarrow \quad n_1 \leq \ell . \end{array}}$$

Here $\mathbf{B}$ denotes stable base locus.

Yet more geometry:

The above theorem appears as a last step of the following result in birational geometry.

Thm (Birkar, 2023) = Conj (McKernan-Shokurov, 2003)

For d, $\varepsilon > 0$ $\exists$ ℓ s.t. if

- $f: X \longrightarrow Z$ is a Fano fibration over smooth curve,
- X ε-lc, dim d

then multiplicities of $f^*z \leq \ell$, $\forall z \in Z$.

And yet more problems:

Conj (Birkar)

For $d, \varepsilon > 0$, $\exists\, t > 0$ s.t. if

- $f: X \to Z$ is a Fano fibration,
- X ε-lc, dim d, $z \in Z$,

Then $\exists$ Cartier divisor $D \geq 0$ through z s.t. $(X, t f^* D)$ is lc.