5-10/12

2022年5月9日 10:23

Reference

- Farrell, Jones, "Topological rigidity for compact nonpositively curved manifolds", 1993

Now show

Th 3.15 (Farrell-Sorcar 2017)

For every $n=4k-2$, where $k\in\mathbb{Z}_{\geq 2}$, $\exists$ compact complex hyp. mfd M^n s.t.

$$\pi_1 \mathcal{T}^{<0}(M) \xrightarrow{F_*} \pi_1 \mathcal{T}(M)$$

is nontrivial.

pf Th 3.15: Recall

Prop 3.24 Given $r\geq 1$ and a compact complex hyp. mfd (N^n, g_c), there $\exists$ finite cover $\hat{N}$ of N and a neg. curved metric g_s on $\hat{N}$ s.t. $(\hat{N}, g_s)$ contains a hyp. geodesic ball $B(p, 2r)$ and $g_s = \hat{g}_c$ on $\hat{N}\setminus B(p, 9r^2)$ where $\hat{g}_c$ is the induced complex hyp. metric on $\hat{N}$.

Now for $n=4k-2$, $k\geq 2$, given compact complex hyp. mfd (N^n, g_c). For $r>>0$, apply 3.24 — How large will be apparent later.

$\Longrightarrow (\hat{N}, \hat{g}_c)$ s.t. 3.24 hold.

Take $(M, g) \triangleq (\hat{N}, \hat{g}_c)$.

Strategy: find $f\in \mathrm{Diff}_0(M)$, s.t. (1) (2) holds:

(1) $\pi_0 \mathrm{Diff}_0(M) \longrightarrow \pi_0 \mathcal{R}^{<0}(M)$

$[f] \longmapsto [f_* g] = [g]$.

(1) $\pi_0 \mathrm{Diff}_0(M) \longrightarrow \pi_0 \widetilde{K}(M)$

$[f] \longmapsto [f_* g] = [g]$.

(2) $[f] \in \pi_0 \mathrm{Diff}_0(M)$.
$[f] \neq 0$

Given $[\Sigma] \in \Theta_{n+1}$ let $\varphi_\Sigma \in \mathrm{Diff}(D^n, \partial)$. s.t.

$$\pi_0 \mathrm{Diff}(D^n, \partial) \xrightarrow{\cong} \Theta_{n+1}$$
$$[\varphi_\Sigma] \longmapsto [\Sigma].$$

Fix $D^n \hookrightarrow M$. $\leadsto$

$$\mathrm{Diff}(D^n, \partial) \xrightarrow{\text{extend by id}} \mathrm{Diff}(M).$$
$$\varphi_\Sigma \longmapsto f_\Sigma$$

Prop 3.27 If M is neg. curved, then $\exists [\Sigma] \in \Theta_{n+1}$ s.t. f_Σ is not isotopic to id.

Let $[\Sigma] \in \Theta_{n+1}$ as in 3.27. Construct $f \in \mathrm{Diff}_0(M)$ as before (pf. Th 3.14).

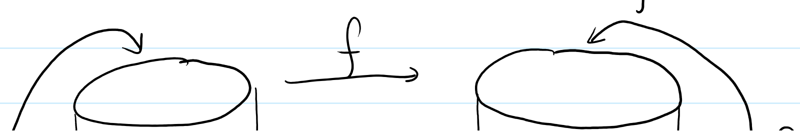
Since (M, g) contains a hyp. ball of large radius

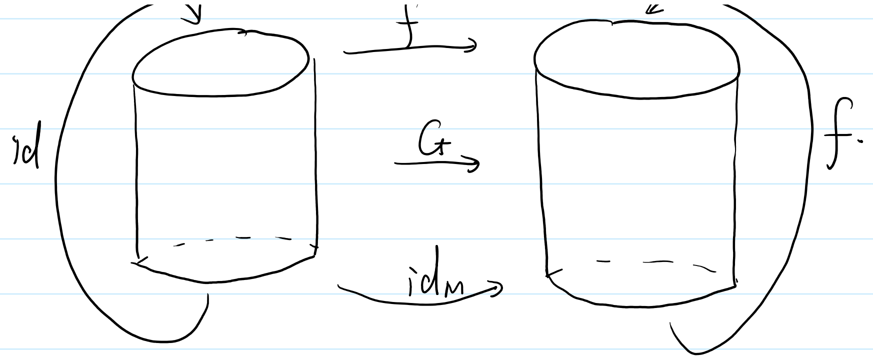
$\xrightarrow{\text{tapering process}}$ (1).

$3.27 \Longrightarrow (2)$. $\square$.

Pf Prop 3.27: Suppose for any $[\Sigma] \in \Theta_{n+1}$, we have
$f \hat{=} f_\Sigma$ is isotopic to id.

Let $G: M \times I \circlearrowleft \cong$ s.t. $G(m, 0) = (m, 0)$
$G(m, 1) = (f(m), 1)$





$G \rightsquigarrow F_0 : M\times S^1 \xrightarrow{\cong} (M\times I)_f$ → mapping torus of f.

On the other hand,
our previous arguments gives

$$F_1 : M\times S^1 \# \Sigma \xrightarrow{\cong} (M\times I)_f$$

Let $\Sigma = \partial W^{4k}$ with W parallel

<u>Prop 3.28</u> If M is neg. curved, then

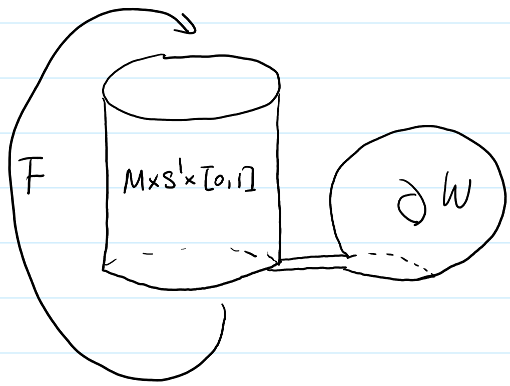
$$F \triangleq F_0^{-1}\circ F_1 : M\times S^1 \# \Sigma \xrightarrow{\cong} M\times S^1$$

is orientation preserving and

$$Q \triangleq (M\times S^1\times[0,1] \natural W)_F$$

$$\approx M\times S^1\times S^1 \# \hat{W}$$

where $\hat{W} = W \cup_\Sigma \mathrm{Cone}(\Sigma)$.



Recall
Prop 3.23': For each $k\geq 2$, $\exists$ prime $p=p(k)$ $(\neq 2)$.
s.t. the following (1) (2) hold:

(1) $p \mid t_k$
(2) If $[\Sigma]\in \Theta_{4k-1}(\partial\pi)$ and $\Sigma = \partial W$ (W parallel.)
N oriented closed mfd, and
$\exists F : N \xrightarrow{\equiv} N\#\Sigma$ s.t. all decomposable
Pontryagin numbers of $Q = (N\times[0,1]\natural W)_F$

are zero, then $[\Sigma] \in \langle p \rangle$.

By Prop 3.23' and 3.28, to get a contradiction, it suffices to show

$$p_{i_1} \cdots p_{i_e}[Q] = 0 \quad \text{for all } e > 1$$

for Q in Prop. 3.28.

Note that

$$Q \approx M \times S^1 \times S^1 \# \hat{W} \triangleq \bar{Q}.$$

Topological invariance of $\mathbb{Q}$-pontrjagin classes

$\Rightarrow$ $p_i(Q) = p_i(\bar{Q})$.

$\Rightarrow$ It suffices to show $p_{i_1} \cdots p_{i_e}[\bar{Q}] = 0$ for $e > 1$.

Denote $T = S^1 \times S^1$.

Consider $\bar{Q} = M \times T \# \hat{W} = M \times T \setminus \mathring{D}^{4k} \cup_{S^{4k-1}} W$

and MV seq: $\quad 0 < i < k$

$$\begin{array}{ccccccc} H^{4i-1}(S^{4k-1}) & \longrightarrow & H^{4i}(\bar{Q}) & \xrightarrow{\cong} & H^{4i}(M\times T\setminus \mathring{D}^{4k}) \oplus H^{4i}(W) & \longrightarrow & H^{4i}(S^{4k-1}) \\ \| & & & & & & \| \\ 0 & & p_i(\bar{Q}) & \longmapsto & (p_i(M\times T \setminus \mathring{D}^{4k}),\ p_i(W)) & & 0 \end{array}$$

$\cong H^{4i}(M\times T)$

$p_i(M\times T)$

W parallel.

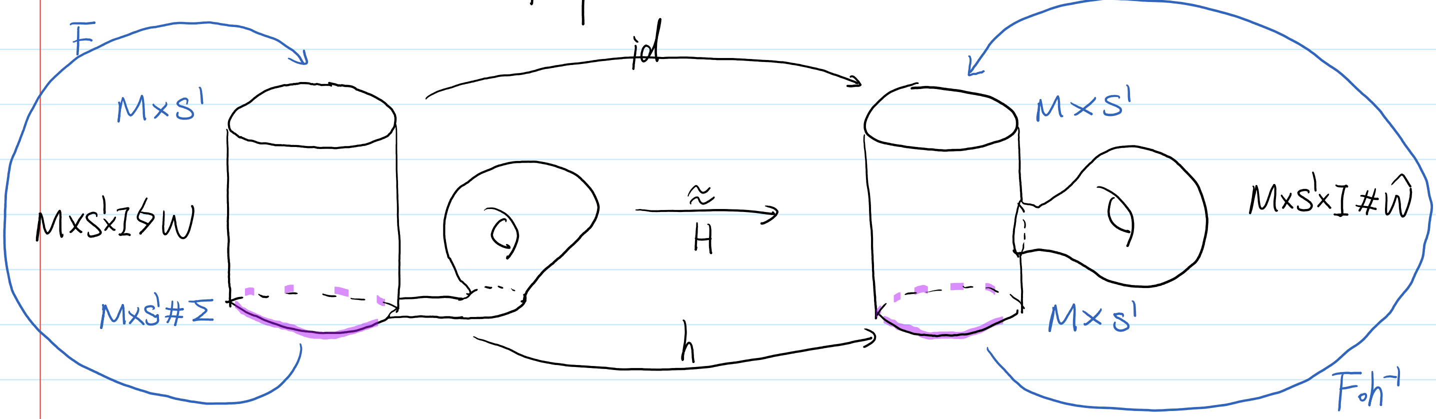
$\xRightarrow{ex}$ $p_{i_1} \cdots p_{i_e}[Q] = p_{i_1} \cdots p_{i_e}[M \times T]$ for $e > 1$.

Since $M\times T = M\times S^1\times S^1 = \partial(M\times D^2\times S^1)$.

$\Longrightarrow p_{i_1}\cdots p_{i_e}[\tilde{Q}] = p_{i_1}\cdots p_{i_e}[M\times T] = 0$ for all $e>1$.

$\square$.

It remains to show Prop 3.28. :



Let $H: M\times S^1\times I \natural W \xrightarrow{\approx} M\times S^1\times I \# \hat{W}$.

s.t. $H|_{Bottom} = h: M\times S^1 \# \Sigma \longrightarrow M\times S^1 \# S^{4k-1} \xrightarrow{\approx} M\times S^1$

id between the

id on $\boxed{M\times S^1 \setminus \mathring{D}^{4k-1}}$

s.t. $H|_{Bottom} = h : M\times S \# \Sigma \underbrace{\longrightarrow}_{\substack{\text{id between the}\\ \text{underlying top. spaces.}}} M\times S \# \Sigma' \cdots \underbrace{\longrightarrow}_{\text{id on } \boxed{M\times S^1 \setminus \mathring{D}^{4k-1}}} M\times S$

$$H|_{Top} = id_{M\times S^1}$$

$$\Longrightarrow (M\times S^1\times[0,1] \cup W)_F \approx \boxed{(M\times S^1\times[0,1] \# \hat{W})_{F\circ h^{-1}}}$$

$$\|?$$

$$\boxed{M\times S^1\times S^1 \# \hat{W}}$$

The following results tells that to show ? it suffices to show

$$F\circ h^{-1} \sim id : M\times S^1 \circlearrowleft$$

which also implies that F is orientation-preserving.

<u>Th 3.29</u> Let N^n be non pos. curved closed mfd of dim ≥ 4, and

$$\alpha, \beta \in Top(N)$$

If $\alpha \sim \beta$, then α, β are topologically pseudo-isotopic

cf. [FJ 93].

<u>Def</u>: $\alpha, \beta \in Top(N)$. If $\exists\ P: N\times I \circlearrowleft \approx$

s.t. $P|_{N\times 0} = \alpha$, $P|_{N\times 1} = \beta$.

then call α, β are topologically pseudo-isotopic.

<u>Prop 3.30</u> Let W be a top. mfd with $\partial W = \partial_1 W \sqcup \partial_2 W$

and $\alpha, \beta : \partial_1 W \xrightarrow{\approx} \partial_2 W$.

If α, β are topologically pseudo-isotopic, then $W_\alpha \approx W_\beta$

where $W_\alpha \triangleq W / x \sim \alpha(x)$, $x \in \partial_1 W$.

leave as an exercise.

Now show $F \circ h^{-1} \sim id : M \times S^1 \circlearrowleft$.

Let
$$\begin{tikzcd} M \times S^1 \arrow[r, "p"] \arrow[dr, "q"'] & M \\ & S^1 \end{tikzcd}$$
be the projections.

It suffices to show Claims 1, 2.

<u>Claim 1</u> $p \circ F \circ h^{-1} \sim p : M \times S^1 \longrightarrow M$.

<u>Claim 2</u> $q \circ F \circ h^{-1} \sim q : M \times S^1 \longrightarrow S^1$

<u>pf Claim 1</u>:

$G(M) \triangleq \{ g : M \xrightarrow{\simeq} M \}$.

$\wr\!\wr$ M aspherical

$\mathrm{Out}(\pi_1 M) \times K(\mathrm{Center}(\pi_1 M), 1)$

$\|$ M neg. curved $\Rightarrow$ $\mathrm{Center}(\pi_1 M)$ trivial.

$\mathrm{Out}(\pi_1 M) \times *$

$\Longrightarrow \pi_1(G(M), id)$ trivial

$p : M \times S^1 \longrightarrow M \quad \leftrightsquigarrow \quad \underline{*} : S^1 \longrightarrow G(M)$

$\qquad\qquad\qquad\qquad\qquad\qquad x \longmapsto id_M$

$p \circ F \circ h^{-1} : M \times S^1 \longrightarrow M \quad \leftrightsquigarrow \quad \alpha : S^1 \longrightarrow G(M)$

$\qquad\qquad\qquad\qquad\qquad\qquad x \longmapsto p \circ F \circ h^{-1}(\cdot, x)$.

$x \longmapsto p\circ F\circ h^{-1}(\cdot, x)$.

$\pi_1(G(M), id)$ trivial $\Longrightarrow \alpha \sim * \Longrightarrow p\circ F\circ h^{-1} \sim p$.

$\Uparrow$

Claim 3

easy to see

$\alpha : S^1 \longrightarrow Map(M, M)$

To see $Im\,\alpha \subset G(M)$, need Claim 3.

<u>Claim 3</u> : the loop $\alpha : S^1 \longrightarrow Map(M,M)$ is based at id_M, i.e. $\exists\, x_0 \in S^1$ s.t.

$$\begin{tikzcd} M\times S^1 \arrow[r, "h^{-1}"] & M\times S^1 \# \Sigma \arrow[r, "F_1"] \arrow[rr, bend left, "F"] & (M\times I)_f \arrow[r, "F_0^{-1}"] & M\times S^1 \\ (m, x_0) \arrow[rrr, mapsto] & & & (m, x_0) \end{tikzcd}$$

for all $m \in M$.

<u>pf Claim 2</u> : Consider

$$\begin{tikzcd} M\times S^1 \arrow[r, "h^{-1}"] \arrow[drr, "q"'] & M\times S^1 \# \Sigma \arrow[r, "F_1"] \arrow[rr, bend left, "F"] & (M\times I)_f \arrow[r, "F_0^{-1}"] \arrow[d, "\bar{q}"] & M\times S^1 \arrow[dl, "q"] \\ & & S^1 & \end{tikzcd}$$

since F_0 is induced by the isotopy

where $\bar{q}$ is the natural projection. To show
$$q\circ F\circ h^{-1} \sim q$$
it suffices to show the two triangles are comm. up to homotopy. Since $M\times S^1$, S^1 are $K(\pi, 1)$, it suffices to show
$$q_\# = \bar{q}_\# \circ (F_1 \circ h^{-1})_\#. \quad \text{(ex.)}$$
where $\#$ denotes the induced homomorphism on π_1.

4. Summary and open problems

4.1 $\mathcal{T}^{<0}(M)$

[Farrell-Ontaneda 09]: $\mathcal{T}^{<0}(M) \not\simeq *$ for many closed real hyp. mfds M of dim ≥ 6

[Sorcar 14, Farrell-Sorcar '17] $\mathcal{T}^{<0}(M) \not\simeq *$ for Gromov-Thurston's mfds / compact complex hyp. mfd of dim ≥ 6.

[Bustamante-Farrell-J 18] $\mathcal{T}^{<0}(M) \not\simeq_{\mathbb{Q}} *$.
(i.e. $\pi_i^{\mathbb{Q}} \mathcal{T}^{<0}(M) \neq 0$ for some $i > 1$)
for many closed real hyp. mfds M. of $\dim \geq 2c_1+2$,
where c_1 is a pos. constant given by Weiss

[Weiss 21] There $\exists$ $c_1, c_2 > 0$ s.t. for $\forall n, k \in \mathbb{Z}_{>0}$ with $n \geq c_1$ and $k < \frac{5}{4}n - c_2$,
the rational Pontryagin class $p_{n+k} \in H^{4n+4k}(BTop(2n))$ $\neq 0$

eg. $c_1 = 83$, $c_2 = 11$.

Q: $T^{<0}(M) \overset{?}{\simeq} *$ for other neg. curved M

eg. $\dim M = 3, 4, 5$

eg. M: neg. curved without large enough hyp. geod. ball.

Q: $T^{<0}(M) \overset{?}{\simeq}_{\mathbb{Q}} *$ for other neg. curved mfd M.

eg. M: neg curved but do support real hyp. metric.

eg. $\dim M$ small.

4.2. $F: T^{<0}(M) \longrightarrow T(M)$

$$F_*: \pi_i T^{<0}(M) \longrightarrow \pi_i T(M)$$

$F_* \backslash M$	real hyp. mfd	Gromov-Thurston's mfd / complex hyp. mfd	neg. curved mfd
trivial	X[1]	X[4,5]	?
injective	X[3]	?	?
surjective	X[2]		X[2]

1. [Farrell-Ontaneda 2009].
2. [Farrell-Ontaneda 2010 a] "On the topology of the space of negatively curved metrics".
3. [Farrell-Ontaneda 2010 b] "Teichmüller spaces and negatively curved fiber bundles"
4. [Sorcar 2014]
5. [Farrell-Sorcar 2017]

Q: How about F_* for $\dim M = 3, 4, 5$?

$$F_*^{\mathbb{Q}}: \pi_i^{\mathbb{Q}} T^{<0}(M) \longrightarrow \pi_i^{\mathbb{Q}} T(M)$$

$F_*^{\mathbb{Q}} \backslash M$	real hyp.	neg. curved
trivial	X[6]	
injective	?	?
surjective	X[7]	X[7]

injective	?	!
surjective	X[7]	X[7]

6. [Bustamante - Farrell - J 18]
7. [Bustamante - Farrell - J 20]

Th Let (M, h) be closed neg. curved Rieman. mfd for $1 \leqslant i < \min\left(\frac{n-10}{2}, \frac{n-8}{3}\right)$ (Igusa's stable range). the following hold:

1. $F_* : \pi_{i+1} T^{<0}(M) \otimes \mathbb{Q} \longrightarrow \pi_{i+1} T(M) \otimes \mathbb{Q}$ is trivial

2. $\pi_i (\mathrm{Diff}_0(M), id) \otimes \mathbb{Q} \longrightarrow \pi_i (\mathcal{R}^{<0}(M), h) \otimes \mathbb{Q}$ is mono.

[Farrell-Hsiang 78] [Farrell-Jones 89]

$$\pi_i(\mathrm{Diff}_0(M), id) \otimes \mathbb{Q} = \begin{cases} \bigoplus_{j=1}^{\infty} H_{i+1-4j}(M; \mathbb{Q}) & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$\pi_i^{\mathbb{Q}} \mathcal{R}^{<0}(M) \neq 0$ for some M

An ingredient of pf of Th above is Morlet's comparison Th for general mfd:

$O(n) \hookrightarrow PM \longrightarrow M^n$: principal tangent bundle of M

$\Gamma\left(PM \times_{O(n)} \frac{Top(n)}{O(n)}\right) \triangleq$ the space of sections of the bundle

$$PM \times_{O(n)} \frac{Top(n)}{O(n)} \longrightarrow M.$$

Th ([Morlet 68] [Burghelea - Lashof 74])
For all smooth closed mfd M^n of dim $n \neq 4$,
$\exists$ a cont. map

$$\mu : \frac{\mathrm{Top}_0(M)}{\mathrm{Diff}_0(M)} \longrightarrow \Gamma\left(PM \times_{O(n)} \frac{\mathrm{Top}(n)}{O(n)}\right)$$

which induces $\cong$ on all π_i for $i>0$.

4.3. Related problems in mfd topology.

To study the map $F: T^{<0}(M) \longrightarrow T(M) = \boxed{B\mathrm{Diff}_0(M)}$

and the fiber bundle $\boxed{\mathrm{Diff}_0(M)} \hookrightarrow R^{<0}(M) \longrightarrow T^{<0}(M)$

} for neg. curved mfd M.

need to understand the topology of $\mathrm{Diff}_0(M)$ completely.

eg. Q: $\pi_i \mathrm{Diff}_0(M) = ?$ outside Igusa's stable range.