Dispersive estimates for non-linear waves in mathematical General Relativity

1 Lecturer:

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2 Mode of the discussion, venue and time, audience:

- Offline, in English, at A513 in the Shuangqing Complex Building A, at Tsinghua University.
- During the Spring semester 2025, starting February 19, 2025. Two sessions per week: every Wednesday, at 11:00–12:30 and then at 14:00–15:30.
- Audience: Graduate, Postdoc, Researcher.

3 Prerequisite:

Basic knowledge from my previous courses on "The Cauchy problem in mathematical General Relativity" and on "Non-linear wave equations in General Relativity", graduate level knowledge in differential geometry and in Riemannian geometry, and basic knowledge in partial differential equations and analysis.

4 Introduction:

This course introduces the tools of analysis for proving dispersive estimates for solutions of non-linear wave equations arising in General Relativity from the point of view of analysis and partial differential equations. The course material builds on one hand, on a previous course that I gave in the Spring 2024 on the Cauchy problem in mathematical General Relativity, and on the other hand, on a previous course that I gave in the Fall 2024 on non-linear wave equations in General Relativity. The goal of this course is to show a method to obtain dispersive estimates for solutions of tensorial coupled non-linear hyperbolic partial differential equations, provided that one exploits the non-linear structure of the wave equations. We shall exhibit how this can be applied in General Relativity for perturbations of the Minkowski space-time governed by the Einstein-Yang-Mills system in the Lorenz gauge and in wave coordinates. We shall study first the simpler case of higher dimensions, and we shall give elements that would allow one to think about the difficulty of the problem in the critical three space dimensions.

5 Keywords:

Energy estimates, non-linear wave equations, hyperbolic partial differential equations, Minkowski vector fields, Klainerman-Sobolev inequality, weighted energy norms, bootstrap argument, decay, Hardy type inequality, commutator term, Grönwall type inequality, Einstein equations, Yang-Mills fields, Einstein-Yang-Mills system, gauge transformations, Minkowski metric, wave coordinates, Lorenz gauge, stability of Minkowski space-time.

6 Syllabus:

1. **Reminders of prerequisites:**
   The Einstein equations, the Yang-Mills equations, the Einstein-Yang-Mills system, the Lorenz gauge, wave coordinates, recasting the Einstein-Yang-Mills system as a coupled system of non-linear hyperbolic partial differential equations, the hyperbolic Cauchy problem, the constraint equations, the gauges invariance of the equations.

2. **Set-up of analysis for proving decay for solutions of non-linear wave equations:**
   - The Minkowski vector fields.
   - Weighted Klainerman-Sobolev inequality.
   - Definition of the norms.
   - The energy norm.
   - The bootstrap argument.
   - The bootstrap assumption.
   - The big O notation.

3. **À priori decay estimates:**
   - The spatial asymptotic behaviour of the fields on the initial hypersurface.
   - Estimates on the time evolution of the fields.

4. **Looking at the structure of the source terms of the coupled non-linear wave equations for the Einstein-Yang-Mills system in the Lorenz gauge and in wave coordinates.**

5. **Using the bootstrap assumption to exhibit the structure of the source terms of the Einstein-Yang-Mills system:**
   - Using the bootstrap assumption to exhibit the structure of the source terms for the Yang-Mills potential.
   - Using the bootstrap assumption to exhibit the structure of the source terms for the metric.
   - The source terms for $n \geq 5$.

6. **Energy estimates.**

7. **A Hardy type inequality.**

8. **The commutator term for $n \geq 4$:**
   - Using the Hardy type inequality to estimate the commutator term.

9. **The energy estimate for $n \geq 4$:**
   - The main energy estimate for the fields for $n \geq 4$.

10. **The proof of global stability of Minkowski space-time for $n \geq 5$:**
    - Using the Hardy type inequality for the space-time integrals of the source terms for $n \geq 5$.
    - Grönwall type inequality on the energy for $n \geq 5$.
    - The proof of the global stability for $n \geq 5$.

7 References:

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