Lecture No 10 April 8, 2022 (Fri) (Lect-9 p.18∼)

10.4 Doob's optional sampling theorem

[Theorem 10.6] $X = (X_t)$: martingale, τ, σ: Markov times such that $\tau \leq \sigma \leq^{\exists} K$ (constant) (i.e., τ, σ are bounded in ω)
$\Longrightarrow E[X_\sigma | \mathcal{F}_\tau] = X_\tau \quad a.s.$ □

[Proof] [Step 1] In case that τ, σ take only finitely many values, this follows from the result in discrete time.

[Step 2] For general case, set

$$\sigma_n(\omega) := \frac{[n\sigma(\omega)]+1}{n}, \tau_n(\omega) := \frac{[n\tau(\omega)]+1}{n},$$

where $[t]$ is Gauss symbol and denotes the integer part of t. Then, these are Markov times.

∵ First show $\{\sigma_n \leq t\} = \{\sigma < \frac{[nt]}{n}\}$. Then, by $\frac{[nt]}{n} \leq t$, we see (RHS)$\in \mathcal{F}_t$ so that we obtain $\{\sigma_n \leq t\} \in \mathcal{F}_t$ for $^{\forall} t \geq 0$.
□

Since τ_n, σ_n take only finitely many values, by Step 1, we have $E[X_{\sigma_n}|\mathcal{F}_{\tau_n}] = X_{\tau_n}$ a.s. In particular, for ${}^{\forall}A \in \mathcal{F}_\tau (\subset \mathcal{F}_{\tau_n})$, we have

$$E[X_{\sigma_n}, A] = E[X_{\tau_n}, A].$$

However, if one can show the uniform integrability of $(X_{\sigma_n}), (X_{\tau_n})$, since $\sigma_n \downarrow \sigma, \tau_n \downarrow \tau$ and X is right-continuous, by taking the limit, we obtain

$$E[X_\sigma, A] = E[X_\tau, A],$$

which implies the conclusion of Theorem 10.6. □

[Step 3] Proof of uniform integrability of $(X_{\sigma_n}), (X_{\tau_n})$.

∵ First we show the uniform integrability of (X_{σ_n}). Since $(|X_t|)_{t\geq 0}$ is a submartingale and $\sigma_n \leq K + \frac{1}{n} \leq K+1$, applying optional sampling theorem for discrete time, we have $E[|X_{K+1}| \,\big|\mathcal{F}_{\sigma_n}] \geq |X_{\sigma_n}|$ so that

$$E[|X_{\sigma_n}|, |X_{\sigma_n}| \geq \lambda] \leq E[|X_{K+1}|, |X_{\sigma_n}| \geq \lambda].$$

Setting $A := \{|X_{\sigma_n}| \geq \lambda\}$, RHS is estimated as follows:

$$\begin{aligned} E[|X_{K+1}|, A] = E[|X_{K+1}|, A \cap \{|X_{K+1}| > M\}] \\ + E[|X_{K+1}|, A \cap \{|X_{K+1}| \leq M\}] \\ \leq E[|X_{K+1}|, |X_{K+1}| > M] + MP(A). \end{aligned}$$

The 1st term is $< \varepsilon$ by taking M large enough. For the 2nd term, by fixing such M, $P(A) \leq \frac{1}{\lambda} E[|X_{\sigma_n}|] \leq \frac{1}{\lambda} E[|X_{K+1}|]$ so that we can estimate it $< \varepsilon$ (uniformly in n) by taking λ large enough. From these, the uniform integrability of (X_{σ_n}) is shown. Similar for (X_{τ_n}). □

10.5 Doob-Meyer decomposition

The next theorem is known. We omit the proof.

[Theorem 10.7] Assume that $X = (X_t)$ is a continuous (i.e. $X_t(\omega)$ is continuous in t for $^\forall\omega$) $(\mathcal{F}_t)$-submartingale satisfying the (Doob local) uniform integrability condition:
(DL) For $^\forall T > 0$, $\{X_\tau\}_{\tau\in\mathcal{S}_T}$ is uniformly integrable,
where $\mathcal{S}_T = \{\tau$: Markov times s.t. $\tau \le T$ a.s.$\}$.
Then, X can be decomposed as $X_t = M_t + A_t, t \ge 0$, where $M = (M_t)$: continuous $(\mathcal{F}_t)$-martingale and $A = (A_t)$: $(\mathcal{F}_t)$–adapted continuous increasing process s.t. $A_0 = 0$.
(increasing means: $A_t \ge A_s$ $a.s.$ for $t > s \ge 0$)

Moreover, this decomposition is unique, i.e. if $X_t = M_t + A_t = M'_t + A'_t$, then $M_t = M'_t, A_t = A'_t$ $^\forall t \ge 0$ a.s. □

[Proof] See Karatzas-Shreve Chapter 1, Theorem 4.10. We discretize time and use Doob decomposition. Then, take limit by noting the uniform integrability condition (DL). It requires some efforts to show the uniform integrability of the approximating sequence (A^n_t) of increasing part. □

Application of Doob-Meyer decomposition

$\mathcal{M}_c^2 := \{M = (M_t)_{t\geq 0}$; square-integrable (i.e. $E[M_t^2] < \infty$, ${}^{\forall}t \geq 0$) continuous $(\mathcal{F}_t)$-martingales s.t. $M_0 = 0$ a.s.$\}$.

Then, for ${}^{\forall}(M_t) \in \mathcal{M}_c^2$, ${}^{\exists 1}A_t$: $(\mathcal{F}_t)$–adapted continuous increasing process s.t. $A_0 = 0$ and $(M_t^2 - A_t)$ is a martingale.

∵ $(X_t := M_t^2)$ is a submartingale and satisfies the uniform integrability condition (DL). Indeed, as in §10.4, for ${}^{\forall}\tau \in \mathcal{S}_T$, we can estimate as

$$E[X_\tau, X_\tau > \lambda] \leq E[X_T, X_\tau > \lambda],$$
$$P(X_\tau > \lambda) \leq \frac{E[X_\tau]}{\lambda} \leq \frac{E[X_T]}{\lambda}.$$

Since $X_T \in L^1$, the absolute continuity of Lebesgue integrals implies "${}^{\forall}\varepsilon > 0$, ${}^{\exists}\delta > 0$ s.t. $P(A) < \delta \Longrightarrow |E[X_T, A]| \leq \varepsilon$". By this, one can prove the condition (DL). Then, we can apply Theorem 10.7 (Doob-Meyer decomposition) for $X_t = M_t^2$. □

We write A_t as $\langle M \rangle_t$ and call it a quadratic variation of (M_t).

The reason that we call $\langle M \rangle_t$ a quadratic variation of (M_t) lies in the following fact.

[Fact] $$\langle M \rangle_t = \lim_{|\Delta| \to 0} \sum_{i=1}^{n} (M_{t_i} - M_{t_{i-1}})^2 \quad \text{in probability,}$$

where $\Delta = \{0 = t_0 < t_1 < \cdots < t_n = t\}$ is a division of $[0, t]$ and $|\Delta| = \max_{1 \le i \le n} (t_i - t_{i-1})$. □

[Remark] If (M_t) is not continuous (in t) but has jumps, similarly to the discrete time case, we need to introduce $[M]_t$ to distinguish from $\langle M \rangle_t$. Recall that, in discrete time, $\langle M \rangle_n$ is predictable, while $[M]_n$ is defined as in Fact without taking limit. These are different. However, for continuous martingales in continuous time, these are the same and we have

$$\langle M \rangle_t = [M]_t.$$

• If $M \in \mathcal{M}_c^2$ and M_t is of bounded variation in t (a.s.), then we have $\langle M \rangle_t = 0$.

☺

$$\sum_{i=1}^{n} (M_{t_i} - M_{t_{i-1}})^2 \leq \max_{1 \leq i \leq n} |M_{t_i} - M_{t_{i-1}}| \times \sum_{i=1}^{n} |M_{t_i} - M_{t_{i-1}}|$$
$$\leq \max_{1 \leq i \leq n} |M_{t_i} - M_{t_{i-1}}| \times V_t(M) \underset{|\Delta| \to 0}{\longrightarrow} 0$$

Here, $V_t(M)$ is the total variation of M on the interval $[0, t]$ and finite by the assumption. On the other hand, by the continuity of M, we have $\max_{1 \leq i \leq n} |M_{t_i} - M_{t_{i-1}}| \to 0$. □


Meyer (from Wikipedia)

Remark on $\langle M \rangle_t = [M]_t$: By the decomposition used in discrete time, we have

$$\begin{aligned} M_t^2 &= M_{t_n}^2 \\ &= M_0^2 + 2\sum_{i=1}^{n} M_{t_{i-1}}(M_{t_i} - M_{t_{i-1}}) + \sum_{i=1}^{n}(M_{t_i} - M_{t_{i-1}})^2. \end{aligned}$$

The 2nd term is a martingale transform and known to converge to a stochastic integral $2\int_0^t M_s dM_s$ (which is a martingale) if M_t is a continuous martingale. The 3rd term is increasing in t. Thus we would expect

$$A_t = \lim_{|\Delta| \to 0} \sum_{i=1}^{n}(M_{t_i} - M_{t_{i-1}})^2$$

so that $\langle M \rangle_t = [M]_t$. □

Hilbert space of square-integrable martingales

Fix $^\forall T > 0$ and define

$$\mathcal{M}^2_{c,T} := \{M = (M_t)_{t\in[0,T]};\ \text{square-integrable continuous } (\mathcal{F}_t)\text{-martingales s.t. } M_0 = 0\ a.s.\}.$$

For $M, N \in \mathcal{M}^2_{c,T}$, set

$$\langle M, N\rangle_t := \frac{1}{4}\big(\langle M+N\rangle_t - \langle M-N\rangle_t\big) \underset{\text{Fact}}{=} \lim_{|\Delta|\to 0} \sum_{i=1}^{n} (M_{t_i} - M_{t_{i-1}})(N_{t_i} - N_{t_{i-1}}) \quad \text{in probability.}$$

Then, $\langle M, N\rangle_t$ is of bounded variation in t (as a difference of two increasing functions) $a.s.$ and $(M_t N_t - \langle M, N\rangle_t)$ is a continuous martingale. P: Check these.

We call $\langle M, N\rangle_t$ a cross variation of M and N.

Define an equivalence relation:

$$M \sim N \quad \underset{\text{def}}{\Longleftrightarrow} M_t = N_t \ {}^{\forall} t \in [0, T] \text{ a.s.}$$
$$\Longleftrightarrow P(M_t = N_t \text{ for } {}^{\forall} t \in [0, T]) = 1$$

(M and N are called indistinguishable.)

We identify equivalent elements in $\mathcal{M}^2_{c,T}$.

[Proposition 10.8] $(\mathcal{M}^2_{c,T}, E[\langle \cdot, \cdot \rangle_T])$ is a real Hilbert space. □