Lecture 20. Khovanov homology and TQFT

homological grading

Last time: Khovanov homology $Kh(L) = \bigoplus_{i,j} Kh^{i,j}(L)$

quantum grading

link diagram D, n crossing

⇝ cube of resolutions

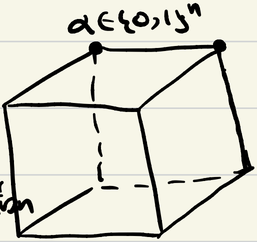
Shift the quantum grading by $|\alpha|$

$$V_\alpha(|D|) = \Big(\bigotimes_{\text{circles in } D_\alpha} V\Big)\{|\alpha|\}$$

$[0,1]^n$

α

D_α: resolution



$|\alpha| = \Sigma \alpha_i = \#1\text{s in } \alpha$

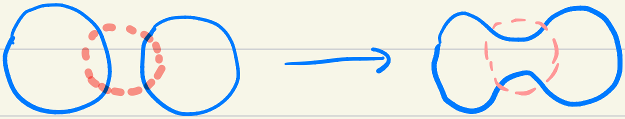
$V = \mathbb{Z}V_+ \oplus \mathbb{Z}V_-$

$(\alpha_0, \alpha_1 \cdots \alpha_i, *, \alpha_{i+1}, \cdots \alpha_n)$

$\xi : \alpha = (\alpha_0, \alpha_1, \cdots, \alpha_i, 0, \alpha_{i+1}, \cdots, \alpha_n) \to \alpha' = (\alpha_0, \cdots, \alpha_i, 1, \alpha_{i+1}, \cdots, \alpha_n)$

$d_\xi : V_\alpha \to V_{\alpha'}$

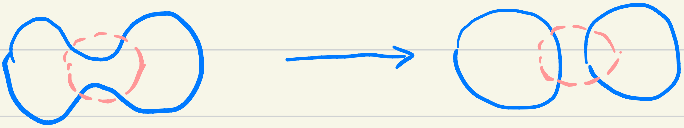
- ξ is a merge: $d_\xi = \otimes \mathrm{Id}_V \otimes (m : V \otimes V \to V)$



$\nearrow \mathbb{Z}[X]/(X^2)$

$m: V_+ \otimes V_+ \mapsto V_+, \; V_+ \otimes V_- \mapsto V_-, \; V_- \otimes V_+ \mapsto V_-, \; V_- \otimes V_- \mapsto 0$

- ξ is a split $d_\xi = (\otimes \mathrm{Id}\, V) \otimes (\Delta : V \to V \otimes V)$



$$\Delta : V_+ \longmapsto V_+ \otimes V_- + V_- \otimes V_+$$
$$V_- \longmapsto V_- \otimes V_-$$

bi-grading of V_+ $(0,+1)$ V_- $(0,-1)$

Chain complex $C^{i,*}(D) = \bigoplus_{|\alpha|=i} V_\alpha(D)$

$$\begin{tikzcd}
 & 01 \arrow[dr] & \\
00 \arrow[ur] \arrow[dr] & & 11 \\
 & 10 \arrow[ur] &
\end{tikzcd}$$

$d : C^{i,j}(D) \to C^{i+1,j}(D)$ $d = \bigoplus_{\substack{\xi:\alpha\to\alpha' \\ |\alpha|=i}} (-1)^{|\xi|} d_\xi$

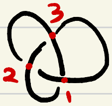
Here $|\xi| = \sum_{i=1}^{i} \alpha_i$ if $\alpha = (\alpha_0, \alpha_1, -- , \alpha_i, 0, \alpha_{i+2}, -- \alpha_n)$

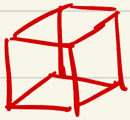
$\alpha' = (\alpha_0, \alpha_1 -- \alpha_i, 1, \alpha_{i+2}, -- \alpha_n)$

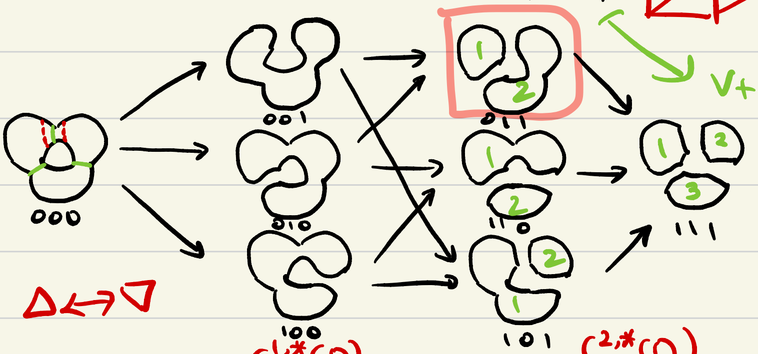
Theorem : $d^2 = 0$ (will prove later).

$Kh^{\bullet,\bullet}(L) = H_*(C^{*,*}(D)[-n_-]\{n_+ - 2n_-\})$

$n_\pm$: # positive /negative crossings in D

Now we set $L = 3_1 =$ 





$V_+ \otimes \Delta(V_+)$

$\parallel$

$V_+\otimes V_+\otimes V_-$

$+ V_+\otimes V_-\otimes V_+$

$V = \mathbb{Z}V_+ \oplus \mathbb{Z}V_-$

$C^{1,*}(D)$ $C^{2,*}(D)$ $C^{3,*}(D)$

$$\begin{tikzcd}
 & V\{1\} \arrow[r, "\Delta"] \arrow[rdd, "\Delta"] & V^{\otimes 2}\{2\} \arrow[rd, "\Delta"] & \\
V^{\otimes 2} \arrow[ru, "m"] \arrow[r, "m"] \arrow[rd, "m"] & V\{1\} \arrow[ru, "-\Delta"] \arrow[r, "\Delta"] & V^{\otimes 2}\{2\} \arrow[r, "\Delta"] & V^{\otimes 3}\{3\} \\
 & V\{1\} \arrow[ru, "-\Delta"] \arrow[r, "-\Delta"'] & V^{\otimes 2}\{2\} \arrow[ru, "-\Delta"] &
\end{tikzcd}$$

$q(V_+) = 1$

$j \backslash i$	0	1	2	3
9		→	→	→$V_+\otimes V_+\otimes V_+$
7			$V_+\otimes V_+$ $V'_+\otimes V'_+$ $V''_+\otimes V''_+$	$V_+\otimes V_+\otimes V_-$ $V_+\otimes V_-\otimes V_+$ $V_-\otimes V_+\otimes V_+$

$V_\pm \otimes V_\pm$

$V'_\pm \otimes V'_\pm$

$V'_\pm \otimes V'_\pm$

5	$V_+\otimes V_+$	V_+ V_+' V_+''	$V_+\otimes V_-$, $V_-\otimes V_+$, $V_+'\otimes V_-'$ $V_-'\otimes V_+'$ $V_+''\otimes V_-''$ $V_-''\otimes V_+''$	$V_+\otimes V_-\otimes V_-$ $V_-\otimes V_+\otimes V_-$ $V_-\otimes V_-\otimes V_+$
3	$V_+\otimes V_-$ $V_-\otimes V_+$	V_- V_-' V_-''	$V_-\otimes V_-$ $V_-'\otimes V_-'$ $V_-''\otimes V_-''$	$V_-\otimes V_-\otimes V_-$
1	$V_-\otimes V_-$			

We can compute the Khovanov homology, for example $j=7$, the chain complex is

$$0 \to \mathbb{Z}^3 \xrightarrow{d} \mathbb{Z}^3 \to 0$$

$$d(V_+\otimes V_+) = V_+\otimes V_+\otimes V_- + V_+\otimes V_-\otimes V_+$$

$$d(V_+'\otimes V_+') = V_+\otimes V_-\otimes V_+ + V_-\otimes V_+\otimes V_+$$

$$d(V_+''\otimes V_+'') = -V_+\otimes V_+\otimes V_- - V_-\otimes V_+\otimes V_+$$

so $d \sim \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$ $\Rightarrow Kh^{2,7}(3_1)=0 \quad Kh^{3,7}(3_1)=\mathbb{Z}/2\mathbb{Z}$

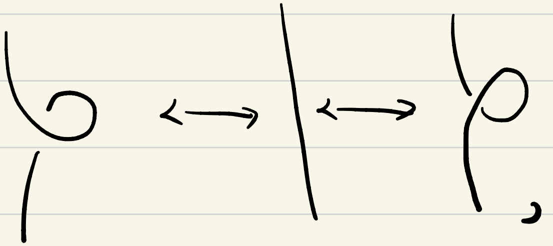
$Kh^{0,7}(3_1)=0 \quad Kh^{1,7}(3_1)=0$

$\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

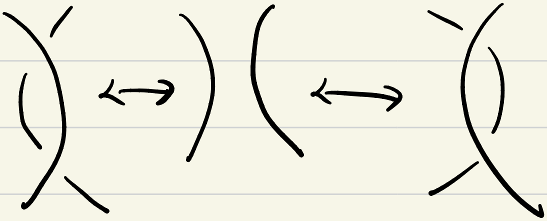
$j \backslash i$	0	1	2	3
9				$\mathbb{Z}$
7				$\mathbb{Z}/2\mathbb{Z}$
5			$\mathbb{Z}$	
3	$\mathbb{Z}$			
1	$\mathbb{Z}$			

Isotopy invariance of Khovanov homology

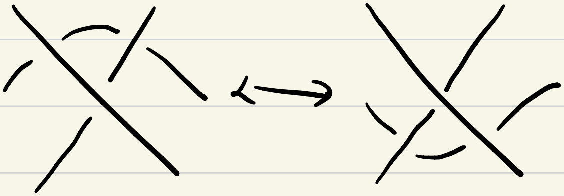
We need to check that $Kh^{\bullet,\bullet}(L)$ is independent with D.
Any two link diagrams D, D' for same L are related by Reidemeister moves



RI



RII



RIII

So we just need to check $Kh^{\bullet,\bullet}(D)$ is unchanged under RI, RII, RIII.

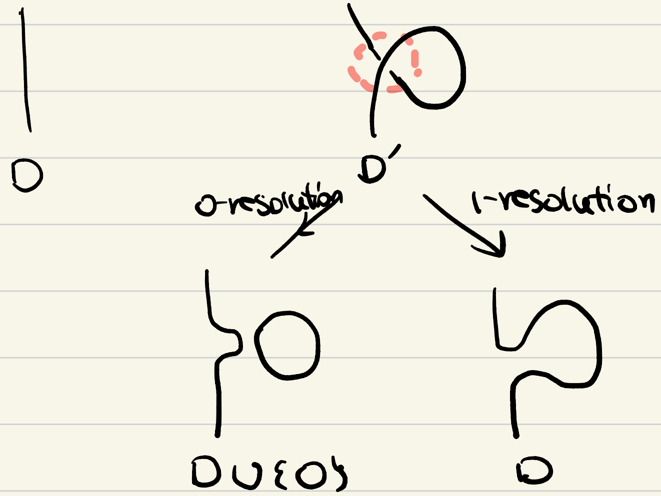
We will just do RI.

Lemma: Let (C,d) be a chain complex and let $C' \subset C$ be a subcomplex. Suppose C' is acyclic ($H_*(C')=0$). Then $H_*(C) \cong H_*(C/C')$. Suppose C/C' is acyclic. Then $H_*(C') \cong H_*(C)$.

Proof: The short exact sequence $0 \to C' \to C \to C/C' \to 0$ induces a long exact sequence $\cdots \to H_*(C') \to H_*(C) \to H_*(C/C') \to \cdots$

$\square$

Now we prove that Kh(D) is unchanged under RI.



Then $C(D') = C(D \sqcup \{O\}) \oplus C(D)$

$= (C(D) \otimes \mathbb{Z}V_-) \oplus (C(D) \otimes \mathbb{Z}V_+) \oplus C(D)$

$= C(D) \oplus C(D) \oplus C(D)$

with differential

$$d_{D'} = \begin{pmatrix} d_D & 0 & 0 \\ 0 & d_D & 0 \\ m_- & m_+ & d_D \end{pmatrix}$$

where $m_- = m(v_-, -) \otimes (\otimes \mathrm{Id}_V)$

$m_+ = m(v_+, -) \otimes (\otimes \mathrm{Id}_V) = \mathrm{Id}_{C(D)}$.

Note $\left(C(D) \oplus C(D), \begin{pmatrix} d_D & 0 \\ \mathrm{Id} & d_D \end{pmatrix}\right)$ is acyclic.

By lemma, $H_*(C(D')) \cong H_*(C(D))$

The proof that $H_*(C(D))$ is unchanged under RII, RIII is similar. (but more delicate.) $\square$.

- **Khovanov homology and TQFT.**

From now on, we work with $\mathbb{Q}$ instead of $\mathbb{Z}$. ($Kh(L) \rightsquigarrow Kh(L;\mathbb{Q})$)

The category of 2-dimensional cobordism, denoted by 2-Cob:

- objects : closed 1-dim manifold. (i.e. $\sqcup S^1$)
- Given objects A, B, a morphism is a cobordism from A to B.

  I.e., an orientable surface M together with a specific diffeomorphism $\bar{A} \sqcup B \xrightarrow{\cong} \partial M$ (boundary parametrization)

(Given M, M', we treat M and M' as the same morphism if $\exists\, M \underset{\varphi}{\xrightarrow{\cong}} M'$ s.t.

$$\begin{tikzcd} & M \arrow[d, "\varphi_{\partial M}"] \\ \bar{A} \sqcup B \arrow[ur, "\cong"] \arrow[r, "\cong"'] & M' \end{tikzcd}$$

commutes.)

Definition : A (1+1)-dim TQFT is a **monoidal** functor

$$F : 2\text{-cob} \longrightarrow \mathrm{Vect}_{\mathbb{Q}}$$

I.e. 1-manifold $\rightsquigarrow$ vector space

cobordism $\rightsquigarrow$ linear map

monidal : $\exists$ "natural" isomorphisms $F(A \sqcup B) \xrightarrow{\cong} F(A) \otimes F(B)$.

s.t. for any $M_1 : A \to B$ $M_2 : A' \to B'$, we have

$$F(M_1 \sqcup M_2) = F(M_1) \otimes F(M_2).$$

Definition A commutative Frobenius algebra over $\mathbb{Q}$ consits of

- A vector space V (finite dimensional)
- linear maps $m: V\otimes V \to V$ (multiplication)
  $\Delta: V \longrightarrow V\otimes V$ (comultiplication)
- unit $1 \in V$ (equivalently, a linear map $\mathbb{Q} \to V$)
  counit $\varepsilon: V \to \mathbb{Q}$

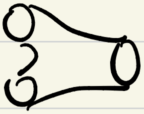
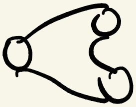
such that :

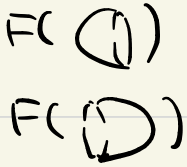
- m, Δ are both commutative and associative.
- $m(1,-)=\mathrm{Id}$ $\quad (\varepsilon \otimes \mathrm{Id}) \circ \Delta = \mathrm{Id}$
- Frobenius law $\quad \Delta \circ m = (m \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \Delta)$

Theorem: There is a bijection

$$\{(1+1)\text{-TQFT over } \mathbb{Q}\}/\text{iso.} \xleftrightarrow{1:1} \{\text{Frobenius algebra over } \mathbb{Q}\}/\text{iso.}$$

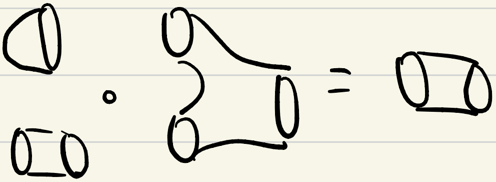
Idea of proof: We make 2 lists :

(1+1)-TQFT F	Frobenius algebra
$F(\emptyset)$	$\mathbb{Q}$
$F(S^1)$	V
$F($  $)$	$m: V\otimes V \to V$
$F($  $)$	$\Delta: V \to V\otimes V$

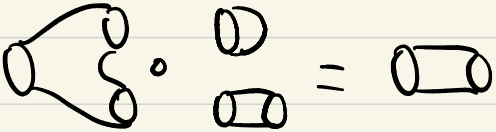


$1 : \mathbb{Q} \to V \quad 1 \mapsto 1$

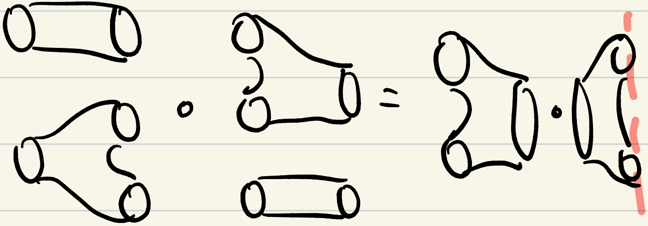
$\varepsilon : V \to \mathbb{Q}$



$m(1,-) = Id$



$(\varepsilon \circ Id) \circ \Delta = Id$



$\Delta \circ m = (m \otimes Id) \circ (Id \otimes \Delta)$

Now we generalize Khovanov homology :

(1+1)-TQFT F / Frobenius algebra $(V, m, \Delta, 1, \varepsilon)$

link diagram D } ⇝ resolution cube

⇝ chain complex C(D)

⇝ homology.

Given link diagram D, we form resolution cube $[0,1]^n$

Each vertex $\alpha \in \{0,1\}^n$, the resolution D_α is a 1-mfd $(\subset \mathbb{R}^2)$

$\overset{F(-)}{\rightsquigarrow} V_\alpha := F(D_\alpha) \in \mathrm{Vect}_R$

Each edge $\xi = (\alpha_1, \dots \alpha_{i-1}, *, \alpha_{i+1}, \dots \alpha_n) : \alpha \to \alpha'$

We have a cobordism $\underset{\underset{\mathbb{R}^2 \times I}{\cap}}{M_\xi} : D_\alpha \to D_{\alpha'}$ E.g.

so $d_\xi := F(M_\xi) : V_\alpha \to V_{\alpha'}$

set $C^i(D) := \bigoplus_{|\alpha|=i} V_\alpha$

$d : C^i(D) \to C^{i+1}(D) \qquad d := \bigoplus_{\substack{\xi : \alpha \to \alpha' \\ |\alpha| = i}} (-1)^{\xi} d_\xi$

claim: $d^2 = 0$

proof:

$$\begin{tikzcd} & \alpha_{01} \arrow[dr, "\xi_{*,1}"] & \\ \alpha_{00} \arrow[ur, "\xi_{0,*}"] \arrow[dr, "\xi_{*,0}"'] & & \alpha_{11} \\ & \alpha_{10} \arrow[ur, "\xi_{1,*}"'] & \end{tikzcd}$$

$\alpha_{kl} = (\dots, k, \dots, l, \dots)$

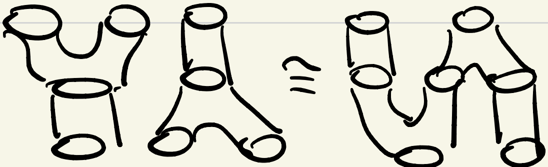
$k, l \in \{0,1\}$

Check: • $(-1)^{|\xi_{0,*}|} \cdot (-1)^{|\xi_{*,1}|} = -(-1)^{|\xi_{*,0}|} \cdot (-1)^{|\xi_{1,*}|}$

• $d_{\xi_{*,1}} \circ d_{\xi_{0,*}} = d_{\xi_{1,*}} \circ d_{\xi_{*,0}}$

↑

This follows from the fact that F is a functor (so it preserve composition. E.g.

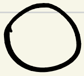


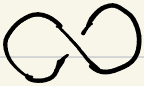
So we can define $Kh_*^F(D) := H_*(C(D), d)$

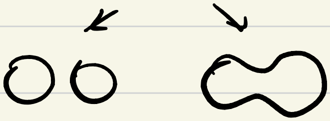
However, for general F, the result $Kh_*^F(D)$ depends on D.

A simple observation:

$Kh_*^F(-)$ is a link invariant $\Rightarrow$ $\dim(\overset{F(S^1)}{\overset{\shortparallel}{V}}) = 2$.

proof: $D =$  $\Rightarrow Kh^F(D) = V$

$D' =$  $\Rightarrow Kh^F(D') = H_*(0 \to V\otimes V \to V \to 0)$



so $\pm \dim(V)$

$= \dim(V\otimes V) - \dim(V)$

$\Rightarrow \dim(V) = 2$ or $0 \leftarrow$ trivial.

(over $\mathbb{Q}$)

2-dimensional Frobenius algebra is completely classified.

It turns out there are essentially 2 TQFTs which gives link invariants

$V_t = \mathbb{Q}[x]/(x^2-t)$ (so $m(1,x) = x$ $\quad m(1,1) = 1$
$m(x,1) = x$ $\quad m(x,x) = t$)

$\Delta : V_t \to V_t \otimes V_t$ $\quad \Delta(1) = 1\otimes x + x \otimes 1$
$\Delta(x) = x\otimes x + t(1\otimes 1)$

$1 \in V_t$ is the unit. $\varepsilon : V_t \to \mathbb{Q}$ $\quad 1 \mapsto 0$, $x \mapsto 1$ counit.

When $t=0$, this gives Khovanov homology

$(v_+ = 1 \quad v_- = x)$

When $t=1$, this gives Lee homology. We will discuss next time.
$(t \neq 0)$

Note: by replacing $\mathbb{Q}$ with more interesting rings (e.g. $\mathbb{Q}[u]$, $\mathbb{F}_2[u]/(u^k)$), we have more TQFTs that gives various generalization of Khovanov homology.

(e.g. equivariant Khovanov homology, Bar-Natan's homology)