

$$G = (V, E)$$

$$\omega \in \{ \bullet, 0 \}^V$$

Bernoulli site perco. on G :

$$\mathbb{P}_p [\omega(x) = \bullet] = p, \quad \mathbb{P}_p [\omega(x) = 0] = 1 - p.$$

indep. for distinct vertices.

$G =$ triangular lattice \mathbb{T}

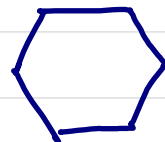
Phase transition

$$\theta(p) = \mathbb{P}_p [0 \xleftrightarrow{B} \infty] \quad \text{increasing in } p.$$

$$p < p_c, \quad \theta(p) = 0; \quad p > p_c, \quad \theta(p) > 0.$$

$$p_c > 0. \quad p_c < 1: \text{ duality.}$$

$$\text{subcritical: } p < p_c, \quad \mathbb{P}_p [0 \leftrightarrow \partial H_n] \leq e^{-cn}.$$

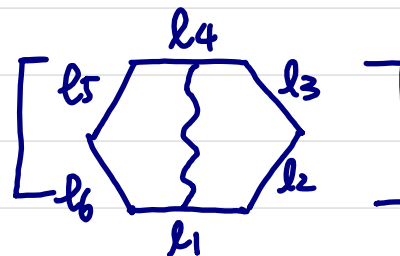


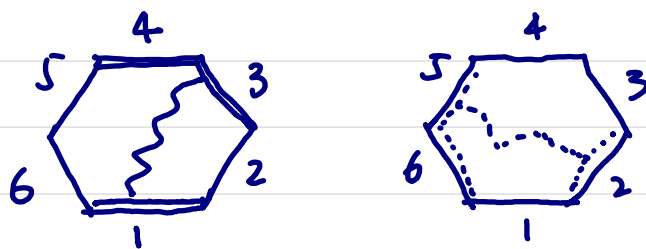
Thm. $p = \frac{1}{2}$. $P_{1/2} [0 \leftrightarrow \partial H_n] \leq n^{-c}$.

Thm. $\theta(\frac{1}{2}) = 0$. $p_c \geq \frac{1}{2}$. $p_c > \frac{1}{2} \times \Rightarrow p_c = \frac{1}{2}$.

RSW estimate. $p = \frac{1}{2}$.

Lemma. $P \left[\begin{array}{c} l_5 \\ \text{---} \\ l_6 \end{array} \begin{array}{c} l_4 \\ \text{---} \\ l_3 \\ \text{---} \\ l_2 \\ \text{---} \\ l_1 \end{array} \right] \geq \frac{1}{9}$.

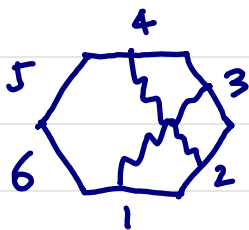


Pf: 

$$P [l_1 \xleftrightarrow{B} l_3 \cup l_4] = \frac{1}{2}.$$

$$P [l_1 \xleftrightarrow{B} l_3] + P [l_1 \xleftrightarrow{B} l_4] \geq \frac{1}{2}.$$

? v

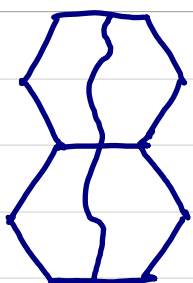


$$P [l_1 \xleftrightarrow{B} l_4] \geq P [l_1 \xleftrightarrow{B} l_3, l_2 \xleftrightarrow{B} l_4]$$

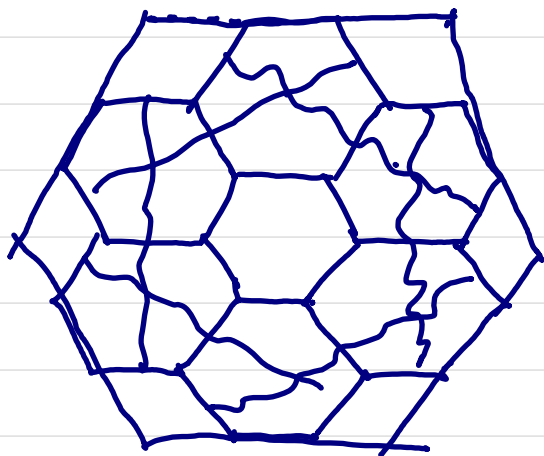
$$\geq P [l_1 \xleftrightarrow{B} l_3]^2$$

$$P [l_1 \xleftrightarrow{B} l_4]^{\frac{1}{2}} + P [l_1 \xleftrightarrow{B} l_4] \geq \frac{1}{2}.$$

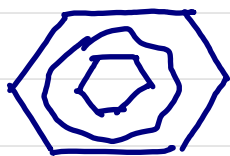
Lemma.

$$\mathbb{P} \left[\text{Diagram} \right] \geq \frac{1}{18^2} c_1 > 0.$$


Cor.



$$A_n := H_{3n} \setminus H_n$$



$$\mathbb{P}[\exists \text{ B circuit in } A_n] \geq c_2 > 0$$

$$\text{Pf. } \mathbb{P}[\exists \text{ B circuit in } A_n]$$

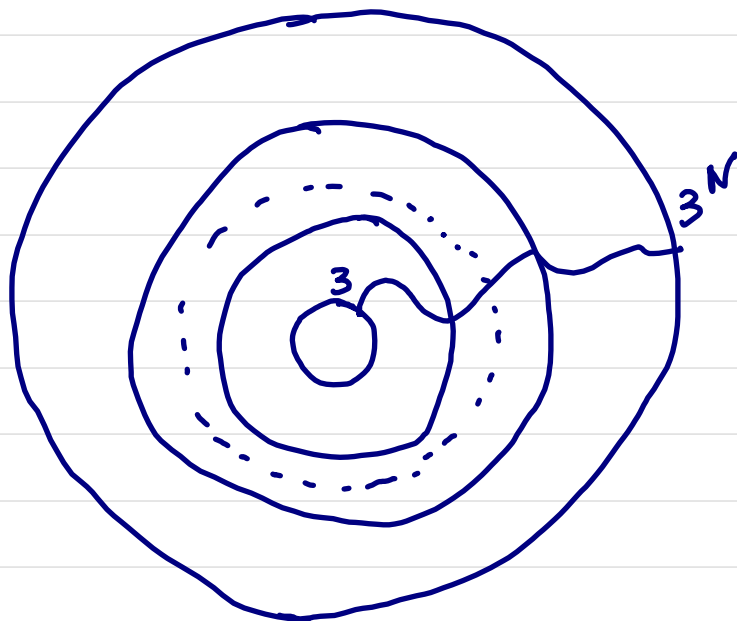
$$\geq \mathbb{P}[\text{six crossings}] \geq c_1^6 > 0.$$

$$c_2 = c_1^6.$$

Pf of Thm.

$$\mathbb{P}[\exists W \text{ circuit in } H_{3^m} \setminus H_{3^{m-1}}] \geq c > 0.$$

$$\mathbb{P}[\partial H_3 \xleftrightarrow{B} \partial H_{3^N}] \leq (1-c)^N.$$



$$\mathbb{P}[0 \xleftrightarrow{B} \partial H_n] \leq n^{-\alpha}, \quad \alpha > 0.$$

$$\theta(\frac{1}{2}) = 0.$$

$$p_c \geq \frac{1}{2}. \quad p_c > \frac{1}{2}. \quad \text{exp. decay.}$$

Contradicts with RSW estimate

$$p_c = \frac{1}{2}.$$

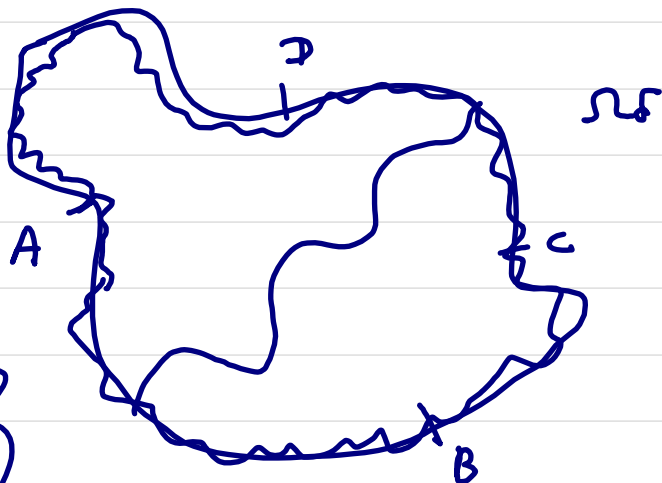
G : honeycomb lattice

G_δ

G^* : triangular.

G_δ^* .

$(\Omega; A, B, C, D)$



$\mathcal{E}_\delta(\Omega; A, B, C, D)$

$$= \left\{ \begin{array}{l} \exists B \text{ crossing in } \Omega_\delta \\ (A_\delta B_\delta) \text{ to } (C_\delta D_\delta) \end{array} \right\}$$

Thm.

- the proba. of \mathcal{E}_δ is convergent:

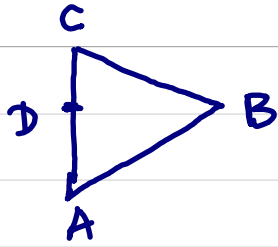
$$P[\mathcal{E}_\delta | \Omega; A, B, C, D] \rightarrow f(\Omega; A, B, C, D) \text{ as } \delta \rightarrow 0.$$

- f is conformal invariant: \forall conformal map ϕ on Ω

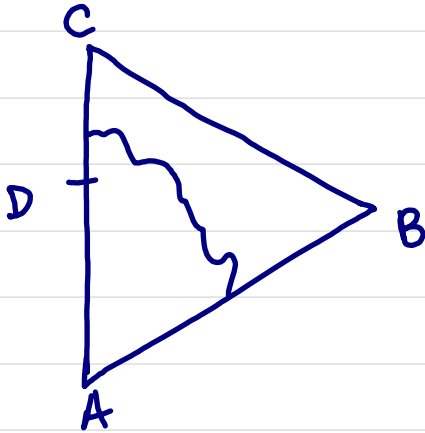
$$f(\phi(\Omega); \phi(A), \phi(B), \phi(C), \phi(D))$$

$$= f(\Omega; A, B, C, D).$$

$(\Omega; A, B, C, D) =$



$$f = \frac{|CD|}{|CA|}.$$



$(\Omega; A, B, C, D)$ quad. (topological rectangle).

Ω : non-empty bounded simply connected domain

$\partial\Omega$: locally connected.

$$\phi: \mathbb{U} \rightarrow \Omega.$$

(Riemann Mapping Thm)

$\partial\Omega$ is a curve

ϕ can be extended continuously to $\overline{\mathbb{U}}$.

$A, B, C, D \in \partial\Omega$. counterclockwise.

$(\Omega_\delta; A_\delta, B_\delta, C_\delta, D_\delta) \subset G_\delta.$

$\rightarrow (\Omega; A, B, C, D)$ in Carathéodory sense.

\exists conformal maps $\phi_\delta: U \rightarrow \Omega_\delta$

$\phi: U \rightarrow \Omega.$

$\phi_\delta \rightarrow \phi$ locally uniformly

$$\phi_\delta^{-1}(A_\delta) \rightarrow \phi^{-1}(A)$$

$$\phi_\delta^{-1}(B_\delta) \rightarrow \phi^{-1}(B)$$

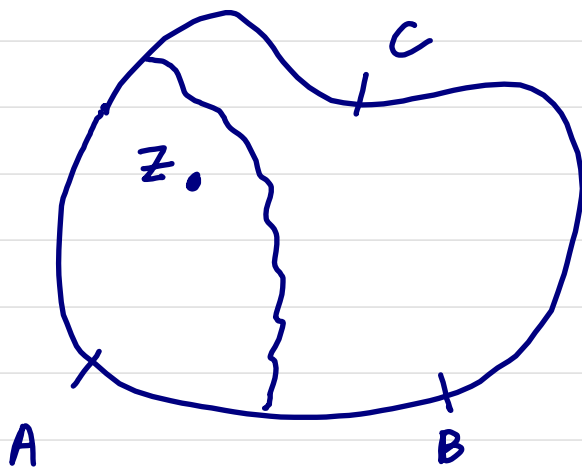
$$\phi_\delta^{-1}(C_\delta) \rightarrow \phi^{-1}(C)$$

$$\phi_\delta^{-1}(D_\delta) \rightarrow \phi^{-1}(D).$$

$z \in \Omega$.

$E_A^\delta(z) = \{ \exists \text{ a } B \text{ path disconnects } A_\delta, z_\delta \text{ from } B_\delta, C_\delta \}$.

$$H_A^\delta(z) = \mathbb{P}[E_A^\delta(z)].$$



$E_B^\delta(z), E_C^\delta(z)$.

$H_B^\delta(z), H_C^\delta(z)$.

$$\tau = e^{\frac{2\pi i}{3}}$$

$$H^\delta(z) := H_A^\delta(z) + \tau H_B^\delta(z) + \tau^2 H_C^\delta(z)$$

$$S^\delta(z) := H_A^\delta(z) + H_B^\delta(z) + H_C^\delta(z).$$

step 1. $\{H_A^\delta, H_B^\delta, H_C^\delta\}_{\delta>0}$ tight / uniform conv.

(RSW estimate \Rightarrow) $|H_A^\delta(x) - H_A^\delta(y)| \leq C|x-y|^\alpha$.

step 2. Holomorphicity.

let H be any subsequential limit of H^δ
 S S^δ .

H and S are holomorphic.

(Main contribution).

step 3. H, S holomorphic Ω
bounded on Ω .

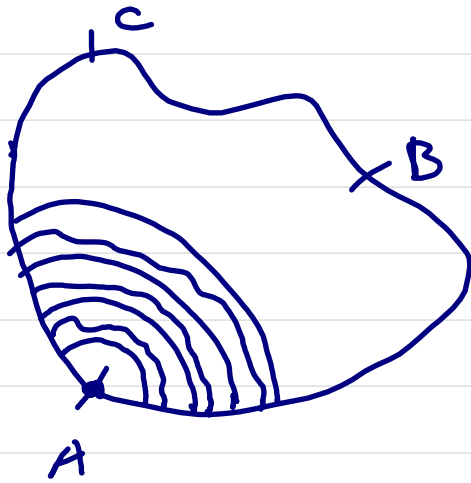
can be continuously extended to $\bar{\Omega}$.

they are uniquely determined by their
boundary value.

S : holomorphic. real-valued.

S is constant. $S(A) = ?$

$$S^\delta(z) = H_A^\delta(z) + H_B^\delta(z) + H_C^\delta(z),$$



$$H_A^\delta(z)$$

$$H_A^\delta(A) \rightarrow 1, \delta \rightarrow 0.$$

$$E_1, E_2, \dots, E_N,$$

$$N = O\left(\frac{1}{\delta}\right)$$

$$P[E_j] \geq c > 0.$$

$$H_A^\delta(A) \geq P\left[\bigcup_{j=1}^N E_j\right] \geq 1 - (1-c)^N \rightarrow 1 \text{ as } \delta \rightarrow 0.$$

$$P\left[\bigcap_{j=1}^N E_j^c\right] \leq (1-c)^N$$

$$H_A^\delta(A) \rightarrow 1, \quad H(A) = 1.$$

$$H_B^\delta(A) \rightarrow 0, \quad H_C^\delta(A) \rightarrow 0.$$

$$S(A) = 1.$$

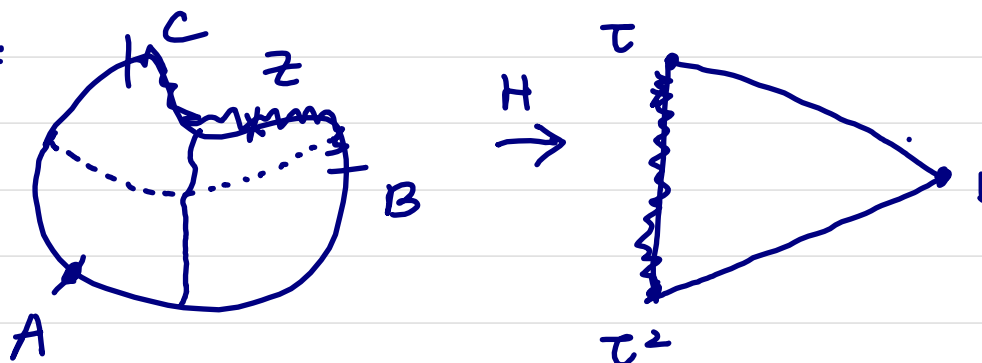
$$S \equiv 1.$$

$$H: H_A(A)=1, H_B(A)=0, H_C(A)=0.$$

$$H = H_A + \tau H_B + \tau^2 H_C.$$

$$H(A) = 1, H(B) = \tau, H(C) = \tau^2.$$

Claim:



consider (BC) . $H(z) = ?$

$$H_A(z) = 0. \quad H_B(z) + H_C(z) = 1$$

$H_B(z)$ moves continuously from 1 to 0
as z moves from B to C .

H induces a one-to-one continuous map
from $[BC]$ to $[\tau\tau^2]$

$(CA), (AB)$.

H • holomorphic in Ω

• extends continuously to $\bar{\Omega}$

• induces a continuous bijection
from $\partial\Omega$ to $\partial\Delta$

H is the conformal map from Ω onto Δ .

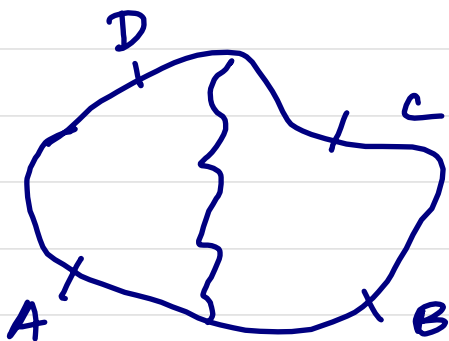
summary: $H_A + H_B + H_C = 1$

$$H_A + \tau H_B + \tau^2 H_C = H$$

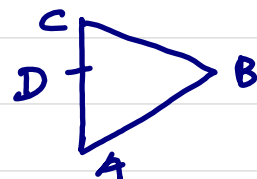
Re. $H_A + H_B + H_C = 1$

$$H_A - \frac{1}{2} H_B - \frac{1}{2} H_C = \operatorname{Re} H.$$

$$H_A = \frac{1}{3} (1 + 2 \operatorname{Re} H)$$



$$H_A(D) = \frac{1}{3} (1 + 2 \operatorname{Re} H)$$



A function $f: \Omega \rightarrow \mathbb{C}$.

holomorphic: $\forall z \in \Omega$, the limit exists

$$f'(z) = \lim_{\varepsilon \rightarrow 0} \frac{f(z+\varepsilon) - f(z)}{\varepsilon}.$$

[Morera lemma] f holomorphic iff

\forall simple closed smooth curve γ , $\oint_{\gamma} f = 0$.

$$H^{\delta}(z) = H_A^{\delta}(z) + \tau H_B^{\delta}(z) + \tau^2 H_C^{\delta}(z).$$

H any sequential limit of H^{δ} .

γ_{δ} approximation of γ :

$$(\gamma_{\delta}(k), 0 \leq k \leq N_{\delta}), \quad N_{\delta} = O\left(\frac{1}{\delta}\right).$$

$$I^{\delta}(\gamma) := \sum \frac{1}{2} (H^{\delta}(\gamma_{\delta}(k)) + H^{\delta}(\gamma_{\delta}(k+1))) \underbrace{(\gamma_{\delta}(k+1) - \gamma_{\delta}(k))}$$

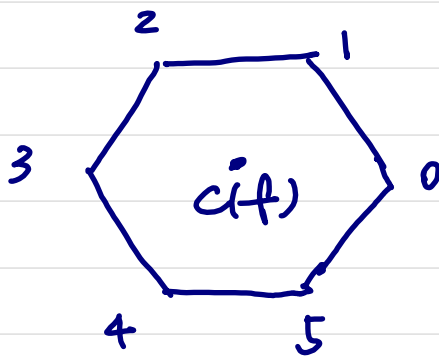
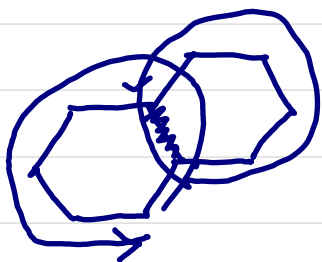
$$\rightarrow \oint_{\gamma} H(z) dz. \quad \text{Goal: } \oint_{\gamma} H = 0.$$

$$I^{\delta}(\gamma) = o(1).$$

$$e = (x, y). \quad H^\delta(e) = \frac{1}{2} (H^\delta(x) + H^\delta(y))$$

$$\partial_e H^\delta = H^\delta(y) - H^\delta(x).$$

$$I^\delta(f) = \sum_{e \in \partial f} e H^\delta(e) = \sum_{f \in \mathcal{F}} \sum_{e \in \partial f} e H^\delta(e)$$

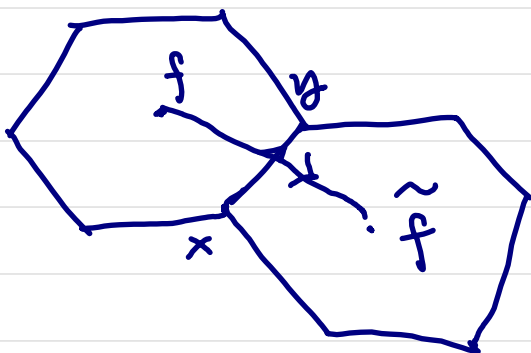


$$\begin{aligned} \sum_{e \in \partial f} e H^\delta(e) &= \sum_{k=0}^5 (x_{k+1} - x_k) \frac{1}{2} (H^\delta(x_{k+1}) + H^\delta(x_k)) \\ \text{"integrate by parts"} \quad &= \sum_{k=0}^5 (x_k - c(f)) \frac{1}{2} (H(x_k) + H(x_{k+1})) \\ &\quad - \sum_{k=0}^5 (x_k - c(f)) \frac{1}{2} (H(x_{k+1}) + H(x_k)) \\ &= \sum_{k=0}^5 (x_k - c(f)) \frac{1}{2} (H(x_{k-1}) - H(x_{k+1})) \\ &= \sum_{k=0}^5 (x_k - c(f)) \frac{1}{2} (H(x_{k-1}) + H(x_{k+1})) \\ &\quad - \sum_{k=0}^5 (x_k - c(f)) \frac{1}{2} (H(x_k) - H(x_{k+1})) \\ &= - \sum_{k=0}^5 \left(\frac{x_k + x_{k+1}}{2} - c(f) \right) (H(x_{k+1}) - H(x_k)) \end{aligned}$$

$$\sum_{e \in \partial f} e H^\delta(e) = - \sum_0^{\infty} \underbrace{\left(\frac{x_k + x_{k+1}}{2} - c(f) \right)}_{O(\delta)} \underbrace{\left(H^\delta(x_{k+1}) - H^\delta(x_k) \right)}_{O(1)}$$

$$I^\delta(\gamma) = \sum_{f \in \mathcal{F}_\delta} \sum_{e \in \partial f} \quad N_\delta = O\left(\frac{1}{\delta}\right)$$

$$\begin{aligned} e = (xy) : & \left(\frac{x+y}{2} - c(f) \right) (H(y) - H(x)) \\ & + \left(\frac{x+y}{2} - c(\tilde{f}) \right) (H(x) - H(y)) \\ & = (c(\tilde{f}) - c(f)) (H(y) - H(x)) \\ & = -e^* \partial_e H^\delta \end{aligned}$$



$$I^\delta(\gamma) = \frac{1}{2} \sum_{e \in \partial \delta} e^* \partial_e H + o(1).$$

$$\text{Goal: } \sum_{e \in \mathcal{E}} e^* \partial_e H = 0.$$

$$H^\delta = H_A^\delta + \tau H_B^\delta + \tau^2 H_C^\delta$$

$$\partial_e H^\delta = \partial_e H_A^\delta + \tau \partial_e H_B^\delta + \tau^2 \partial_e H_C^\delta$$

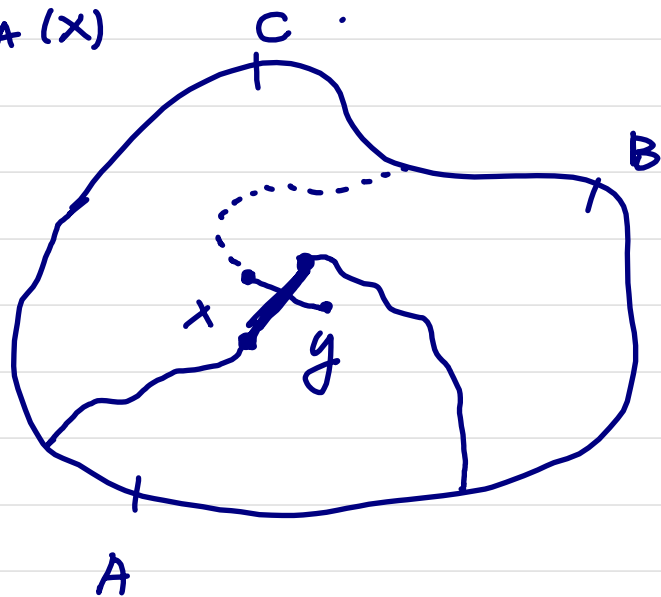
$$\partial_e H_A = H_A(y) - H_A(x)$$

$$= \mathbb{P}[E_A(y) \setminus E_A(x)]$$

$$- \mathbb{P}[E_A(x) \setminus E_A(y)]$$

$$P_A(e) := \mathbb{P}[E_A(y) \setminus E_A(x)]$$

$$e = (x, y).$$



$$\partial_e H_A = P_A(e) - P_A(-e).$$

Lemma. [color switching]

$$P_A(e) = P_B(\tau e) = P_C(\tau^2 e).$$

$$\sum_{e \in \mathfrak{S}} e^* \partial_e H^\delta$$

$$= 2 \sum e^* (P_A(e) + \tau P_B(e) + \tau^2 P_C(e))$$

color switching

$$\stackrel{\curvearrowright}{=} 2 \sum e^* (P_A(e) + \tau \underbrace{P_A(\tau^2 e)} + \tau^2 P_A(\tau e))$$

$$= 2 \sum \underbrace{(e^* + \tau (\tau e)^* + \tau^2 (\tau^2 e)^*)}_{\substack{\parallel \\ 0}} P_A(e)$$

$$\tau = e^{\frac{2\pi i}{3}}$$

$$\sum_{e \in \mathfrak{S}} e^* \partial_e H^\delta = 0.$$

$$I^\delta(\gamma) = 0(1).$$

$$\downarrow$$

$$\oint_\gamma H = 0.$$