Comments about last lecture

- It might be interesting to do exercises 2.1, 2.2, 2.3, 2.4, 2.5 in my book on $SL_2(\mathbb{F}_q)$:
  - → quotients
  - → $\mathbb{F}_q$-structures
  - → Exercise 2.2 is more difficult but gives another proof of Theorem 4.6 (a) $\quad (\mathcal{Y}/G \simeq \mathbb{A}^1(\mathbb{F}))$.
- Find the mistake in my book, §2.5.1. :)
- Compute $\bar{\mathcal{Y}}/G$, $\bar{\mathcal{Y}}/\mu_{q+1}$, $\bar{\mathcal{Y}}/U$.
- Complement to Theorem 3.10. (e) $\mathrm{Tr}^*_{X \times X'}(f \times f') = \mathrm{Tr}^*_X(f)\, \mathrm{Tr}^*_{X'}(f')$
- Exercise 4.11: Let M be a $K\Gamma$-module and let $\varphi \in \mathrm{End}_{K\Gamma}(M)$.

  Show that
$$\mathrm{Tr}(\varphi, M^\Gamma) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \mathrm{Tr}(\gamma\varphi, M).$$

$$\boxed{\langle R(\alpha),\ R'(\theta)\rangle = 0}$$

Harish-Chandra induction.

$\mathcal{R} : KT\text{-mod} \longrightarrow KG\text{-mod}$

$\quad\ \ \overset{\wr\!\wr}{K\mu_{q-1}}\text{-mod}$

$R : \mathrm{Class}(T) \longrightarrow \mathrm{Class}(G)$

<u>Prop. 2.4.</u> $R(\alpha)(1) = (q+1)\,\alpha(1)$

<u>Mackey formula 2.5</u> $\langle R(\alpha), R(\beta)\rangle_G = \langle \alpha, \beta\rangle_T + \langle \alpha, {}^s\beta\rangle_T$

<u>Theo. 2.6.</u> Let $\alpha, \beta \in \mathrm{Irr}(T) = \mathrm{Hom}(T, K^\times)$.

(a) $R(\alpha) = R(\alpha^{-1})$

(b) $\langle R(\alpha), R(\beta)\rangle = 0$ if $\beta \neq \alpha^{\pm 1}$

(c) If $\alpha^2 \neq 1$, then $R(\alpha) \in \mathrm{Irr}(G)$

<u>Example 2.7.</u> $R(1_T) = 1_G + St$

<u>Example 2.8.</u> $R(\alpha_0) = R(\alpha_0)^+ + R(\alpha_0)^-$

$$(2.9)\quad R(\chi_{KT}) = \underset{\text{dim. }1}{1_G} + \underset{\text{dim. }q}{St} + \underset{\text{dim. }\frac{q+1}{2}}{R(\alpha_0)^+} + \underset{\text{dim. }\frac{q+1}{2}}{R(\alpha_0)^-} + 2\sum_{\alpha \in (\mathrm{Irr}\,T \setminus \{1,\alpha_0\})/_{\mathrm{INV}}} \underset{\text{dim } q+1}{R(\alpha)}$$

$\frac{q+5}{2}$ distinct irreducible characters.

Deligne-Lusztig induction

$\mathcal{R}'_i : KT'\text{-mod} \longrightarrow KG\text{-mod}$

$\quad\ \ \overset{\wr\!\wr}{K\mu_{q+1}}\text{-mod}$

$R' = R'_1 - R'_2 : \mathrm{Class}(T') \longrightarrow \mathrm{Class}(G)$

$R'(\theta)(1) = (q-1)\,\theta(1)$

<u>Mackey formula.</u> $\langle R'(\theta), R'(\eta)\rangle_G = \langle \theta, \eta\rangle_{T'} + \langle \theta, {}^{s'}\eta\rangle$

<u>Theo.</u> Let $\theta, \eta \in \mathrm{Irr}(T') = \mathrm{Hom}(T', K^\times)$

(a) $R'(\theta) = R'(\theta^{-1})$

(b) $\langle R'(\theta), R'(\eta)\rangle = 0$ if $\theta \neq \eta^{\pm 1}$

(c) If $\theta^2 \neq 1$, then $R'(\theta) \in \mathrm{Irr}\,G$

<u>Example.</u> $R'(1_{T'}) = -1_G + St$

<u>Example.</u> $R'(\theta_0) = R'(\theta_0)^+ + R'(\theta_0)^-$

$$R'(\chi_{KT'}) = -\underset{\text{dim. }1}{1_G} + \underset{\text{dim. }q}{St} + \underset{\text{dim. }\frac{q-1}{2}}{R'(\theta_0)^+} + \underset{\text{dim }\frac{q-1}{2}}{R'(\theta_0)^-} + 2\sum_{\theta \in (\mathrm{Irr}\,T \setminus \{1,\theta_0\})/_{\mathrm{INV}}} \underset{\text{dim. }q-1}{R'(\theta)}$$

$\frac{q+3}{2}$ distinct NEW irreducible characters

Mackey formula 4.12. $\langle R'(\theta), R'(\eta)\rangle_G = \langle \theta, \eta\rangle_T + \langle \theta, {}^{s'}\eta\rangle_T$

Proof. We may assume that $\theta, \eta \in \operatorname{Irr} T' = \operatorname{Hom}_{gp}(T', K^\times)$

$$\langle R'(\theta), R'(\eta)\rangle_G = \frac{1}{|G|}\sum_{g\in G} R'(\theta)(g)\, R'(\eta)(g^{-1}) = \frac{1}{|G|}\sum_{g\in G} R'(\theta)(g)\, R'(\eta)(g) \quad \text{(by cor 4.10)}$$

$$= \frac{1}{|G|}\sum_{g\in G} \frac{1}{(q+1)^2}\sum_{\xi,\xi'\in\mu_{q+1}} \operatorname{Tr}^*_Y(g,\xi)\, \operatorname{Tr}^*_Y(g,\xi')\ \theta(\xi^{-1})\, \eta(\xi'^{-1})$$

$$= \frac{1}{(q+1)^2}\sum_{\xi,\xi'\in\mu_{q+1}} \theta(\xi^{-1})\,\eta(\xi'^{-1}) \left(\underbrace{\frac{1}{|G|}\sum_{g\in G} \operatorname{Tr}^*_Y(g,\xi)\, \operatorname{Tr}^*_Y(g,\xi')}_{f(\xi,\xi')} \right)$$

$$f(\xi,\xi') = \frac{1}{|G|}\sum_{g\in G} \operatorname{Tr}^*_{Y\times Y}(g,\xi,\xi') \quad \text{by Theo 3.10(e)}$$

$$= \sum_{i\geq 0} (-1)^i \left(\frac{1}{|G|}\sum_{g\in G} \operatorname{Tr}\big((g,\xi,\xi'), H^i_c(Y\times Y)\big)\right)$$

$$= \sum_{i\geq 0} (-1)^i \operatorname{Tr}\big((\xi,\xi'), H^i_c(Y\times Y)^G\big) \quad \text{by exercise 4.11.}$$

$$= \sum_{i\geq 0} (-1)^i \operatorname{Tr}\big((\xi,\xi'), H^i_c(Y^2/G)\big) \quad \text{by theo. 3.7(i)}$$

$$= \operatorname{Tr}^*_{Y^2/G}(\xi,\xi')$$

$$Y^2 = \{(x,y,z,t)\in \mathbb{A}^4(\mathbb{F}) \mid xy^q - yx^q = zt^q - tz^q = 1\} = Y^2_0 \,\dot\cup\, Y^2_{\neq 0}$$

closed, open

where $Y^2_0 = \{(x,y,z,t)\in Y^2 \mid xt - yz = 0\}$

$Y^2_{\neq 0} = \{(x,y,z,t)\in Y^2 \mid xt - yz \neq 0\}$

Both $(G\times\mu_{q+1}\times\mu_{q+1})$-stable

So $f(\xi,\xi') = \mathrm{Tr}^*_{Y_0^2/G}(\xi,\xi') + \mathrm{Tr}^*_{Y^2_{\neq 0}/G}(\xi,\xi')$ by Theo 3.10 (a).

- $\mu_{q+1} \times Y \longrightarrow Y_0^2$ is an isomorphism of varieties. It is $(G\times\mu_{q+1}\times\mu_{q+1})$-equiv.
  $(a, v) \longmapsto (v, av)$ if we set $(g,\xi,\xi')\cdot(a,v) = (\xi^{-1}\xi' a, \xi g\cdot v)$

  So $Y_0^2/G \simeq \mu_{q+1} \times Y/G \simeq \mu_{q+1} \times \mathbb{A}^1(\mathbb{F})$ by 4.6 (a). So

(M1) $\mathrm{Tr}^*_{Y_0^2/G}(\xi,\xi') = \mathrm{Tr}^*_{\mu_{q+1}}(\xi,\xi') \cdot \mathrm{Tr}^*_{\mathbb{A}^1(\mathbb{F})}(\xi^2) = \begin{cases} q+1 & \text{if } \xi^{-1}\xi' = 1 \\ 0 & \text{otherwise} \end{cases}$

(by Theo 3.10 (e) and 3.11. (g))

- Let $\Psi : Y^2_{\neq 0} \longrightarrow \mathcal{V} = \{(u,a,b) \in \mathbb{F}^\times \times \mathbb{F} \times \mathbb{F} \mid u^{q+1} - ab = 1\}$
  $(x,y,z,t) \longmapsto (xt - yz,\ xt^q - yz^q,\ x^q t - y^q z)$

Then Ψ is a morphism between smooth varieties which is constant on G-orbits.

<u>Exercise*:</u> Ψ induces an isomorphism $\overline{\Psi} : Y^2_{\neq 0}/G \longrightarrow \mathcal{V}$ (use Prop 4.5)

Through $\overline{\Psi}$, the action of (ξ,ξ') on $\mathcal{V}$ is given by $(\xi,\xi')(u,a,b) = (\xi\xi' u, \xi\xi'^{-1} a, \xi^{-1}\xi' b)$

So $\mathrm{Tr}^*_{Y^2_{\neq 0}/G}(\xi,\xi') = \mathrm{Tr}^*_{\mathcal{V}}(\xi,\xi')$

Now, the connected group $\mathbb{F}^\times$ acts on $\mathcal{V}$ through: $\lambda\cdot(u,a,b) = (u, \lambda a, \lambda^{-1} b)$ and this action commutes with (ξ,ξ'): so $\mathrm{Tr}^*_{\mathcal{V}}(\xi,\xi') = \mathrm{Tr}_{\mathcal{V}^{\mathbb{F}^\times}}(\xi,\xi')$ (by Theo. 3.10 (c)).
But $\mathcal{V}^{\mathbb{F}^\times} = \mu_{q+1} \times \{0\} \times \{0\}$ so

(M2) $\mathrm{Tr}^*_{Y^2_{\neq 0}/G}(\xi,\xi') = \mathrm{Tr}^*_{\mu_{q+1}}(\xi,\xi') = \begin{cases} q+1 & \text{if } \xi\xi' = 1 \\ 0 & \text{otherwise.} \end{cases}$; $f(\xi,\xi') = (q+1)(\delta_{\xi,\xi'} + \delta_{\xi^{-1},\xi'})$

$$\langle R'(\theta), R'(\eta)\rangle_G = \frac{1}{(q+1)^2} \sum_{\xi, \xi' \in \mu_{q+1}} \theta(\xi^{-1})\eta(\xi'^{-1}) f(\xi, \xi')$$
$$= \frac{1}{q+1} \sum_{\xi, \xi' \in \mu_{q+1}} \theta(\xi^{-1})\eta(\xi'^{-1}) \left(\delta_{\xi,\xi'} + \delta_{\xi,\xi'^{-1}}\right)$$
$$= \frac{1}{q+1} \sum_{\xi \in \mu_{q+1}} \left(\theta(\xi^{-1})\eta(\xi^{-1}) + \theta(\xi^{-1})\eta(\xi)\right)$$
$$= \langle \theta, {}^{s'}\eta\rangle_{T'} + \langle \theta, \eta\rangle_{T'} . \blacksquare$$

Theorem 4.13. Let $\theta, \eta \in \mathrm{Irr}(T')$. Then

(a) $\langle R'(\theta), R'(\eta)\rangle_G = 0$ if $\theta \neq \eta^{\pm 1}$.

(b) If $\theta \neq 1$, then $R'(\theta) = R'_1(\theta) \in \mathbb{N}\,\mathrm{Irr}\,G$

(c) If $\theta^2 \neq 1$, then $R'(\theta) \in \mathrm{Irr}\,G$.

Proof. (a) is a direct consequence of the Mackey formula 4.12.

(b) Since Y is irreducible, we have $H^2_c(Y) = K$, with trivial action of $G \times \mu_{q+1}$ (by Theo 3.7(c)). So $H^2_c(Y) \otimes_{KT'} E_\theta = 0$ if $\theta \neq 1$.

(c) is a direct consequence of (b) and the Mackey formula 4.12. $\blacksquare$

Example 4.14. $q+1$ is even so μ_{q+1} admits a unique linear character $\theta_0 \neq 1$ such that $\theta_0^2 = 1$. So

$$R'(\theta_0) \in \mathbb{N}\,\mathrm{Irr}\,G \qquad (\text{see } 4.13(b))$$

and

$$\langle R'(\theta_0), R'(\theta_0)\rangle = 2 \qquad (\text{see } 4.12)$$

Hence $R'(\theta_0) = R'(\theta_0)^+ + R'(\theta_0)^-$

with $R'(\theta_0)^\pm \in \mathrm{Irr}\,G$ and $R'(\theta_0)^+ \neq R'(\theta_0)^-$. $\blacksquare$

(4.15) $$R'(\chi_{KT'}) = -\overbrace{1_G}^{\text{dim. }1} + \overbrace{\mathrm{St}}^{\text{dim. }q} + \underbrace{R'(\theta_0)^+}_{\text{dim. }\frac{q-1}{2}} + \underbrace{R'(\theta_0)^-}_{\text{dim. }\frac{q-1}{2}} + 2 \sum_{\theta \in ((\mathrm{Irr}\,T')\setminus\{1_{T'}, \theta_0\})/\mathrm{INV}} \underbrace{R'(\theta)}_{\text{dim } q-1}$$

Mixed Mackey formula 4.13. $\langle R(\alpha), R'(\theta)\rangle_G = 0$

Proof. We may assume that $\alpha \in \mathrm{Irr}\, T$ and $\theta \in \mathrm{Irr}\, T'$. $\left(R(\theta) = R'_1(\theta) - R'_2(\theta)\right)$

$$\langle R(\alpha), R'_1(\theta)\rangle_G = \langle \alpha, {}^*R\, R'_1(\theta)\rangle_T \qquad \text{(adjunction for HC-induction)}$$

$$\begin{aligned}
{}^*\mathcal{R}\, \mathcal{R}'_1(E_\theta) &= K[U\backslash G] \otimes_{KG} \left(H^1_c(Y) \otimes_{KT'} E_\theta\right) \\
&= \left(K[U\backslash G] \otimes_{KG} H^1_c(Y)\right) \otimes_{KT'} E_\theta \\
&= H^1_c(Y)^U \otimes_{KT'} E_\theta && \text{(by Prop. 2.4)} \\
&= H^1_c(Y/U) \otimes_{KT'} E_\theta && \text{(3.10 (i))} \\
&= H^1_c(\mathbb{A}^1(\mathbb{F}) \smallsetminus \{0\}) \otimes_{KT'} E_\theta && \text{(Theo 4.6 (b))}
\end{aligned}$$

Similarly ${}^*\mathcal{R}\, \mathcal{R}'_2(E_\theta) = \cdots = H^2_c(\mathbb{A}^1(\mathbb{F}) \smallsetminus \{0\}) \otimes_{KT'} E_\theta$.

Through the isomorphism $Y/U \xrightarrow{\bar{\nu}} \mathbb{A}^1(\mathbb{F}) \smallsetminus \{0\}$, $\underset{\substack{\| \\ T}}{\mu_{q-1}} \times \underset{\substack{\| \\ T'}}{\mu_{q+1}}$ acts on $\mathbb{A}^1(\mathbb{F}) \smallsetminus \{0\}$ by homotheties: so this action is the restriction of the action of $\mathbb{F}^\times$ by homotheties: so $\mu_{q-1} \times \mu_{q+1}$ acts trivially on $H^i_c(\mathbb{A}^1(\mathbb{F}) \smallsetminus \{0\})$ (Theo 3.7 (j)).

So $\underset{(KT,KT')\text{-bimodule}}{H^1_c(Y/U)} \simeq H^2_c(Y/U) \simeq K_{T \times T'}$.

So ${}^*R\, R'_1(\theta) = {}^*R\, R'_2(\theta)$ so ${}^*R\, R'(\theta) = 0$. ∎

$$\langle R(1_T), R'(1_{T'})\rangle_G = \langle 1_G + \mathrm{St}, -1_G + \mathrm{St}\rangle = 0$$

4.F. Conclusion.

$$\text{(4.16)} \quad \mathrm{Irr}(G) = \Big\{ 1_G,\ St,\ R'(\alpha_0)^{\pm},\ \underset{\substack{\alpha \in \mathrm{Irr}\,T/_{\mathrm{INV}} \\ \alpha^2 \neq 1}}{R(\alpha)},\ R'(\theta_0)^{\pm},\ \underset{\substack{\theta \in \mathrm{Irr}\,T'/_{\mathrm{INV}} \\ \theta^2 \neq 1}}{R'(\theta)} \Big\}$$

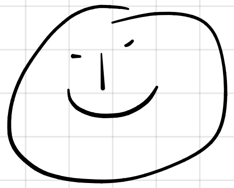
$$\text{(4.17)} \quad \chi(1) = \begin{cases} 1 & \text{if } \chi = 1_G \\ q & \text{if } \chi = St \\ (q+1)/2 & \text{if } \chi = R(\alpha_0)^{\pm} \\ q+1 & \text{if } \chi = R(\alpha) \\ (q-1)/2 & \text{if } \chi = R'(\theta_0)^{\pm} \\ q-1 & \text{if } \chi = R'(\theta) \end{cases}$$

Proof. Let $d_{\pm} = R(\alpha_0)^{\pm}(1)$; $d'_{\pm} = R'(\theta_0)^{\pm}(1)$.

$$|G| = \sum_{\chi \in \mathrm{Irr}\,G} \chi(1)^2 \quad \text{and} \quad d_+ + d_- = q+1 \quad \text{and} \quad d'_+ + d'_- = q-1.$$

CHECK: This forces $d_{\pm} = \frac{q+1}{2}$ and $d'_{\pm} = \frac{q-1}{2}$. ■

Exercise. What happens if q is even?



4.6. Small values of q.

$$(4.17) \qquad \chi(1) = \begin{cases} 1 & \text{if } \chi = 1_G \\ q & \text{if } \chi = St \\ (q+1)/2 & \text{if } \chi = R(\alpha_0)^{\pm} \\ q+1 & \text{if } \chi = R(\alpha) \\ (q-1)/2 & \text{if } \chi = R'(\theta_0)^{\pm} \\ q-1 & \text{if } \chi = R'(\theta) \end{cases}$$

- $q = 3$: dim. 1, 3, 2, 2, 1, 1, 2

three linear characters!

$SL_2(\mathbb{F}_3) \hookrightarrow SL_2(\mathbb{C})$

By reduction modulo a prime number (not easy), you get $SL_2(\mathbb{F}_3) \hookrightarrow SL_2(\mathbb{F}_\ell)$ for any odd prime number ℓ.

- $q = 5$: dim. 1, 5, 3, 3, 6, 2, 2, 4, 4
  - $SL_2(\mathbb{F}_3) \hookrightarrow SL_2(\mathbb{F}_5)$ of index 5 $\Rightarrow$ $PSL_2(\mathbb{F}_5) \xrightarrow{\sim} A_5$: $1 \to \mu_2 \to SL_2(\mathbb{F}_5) \to A_5 \to 1$.
  - $SL_2(\mathbb{F}_5) \hookrightarrow SL_2(\mathbb{C})$:

By reduction modulo a prime number (not easy), you get

$$SL_2(\mathbb{F}_5) \hookrightarrow SL_2(\mathbb{F}_\ell) \quad \text{if } \ell \equiv \pm 1 \mod 10$$
$$SL_2(\mathbb{F}_5) \hookrightarrow SL_2(\mathbb{F}_{\ell^2}) \quad \text{if } \ell \equiv \pm 3 \mod 10$$

In particular, $SL_2(\mathbb{F}_5) \hookrightarrow SL_2(\mathbb{F}_9)$ of index 6 :

$\Rightarrow SL_2(\mathbb{F}_9) \longrightarrow S_6 \quad \rightsquigarrow \quad PSL_2(\mathbb{F}_9) \simeq A_6$

• $\underline{q=7}$: dim. 1, 7, 4, 4, 8, 8, 3, 3, 6, 6, 6

$$PSL_2(\mathbb{F}_7) \hookrightarrow SL_3(\mathbb{C})$$

(order 168)

$$(4.17) \qquad \chi(1) = \begin{cases} 1 & \text{if } \chi = 1_G \\ q & \text{if } \chi = St \\ (q+1)/2 & \text{if } \chi = R(\alpha_0)^{\pm 1} \\ q+1 & \text{if } \chi = R(\alpha) \\ (q-1)/2 & \text{if } \chi = R'(\theta_0)^{\pm} \\ q-1 & \text{if } \chi = R'(\theta) \end{cases}$$

Let Γ denote the image in $SL_3(\mathbb{C})$: Γ acts on $\mathbb{C}^3$ so it acts $\mathbb{C}[X_1, X_2, X_3]$ and $|\Gamma| = 168$.

It can be proved that $\mathbb{C}[X_1, X_2, X_3]^\Gamma$ admits an homogeneous element $f \neq 0$ of degree 4. Let

$$\mathcal{C} = \{ [x_1, x_2, x_3] \in \mathbb{P}^2(\mathbb{C}) \mid f(x_1, x_2, x_3) = 0 \}$$

It can be proved that: $\mathcal{C}$ is smooth irreducible

so $$g(\mathcal{C}) = \frac{(4-1)(4-2)}{2} = 3$$

<u>Hurwitz</u>. $$|\mathrm{Aut}(\mathcal{C})| \leqslant 84(g(\mathcal{C})-1) = 168$$

So $\mathrm{Aut}(\mathcal{C}) = \Gamma \simeq PSL_2(\mathbb{F}_7)$.

$\mathcal{C}$ is the Klein curve.