Quantization and Index Theory

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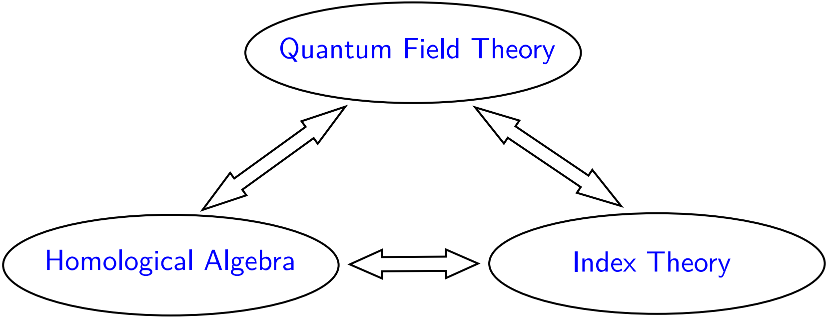
2024 Current Developments in Mathematics and Physics @ Beijing


Quantum Field Theory
Homological Algebra
Index Theory

Motivation: Quantum Mechanics and Index Theorem

Topological QM leads to a path integral on the loop space

$$\int_{\mathrm{Map}(S^1,X)} e^{-S/\hbar} \quad \overset{\hbar\to 0}{\Longrightarrow} \quad \int_X (\text{curvatures})$$

Topological nature implies the exact semi-classical limit $\hbar \to 0$, which localizes the path integral to constant loops.

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- LHS= the analytic index
- RHS= the topological index

This is the physics "derivation" of Atiyah-Singer Index Theorem (**Alvarez-Gaumé, Friedan-Windey, Witten**)

Algebraic Index Theorem

Given a deformation quantization $\mathcal{A}_\hbar(M) = (C^\infty(M)[[\hbar]], \star)$ on a symplectic manifold (M, ω), there exists a unique linear map

$$\mathrm{Tr} : \mathcal{A}_\hbar(M) \to \mathbb{C}((\hbar))$$

satisfying a normalization condition and the trace property

$$\mathrm{Tr}(f \star g) = \mathrm{Tr}(g \star f).$$

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Then

$$\mathrm{Tr}(1) = \int_M e^{\omega_\hbar/\hbar} \hat{A}(M).$$

This is the algebraic index theorem which was first formulated by **Fedosov** and **Nest-Tsygan** as the algebraic analogue of Atiyah-Singer index theorem.

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Replace S^1 by an elliptic curve E:

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In this talk, we will explain a chiral algebraic index theory as an algebraic analogue of this elliptic index.

Let us first explain how some notions of homological algebra in noncommutative geometry arise naturally in quantum field theory.

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Hochschild-Kostant-Rosenberg = Renormalization Group Flow

A: associative algebra. Hochschild chain complex

$$(C_\bullet(A), b) = \cdots C_p(A) \xrightarrow{b} C_{p-1}(A) \to \cdots \to C_1(A) \xrightarrow{b} C_0(A)$$

where

$$C_p(A) := A^{\otimes p+1}$$

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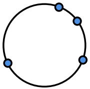
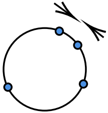
The Hochschild differential b is

$$b(a_0 \otimes \cdots \otimes a_p) = a_0 a_1 \otimes \cdots \otimes a_p - a_0 \otimes a_1 a_2 \otimes \cdots \otimes a_p + \cdots + (-1)^{p-1} a_0 \otimes a_1 \otimes \cdots \otimes a_{p-1} a_p + (-1)^p a_p a_0 \otimes \cdots \otimes a_{p-1}.$$

b  $=$ $\sum$ 

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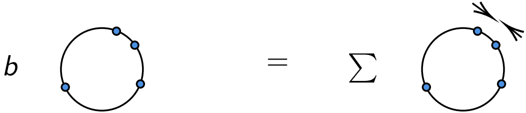

Hochschild Homology $HH_\bullet(A) = H_\bullet(C_\bullet(A), b)$

Hochschild-Kostant-Rosenberg (HKR) Theorem

Let $A = k[x_1, \cdots, x_n]$. The HKR Theorem says

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It can be realized by an explicit HKR map of chain complexes

$$\sigma : (C_\bullet(A), b) \to (\Omega^\bullet(A), 0)$$

$$a_0 \otimes a_1 \otimes \cdots \otimes a_p \to a_0 da_1 \wedge \cdots \wedge da_p$$

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In general, Hochschild chain complex describes the analogue of differential forms in noncommutative geometry.

We will describe a realization of quantized HKR below that is related to the index theorem.

Topological Quantum Mechanics and Algebraic Index

We consider the AKSZ construction of TQM on the standard phase space $\mathbb{R}^{2n}$. The space of field is described by

$$(\mathbb{X}^1, \cdots, \mathbb{X}^n, \mathbb{P}_1, \cdots, \mathbb{P}_n), \qquad \mathbb{X}^i, \mathbb{P}_i \in \Omega^\bullet_{S^1}.$$

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For any $\mathcal{O} \in \mathbb{R}[x^i, p_i]$, we define an observable

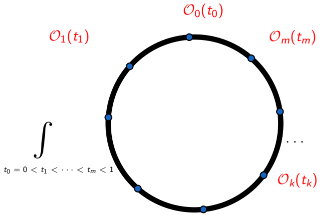
$$\mathcal{O}(\mathbb{X}^i, \mathbb{P}_i) = \mathcal{O}^{(0)}(t) + \mathcal{O}^{(1)}(t)dt.$$

Here t is the coordinate on S^1.

Consider the correlation via configuration space integral

$$\langle \mathcal{O}_0 \otimes \mathcal{O}_1 \cdots \otimes \mathcal{O}_m \rangle_{1d} \qquad\qquad \mathcal{O}_i \in \mathbb{R}[x^i, p_i]$$

$$:= \int_{t_0=0<t_1<\cdots<t_m<1} \left\langle \mathcal{O}_0^{(0)}(t_0)\mathcal{O}_1^{(1)}(t_1)\cdots\mathcal{O}_m^{(1)}(t_m) \right\rangle_{free}$$



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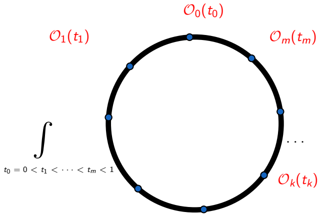

This can be viewed as a BV pushforward to zero modes.

Theorem (**Gui-L-Xu**, CMP 2021)

Let $W_{2n}^{\hbar}$ = Weyl algebra. The correlation map in TQM intertwines

$$\begin{array}{ccc} \langle - \rangle_{1d} : C_\bullet(W_{2n}^{\hbar}) & \to & \Omega_{2n}^{\bullet}((\hbar)) \\ b & \to & \hbar\Delta = \hbar\mathcal{L}_{\Pi} \\ B & \to & d_{2n} \end{array}$$

It particular, it gives an explicit formula of quantized HKR map.

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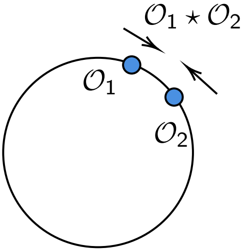
Such map can be glued on a symplectic manifold X, leading to

- a trace map on deformation quantized algebra (**Feigin-Felder-Shoikhet** formula).
- TQM $\Longrightarrow$ algebraic index theorem (**Grady-Li-L** 2017, **Gui-L-Xu** 2021)
- TQM on symplectic orbifolds (**L-Peng** 2024)

2d Chiral CFT and elliptic chiral index

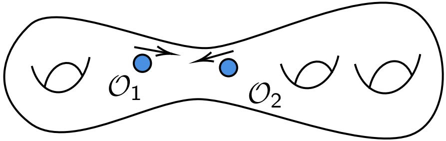
1d TQM	2d Chiral CFT
S^1	Σ
Associative algebra	Vertex operator algebra

Associative product



Operator product expansion

$$\mathcal{O}_1(z)\mathcal{O}_2(w) \sim \sum_n \frac{\mathcal{O}_{1(n)}\mathcal{O}_2(w)}{(z-w)^{n+1}}$$



Example: $\beta\gamma - bc$ system

The VOA $\mathcal{V}^{\beta\gamma - bc}$ of $\beta\gamma - bc$ system is the chiral analogue of Weyl/Clifford algebra.

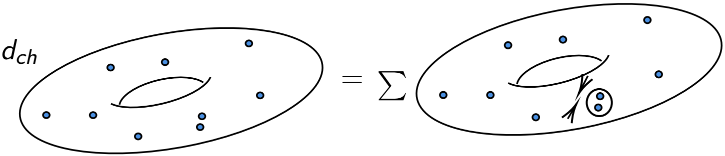
$$\beta(z)\gamma(w) \sim \frac{1}{z-w} + \text{reg.} \qquad b(z)c(w) \sim \frac{1}{z-w} + \text{reg.}$$

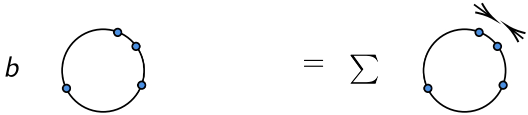
It gives rise to a chiral algebra (in the sense of Beilinson and Drinfeld) $\mathcal{A}^{\beta\gamma - bc} = \mathcal{V}^{\beta\gamma - bc} \otimes_{\mathcal{O}_x} \omega_X$ on a Riemann surface $X = \Sigma$.

Elliptic chiral homology

- In [**Zhu**, 1994], **Zhu** studied the space of genus 1 conformal block (the 0-th elliptic chiral homology) and establish the modular invariance for certain class of VOA.
- **Beilinson** and **Drinfeld** define the chiral homology for general algebraic curves using the Chevalley-Cousin complex.
- **Ekeren-Heluani**: explicit complex expressing the 0th and 1st elliptic chiral homology.

Intuitively, the chiral differential in the chiral complex looks like a 2d chiral analogue of the Hochschild differential b.





Theorem (Gui-L, 2021)

Let $E_\tau = \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\tau$. *We can construct an explicit map*

$$\langle - \rangle_{2d} : C^{\mathrm{ch}}(E_\tau, \mathcal{A}^{\beta\gamma - bc}) \to \mathcal{A}_{2d}((\hbar))$$

which intertwines the chiral differential d_{ch} *with* $\hbar\Delta$

$$\langle \mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n \rangle_{2d} := \fint_{E_\tau^n} \langle \mathcal{O}_1(z_1) \cdots \mathcal{O}_n(z_n) \rangle.$$

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- The BV trace map leads to Witten genus.

Elliptic chiral index (after Douglas-Dijkgraaf)

The partition function of a chiral deformation of conformal field theory by a chiral lagrangian $\mathcal{L}$ is given by

$$\left\langle e^{\frac{1}{\hbar}\int_\Sigma \mathcal{L}} \right\rangle_{2d}.$$

If we quantize the theory on the elliptic curve E_τ,

$$\boxed{\lim_{\bar\tau\to\infty} \left\langle e^{\frac{1}{\hbar}\int_{E_\tau} \mathcal{L}} \right\rangle_{2d} = \mathrm{Tr}_{\mathcal{H}}\, q^{L_0-\frac{c}{24}} e^{\frac{1}{\hbar}\oint dz\mathcal{L}}, \quad q = e^{2\pi i\tau}}$$

where the operation $\lim\limits_{\bar\tau\to\infty}$ sends

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This can be viewed as a chiral algebraic index. The regularized integral

- precisely explains $\bar\tau \to \infty$ [**L-Zhou** 2021]
- the holomorphic anomaly equation [**L-Zhou**, 2023]
- the contact equation [**Gui-L-Tang**, in preparation]

Algebraic Index vs Elliptic Chiral Index

1d TQM	2d Chiral CFT
Associative algebra	Vertex operator algebra
Hochschild homology	Chiral homology
QME: $(\hbar\Delta + b)\langle - \rangle_{1d} = 0$	QME: $(\hbar\Delta + d_{ch})\langle - \rangle_{2d} = 0$
$\langle \mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n \rangle_{1d} =$integrals on the compactified configuration spaces of S^1	$\langle \mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_n \rangle_{2d} =$regularized integrals of singular forms on Σ^n
Algebraic Index	Elliptic Chiral Algebraic Index

Joint work with **Zhengping Gui**. arXiv:2112.14572 [math.QA]

Application: Higher genus mirror symmetry

Quantum B-twisted topological string-field theory on general Calabi-Yau is formulated by **Costello-L** (2012, 2015, 2016) generalizing **Bershadsky-Cecotti-Ooguri-Vafa**'s Kodaira-Spencer gravity on Calabi-Yau 3-folds. We call it

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It is conjectured to be mirror to higher genus Gromov-Witten invariants of the mirror in the holomorphic limit.

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It is conjectured to be mirror to higher genus Gromov-Witten invariants of the mirror in the holomorphic limit.

It has an extension by coupling with holomorphic Chern-Simons theory in the large N limit, leading to open-closed BCOV theory and twisted supergravity [**Costello-L**, ATMP 2020].

Quantum BCOV theory on elliptic curves is completely solved (**L**, JDG 2023) by the chiral deformation of free chiral boson

$$S = \int \partial\phi \wedge \bar{\partial}\phi + \sum_{k\geq 0} \int \eta_k \frac{W^{(k+2)}(\partial_z \phi)}{k+2}$$

where

$$W^{(k)}(\partial_z \phi) = (\partial_z \phi)^k + O(\hbar)$$

are the bosonic realization of the $W_{1+\infty}$-algebra.

Higher genus mirror symmetry on elliptic curves

- The chiral index of quantum BCOV theory is

$$\mathrm{Ind}^{\mathrm{BCOV}}(E_\tau) = \mathrm{Tr}\, q^{L_0 - \frac{1}{24}} e^{\frac{1}{\hbar} \sum_{k \geq 0} \oint_A \eta_k \frac{W^{(k+2)}}{k+2}}$$

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- The chiral index coincides with the stationary Gromov-Witten invariants on the mirror elliptic curve computed by **Dijkgraaf** and **Okounkov-Pandharipande**.

$$\boxed{\mathrm{Ind}^{\mathrm{BCOV}}(E_\tau) = \langle \mathrm{Stationary}\rangle_E^{\mathrm{GW}}}$$

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In this case, we find [**L**, JDG 2023]

Quantum Mirror Symmetry=Boson-Fermion Correspondence.

Theorem: The following statements hold

(1) Happy 75th birthday, Yau!

(2) Happy 15th birthday, YMSC!

Thank you!