2021-9-27 Kähler geometry

$\boxed{\text{Thm}}$

$$R_{i\bar{j}k\bar{l}} = \frac{\partial^2 g_{k\bar{l}}}{\partial z^i \partial \bar{z}^j} - g^{p\bar{q}} \frac{\partial g_{k\bar{q}}}{\partial z^i} \frac{\partial g_{p\bar{l}}}{\partial \bar{z}^j}$$

$$R_{i\bar{j}k\bar{l}} = g\left(\frac{\partial}{\partial z^k}, \left(\nabla_{\frac{\partial}{\partial z^i}} \nabla_{\frac{\partial}{\partial \bar{z}^j}} - \nabla_{\frac{\partial}{\partial \bar{z}^j}} \underbrace{\nabla_{\frac{\partial}{\partial z^i}}\right) \frac{\partial}{\partial \bar{z}^l}}_{0} \right)$$

$$= g\left(\frac{\partial}{\partial z^k}, \nabla_{\frac{\partial}{\partial z^i}} \left(\Gamma^{\bar{q}}_{\bar{j}\bar{l}} \frac{\partial}{\partial \bar{z}^q}\right)\right) = \left(\frac{\partial}{\partial z^i} \Gamma^{\bar{q}}_{\bar{j}\bar{l}}\right) g_{k\bar{q}} \quad (1)$$

$$\Gamma^{\bar{q}}_{\bar{j}\bar{l}} = \overline{\Gamma^{q}_{jl}} = \overline{g^{q\bar{p}} \frac{\partial g_{l\bar{p}}}{\partial z^j}} = g^{p\bar{q}} \frac{\partial g_{p\bar{l}}}{\partial \bar{z}^j}$$

$$\therefore R_{i\bar{j}k\bar{l}} = g_{k\bar{q}} \frac{\partial}{\partial z^i} \left(g^{p\bar{q}} \frac{\partial g_{p\bar{l}}}{\partial \bar{z}^j} \right)$$

$$= \frac{\partial^2 g_{k\bar{l}}}{\partial z^i \partial \bar{z}^j} - g_{k\bar{q}} g^{p\bar{s}} g^{t\bar{q}} \frac{\partial g_{t\bar{s}}}{\partial z^i} \frac{\partial g_{p\bar{l}}}{\partial \bar{z}^j}$$

$$= \frac{\partial^2 g_{k\bar{l}}}{\partial z^i \partial \bar{z}^j} - g^{p\bar{s}} \frac{\partial g_{k\bar{s}}}{\partial z^i} \frac{\partial g_{p\bar{l}}}{\partial \bar{z}^j}$$

Majority of Kähler geometers use $R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{k\bar{l}}}{\partial z^i \partial \bar{z}^j} + \cdots$ — (2)

Def (M, g) Riemannian mfd.

$e_1, \dots, e_n$ orthonormal frame of TM.
local.

$$\mathrm{Ric}(X,Y) = \sum_{i=1}^{n} R(X, e_i, Y, e_i)$$

$$\mathrm{Ric}(Y, X) = \mathrm{Ric}(X, Y)$$

Ric is symmetric

$$\mathrm{Ric} = R_{ij}\, dx^i \otimes dx^j \qquad R_{ij} = R_{ji}$$

Ric is called the Ricci curvature or Ricci tensor.

$$R_{ij} = g^{kl} R_{ikjl}$$

$$= R_{ikj}{}^{k} = R_{i}{}^{k}{}_{jk}$$

(M, g) Kähler mfd.

$$\underbrace{R_{i\bar{j}}{}^{k}{}_{k}}_{} + \underbrace{R_{i}{}^{k}{}_{k\bar{j}}}_{= -R_{i}{}^{k}{}_{\bar{j}k}} + \overbrace{R_{ik\bar{j}}{}^{k}}^{=0} = 0$$

$$\frac{1}{\sqrt{2}}(e - iJe)$$

$\therefore R_{i\bar{j}}{}^{k}{}_{k} = R_{i}{}^{k}{}_{\bar{j}k} = R_{i\bar{j}}$

Ricci curvature for Kähler

Majority of Kähler geometers use with (2)

$R_{i\bar{j}} = R_{i\bar{j}k}{}^{k}$ the same as ours.

<u>Prop</u> $R_{i\bar{j}} = -\dfrac{\partial^2}{\partial z^i \partial \bar{z}^j} \log \det (g_{k\bar{l}})$

$\because$ In general for invertible matrix

$d(\det A) = \det A \cdot \operatorname{tr}(A^{-1} dA)$

Exercise

So $\Gamma_{ij}^{j} = g^{j\bar{k}} \dfrac{\partial g_{j\bar{k}}}{\partial z^i}$

$= \dfrac{\partial}{\partial z^i} \log \det (g_{k\bar{l}})$

(1)

$\therefore R_{i\bar{j}}{}^{k}{}_{k} = -g^{k\bar{l}} R_{i\bar{j}k\bar{l}} = -g^{k\bar{l}} \dfrac{\partial \Gamma_{\bar{j}\bar{l}}^{\bar{q}}}{\partial z^i} g_{k\bar{q}}$

$= -\dfrac{\partial}{\partial z^i} \Gamma_{\bar{j}\bar{l}}^{\bar{l}} = -\dfrac{\partial}{\partial z^i}\dfrac{\partial}{\partial \bar{z}^j} \log \det g$ ☺

$$\Theta = \underbrace{\bar\partial \theta}_{d\theta + \theta \wedge \theta} = \left(R^i{}_{jk\bar{l}} \, dz^k \wedge d\bar{z}^l \right)$$

curvature form (2-form)
matrix.

Def $\det\left(I + t \frac{\sqrt{-1}}{2\pi} \Theta \right)$ $m = \dim_{\mathbb{C}} M$.

$$= 1 + t c_1 + t^2 c_2 + \cdots + t^m c_m$$

Fact $c_p(g)$ is a real closed $2p$-form.

de Rham class $[c_p(g)]$ is indep of g.

We do it only for $p = 1$. (later)

Def $[c_p(g)] \overset{\text{def}}{=} c_p(M) \in H^{2p}_{DR}(M)$

is called the p-th Chern class.

Look at the case $p=1$.

$$c_1(g) = \frac{\sqrt{-1}}{2\pi}\,\mathrm{tr}(\Theta) = \frac{\sqrt{-1}}{2\pi}\,\underbrace{R^{i}{}_{i k \bar{l}}}_{R_{k\bar{l}}{}^{i}{}_{i} \,=\, R_{k\bar{l}}}\, dz^k \wedge d\bar{z}^l$$

$$= -\frac{\sqrt{-1}}{2\pi}\,\partial\bar{\partial}\log\det(g_{i\bar{j}})$$

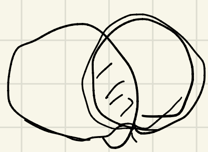
If g' is another Kähler form, then

$$-\frac{\sqrt{-1}}{2\pi}\,\partial\bar{\partial}\log\det(g_{i\bar{j}}) - \left(-\frac{\sqrt{-1}}{2\pi}\,\partial\bar{\partial}\log\det(g'_{i\bar{j}})\right)$$

$$= -\frac{\sqrt{-1}}{2\pi}\,\partial\bar{\partial}\log\frac{\det(g_{i\bar{j}})}{\det(g'_{i\bar{j}})}$$

<u>Lemma</u>

$\dfrac{\det(g_{i\bar{j}})}{\det(g'_{i\bar{j}})} \in C^\infty(M)$, i.e. indep of local coordinate.

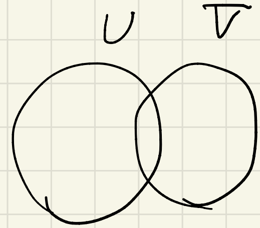


proof (6)

If $z^1, \dots, z^m$ is holo coord on U.

$w^1, \dots, w^m$ is holo coord on V

$U \cap V \neq \phi$



$$g\left(\frac{\partial}{\partial w^i}, \overline{\frac{\partial}{\partial w^j}}\right) = g\left(\frac{\partial z^k}{\partial w^i}\frac{\partial}{\partial z^k}, \overline{\frac{\partial z^l}{\partial w^j}\frac{\partial}{\partial z^l}}\right)$$

$$\therefore \det\left(g\left(\frac{\partial}{\partial w^i}, \overline{\frac{\partial}{\partial w^j}}\right)\right) = \cancel{\left|\det\left(\frac{\partial z^i}{\partial w^j}\right)\right|^2} \det\left(g\left(\frac{\partial}{\partial z^i}, \overline{\frac{\partial}{\partial z^j}}\right)\right)$$

In the same way

$$\det\left(g'\left(\frac{\partial}{\partial w^i}, \overline{\frac{\partial}{\partial w^j}}\right)\right) = \cancel{\left|\det\left(\frac{\partial z^i}{\partial w^j}\right)\right|^2} \det\left(g'\left(\frac{\partial}{\partial z^i}, \overline{\frac{\partial}{\partial z^j}}\right)\right)$$

$$\therefore \frac{\det\left(g\left(\frac{\partial}{\partial w^i}, \overline{\frac{\partial}{\partial w^j}}\right)\right)}{\det\left(g'\left(\frac{\partial}{\partial w^i}, \overline{\frac{\partial}{\partial w^j}}\right)\right)} = \frac{\det\left(g\left(\frac{\partial}{\partial z^i}, \overline{\frac{\partial}{\partial z^j}}\right)\right)}{\det\left(g'\left(\frac{\partial}{\partial z^i}, \overline{\frac{\partial}{\partial z^j}}\right)\right)}$$

Thus

$$c_1(g) - c_1(g') = \overset{\partial + \bar\partial}{\widehat{d}} \left(-\frac{\sqrt{-1}}{2\pi} \bar\partial \log \frac{\det g}{\det g'} \right)$$

$$\therefore \quad [c_1(g)] = [c_1(g')]$$

Lemma : If α is a real exact (p,p)-form then $\exists$ real $(p-1, p-1)$-form β s.t.

$$\alpha = i \partial \bar\partial \beta.$$

In particular for $p = 1$, i.e. for exact $(1,1)$ α, we have $(0,0)$-form i.e. a smooth function $\varphi \in C^\infty(M)$ such that $\alpha = i \partial \bar\partial \varphi$.

(This is called the $\partial\bar\partial$-lemma.)

This is an application of Hodge theory

See A. Futaki Lecture Notes springer.

S. Kobayashi : Transformation groups in differential geometry Springer, Appendix.