Combinatorics Lecture 12, 2022/7/5

§ Algebraic methods in Combinatorics

§1. Odd/even Town

Question: A town has n residents. They want to form some clubs satisfying the following:

(1) Each club has an odd number of members,

(2) Each two clubs must share an even number of members.

i.e. - Let $A_i = \{i\} \Rightarrow n$ clubs.

- n = even, $A_i = [n] \setminus \{i\} \Rightarrow n$ clubs.

Thm 1 (Odd/even Town) Let $\mathcal{F} \subseteq 2^{[n]}$ be a family satisfying the follows:

(1) $\forall A \in \mathcal{F}$, $|A|$ is odd,

(2) $\forall A \neq B \in \mathcal{F}_1$, $|A \cap B|$ is even.

Then $|\mathcal{F}_1| \leq n$.

Pf. For $A \in \mathcal{F}_1$, we define an indicator vector $\vec{1}_A \in \{0,1\}^n$ by letting

$$\vec{1}_A(i) = \begin{cases} 1, & i \in A \\ 0, & \text{otherwise.} \end{cases}$$

We aim to show that all vectors $\vec{1}_A$'s

(*) for $A \in \mathcal{F}_1$ are linearly independent over $\mathbb{F}_2 = \{0,1\}$.

Consider $\sum_{A \in \mathcal{F}_1} \alpha_A \vec{1}_A = \vec{0}$, $\alpha_A \in \{0,1\}$.

Note that

$$\Rightarrow \begin{cases} \vec{1}_A \cdot \vec{1}_A = |A| \equiv 1 \mod 2 \\ \vec{1}_A \cdot \vec{1}_B = |A \cap B| \equiv 0 \mod 2 \end{cases}$$

$\forall A \neq B \in \mathcal{F}_1$.

Then we have: $\forall A \in \mathcal{F}_1$

$$\vec{0} \cdot \vec{1}_A = \left(\sum_{B \in \mathcal{F}_1} \alpha_B \cdot \vec{1}_B\right) \vec{1}_A$$

$$\equiv \alpha_A \vec{1}_A \cdot \vec{1}_A \equiv \alpha_A \pmod 2$$

$\Rightarrow \alpha_A = 0$. This proves $(*)$.

So the number of the linearly independent vectors $\vec{1}_A$'s for $A \in \mathcal{F}$

is at most the dimension of $\{0,1\}^n$.

$\Rightarrow |\mathcal{F}| \leq n$. $\square$

Thm 2 (Even/odd town) Let $\mathcal{F} \subseteq 2^{[n]}$

be a family such that

(1) $|A|$ is even for $\forall A \in \mathcal{F}$,

(2) $|A \cap B|$ is odd for $\forall A \neq B \in \mathcal{F}$.

Then $|\mathcal{F}| \leq n$.

First we show a weak bound

Lemma 1. Such $\mathcal{F}$ has $|\mathcal{F}| \leq n+1$.

Pf: $\mathcal{F}^* = \{ A \cup \{n+1\} : \forall A \in \mathcal{F} \}$

$\Rightarrow \mathcal{F}^* \subseteq 2^{[n+1]}$ satisfies the odd/even town conditions. By Thm 1,

$|\mathcal{F}| = |\mathcal{F}^*| \leq n+1$. ∎

<u>pf of Thm 2</u>. By Lemma 1, it suffices to show that such $\mathcal{F}$ cannot have $|\mathcal{F}| = n+1$.

Suppose for a contradiction that $\boxed{\mathcal{F} = \{A_1, \ldots, A_{n+1}\}}$ satisfies <u>even/odd</u> conditions.

As before, we define $\vec{\mathbb{1}}_{A_i} \in \{0,1\}^n$.

So we have $n+1$ vectors in the n-dimensional space $\{0,1\}^n$. $\Rightarrow$ They must be linearly dependent

$\Rightarrow \exists\ \alpha_i \in \{0,1\}$ for $1 \leq i \leq n+1$

which are NOT all 0's with

$$\boxed{\sum_{i=1}^{n+1} \alpha_i \vec{\mathbb{1}}_{A_i} = \vec{0}}.$$

Note that $\vec{\mathbb{1}}_{A_i} \cdot \vec{\mathbb{1}}_{A_i} = |A_i| = 0 \mod 2$

$\Big\{ \vec{\mathbb{1}}_{A_i} \cdot \vec{\mathbb{1}}_{A_j} = |A_i \cap A_j| = 1 \mod 2.$

Then for any $1 \leq j \leq n+1$,

$$0 = \vec{0} \cdot \vec{\mathbb{1}}_{A_j} = \left(\sum_{i=1}^{n+1} \alpha_i \vec{\mathbb{1}}_{A_i} \right) \cdot \vec{\mathbb{1}}_{A_j}$$

$$= \left(\sum_{i=1}^{n+1} \alpha_i \right) - \alpha_j$$

$\Rightarrow \forall j, \quad \alpha_j = \sum_{i=1}^{n+1} \alpha_i$ is a constant.

$\Rightarrow \forall j, \quad \alpha_j = 1$ (as they cannot be all 0)

$\Rightarrow \quad 1 = n+1 \mod 2 \Rightarrow$ n is even.

Moreover, $\displaystyle\sum_{i=1}^{n+1} \vec{\mathbb{1}}_{A_i} = \vec{0}$. (A)

Consider $\mathcal{F}_1^c = \{ A^c : \forall A \in \mathcal{F}_1 \}$

Claim: $\mathcal{F}_1^c$ also satisfies the even/odd condition

- $|A^c| = n - |A|$ is even for $\forall A^c \in \mathcal{F}_1^c$.
- $|A^c \cap B^c| = \underbrace{n}_{\text{even}} - \underbrace{|A|}_{\text{even}} - \underbrace{|B|}_{\text{even}} + \underbrace{|A \cap B|}_{\text{odd}}$

  is odd for $\forall A^c \neq B^c \in \mathcal{F}_1$.

Repeating the previous proof for $\mathcal{F}_1^c$, we get

$$\sum_{i=1}^{n+1} \vec{1}_{A_i^c} = \vec{0} \qquad \text{(B)}$$

Now (A) + (B),

$$\vec{0} = \sum_{i=1}^{n+1} \left(\vec{1}_{A_i} + \vec{1}_{A_i^c} \right) = \sum_{i=1}^{n+1} \vec{1}$$

$$= (n+1)\,\vec{1} = \vec{1},$$

this is a contradiction!

so we proved Thm 2. ☒

<u>Exercise</u>: Construct $\mathcal{F}$ for Thm 2

with $|\mathcal{F}| = n$.

§2. Fisher's inequality

<u>Thm 3</u>. For a fixed k, let $\boxed{\mathcal{F} \subseteq 2^{[n]}}$ be a family such that $|A \cap B| = k$ $\forall A \neq B \in \mathcal{F}$,

Then $|\mathcal{F}| \leq n$.

Pf: Consider $\vec{I}_A$, for $A \in \mathcal{F}$, over $\mathbb{R}$

$\Rightarrow$ all vectors $\vec{I}_A$ are linearly independent over $\mathbb{R}$.

<u>Proof Idea</u>.. $\sum \alpha_A \vec{I}_A = \vec{0}$

compute $0 = \left(\sum_{A \in \mathcal{F}} \alpha_A \vec{I}_A\right) \cdot \left(\sum_{A \in \mathcal{F}} \alpha_A \vec{I}_A\right)$ ☒

Corollary (De Bruijn–Erdős Thm) Suppose P is a set of n points in $\mathbb{R}^2$. Then they are in a line, or they can define at least n lines.

Pf: Let L be the family of all lines in $\mathbb{R}^2$ defined by the set $P = \{x_1, \dots, x_n\}$.

For $\forall i \in [n]$, let

$$L_i = \{\text{all } \ell \in L : \ell \text{ passes } x_i\}.$$

Note that $|L_i \cap L_j| = 1$, $\forall i \neq j$
(as there is a unique line passing through x_i & x_j.)

We observe that

$(\exists\, i \neq j \text{ with } L_i = L_j)$ if and only if

all n points lie in a line. ✓

We may assume that this case doesn't occur, i.e. $L_i \neq L_j$ for $\forall i \neq j$.

Let $\mathcal{F} = \{ L_i : i \in [n] \} \subseteq 2^L$.

Then we see $\mathcal{F}$ satisfies the condition of fisher's inequality $\Longrightarrow n = |\mathcal{F}| \leq |L| \quad \boxed{\Xi}$.

<u>Corollary</u>: Let G be a graph whose vertices are triples in $\binom{[k]}{3}$ such that

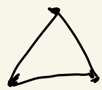
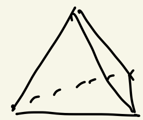
- $V(G) = \binom{[k]}{3}$,
- for $A, B \in \binom{[k]}{3}$, $AB \in E(G)$ if and only if $|A \cap B| = 1$.

Then G doesn't contain any clique or independent set of size $k+1$. $\left(\Longrightarrow R(k+1, k+1) > \binom{k}{3} \right)$

<u>Pf</u>. <u>Exercise</u> $\boxed{\Xi}$

§ 3. Distance problems

<u>Problem 1</u>. What is the maximum of points in $\mathbb{R}^n$ such that the distance between any two points is 1 ?

i.e. $n=2$  , $n=3$ 

<u>Thm 4</u>. There are at most $n+1$ such points.

(moreover, such $n+1$ points can be found)

Exercise

Pf. Omit.

<u>Problem 2</u>. We say a set is a <u>2-distance set</u> if the pairwise distance is either c or d.

What is the max size of a 2-distance set in $\mathbb{R}^d$?

<u>Thm 5</u> All 2-distance sets in $\mathbb{R}^n$ have at most $\frac{1}{2}(n+1)(n+4)$ points.

<u>Lemma</u>. For $i \in [n]$, Let $f_i : \Omega \to \mathbb{F}$ be a polynomial, where $\mathbb{F}$ is a field. If there exist $v_1, v_2, \dots, v_n \in \Omega$ such that

$$\begin{cases} f_i(v_i) \neq 0 & \forall i \in [n] \\ f_i(v_j) = 0 & \boxed{\text{for } \forall\ j < i,} \end{cases}$$

✓

then $f_1, f_2, \dots, f_n$ are linearly independent in the <u>linear space spanned by $f_1, f_2, \dots, f_n$.</u>

<u>pf of Thm 5</u>. Let $A = \{\vec{a}_1, \dots, \vec{a}_m\}$ be a 2-distance set in $\mathbb{R}^n$, with distances c & d.

For $\forall\ i \in [m]$, define $f_i : \mathbb{R}^n \to \mathbb{R}$

by
$$f_i(\vec{x}) = \left(\|\vec{x} - \vec{a}_i\|^2 - c^2\right) \cdot \left(\|\vec{x} - \vec{a}_i\|^2 - d^2\right).$$

So each f_i is a polynomial with n variable of degree 4.

$$\Rightarrow \begin{cases} f_i(\vec{a}_i) = c^2 d^2 > 0 \\ f_j(\vec{a}_i) = \left(\|\vec{a}_i - \vec{a}_j\|^2 - c^2\right) \cdot \left(\|\vec{a}_i - \vec{a}_j\|^2 - d^2\right) = 0 \\ \quad \text{for } \forall\, i \neq j . \end{cases}$$

By Lemma, we see that $f_1, f_2, \cdots, f_m$ are linearly independent in the linear space

$S \triangleq \text{span}(f_1, \cdots, f_m)$.

clearly, $\dim(S) = m$

Let $\vec{x} = (x_1, \cdots, x_n)$ & $\vec{a}_i = (a_{i1}, \cdots, a_{in})$.

Then
$$f_j(\vec{x}) = \left(\sum_{i=1}^{n} (x_i - a_{ji})^2 - c^2\right) \cdot \left(\sum_{i=1}^{n} (x_i - a_{ji})^2 - d^2\right)$$

$$= \left(\sum_i x_i^2 - 2 \sum_i x_i a_{ji} + \sum_i a_{ji}^2 - c^2 \right)$$

$$\cdot \left(\sum_i x_i^2 - 2 \sum_i x_i a_{ji} + \sum_i a_{ji}^2 - d^2 \right)$$

can be expressed as the linear combination of the following monomials:

$$B = \left\{ \left(\sum_i x_i^2\right)^2,\ x_j \left(\sum_i x_i^2\right),\ x_j x_\ell,\ x_j^2,\ x_j,\ 1 \right\} \text{ for all } j \neq \ell.$$

There are $1 + n + \binom{n}{2} + n + n + 1 = \frac{1}{2}(n+1)(n+4)$ monomials in B and $\forall f_i \in \operatorname{Span} B$.

$$\Rightarrow \quad S \subseteq \operatorname{Span} B$$

$$\Rightarrow \quad m = \dim(S) \leq \dim(\operatorname{Span}(B))$$

$$\leq |B| = \frac{1}{2}(n+1)(n+4)$$

∎

Rmk. Let a_n be the maximum size of a 2-distance set in $\mathbb{R}^n$.

We just proved: $a_n \le \frac{1}{2}(n+1)(n+4)$

The current best bounds for a_n are

$$\frac{1}{2}\binom{n+1}{2} \le a_n \le \frac{1}{2}\binom{n+2}{2}$$

Exercise: $a_2 = 5$.

§4 L-intersecting family

Def. Consider a set $L \subseteq \{0,1,2,\dots,n\}$. We say a family $\mathcal{F} \subseteq 2^{[n]}$ is L-intersecting if $|A \cap B| \in L$ for $\forall A \neq B \in \mathcal{F}$.

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i.e. $L = \{2,3\}$. $\forall A \neq B \in \mathcal{F}$
$|A \cap B| = 2$ or 3

A Lecture Note by Babai-Frankl:
Linear algebra methods in Combinatorics

<u>Thm 6</u> (Frankl-Wilson, 1981) If $\mathcal{F} \subseteq 2^{[n]}$ is an L-intersecting, then $|\mathcal{F}| \leq \sum_{k=0}^{|L|} \binom{n}{k}$.

<u>pf</u>. Let $\mathcal{F} = \{A_1, \dots, A_m\}$, where $|A_1| \leq |A_2| \leq \cdots \leq |A_m|$.

For $i \in [m]$, define $f_i(\vec{x}) : \mathbb{R}^n \to \mathbb{R}$ by

$$f_i(\vec{x}) = \prod_{\substack{\ell \in L \\ \ell < |A_i|}} (\vec{x} \cdot \vec{\mathbb{1}}_{A_i} - \ell)$$

has degree at most $|L|$

Consider the vectors $\vec{\mathbb{1}}_{A_1}, \dots, \vec{\mathbb{1}}_{A_m} \in \{0,1\}^n$.

Then we have

- $f_i(\vec{\mathbb{1}}_{A_i}) = \prod_{\ell \in L,\, \ell < |A_i|} (\vec{\mathbb{1}}_{A_i} \cdot \vec{\mathbb{1}}_{A_i} - \ell)$

$$= \prod_{\ell \in L, \ell < |A_i|} (|A_i| - \ell) \neq 0.$$

- $\forall j < i$,

$$f_i(\vec{1}_{A_j}) = \prod_{\ell \in L, \ell < |A_i|} (\vec{1}_{A_j} \cdot \vec{1}_{A_i} - \ell)$$

$$= \prod_{\ell \in L, \ell < |A_i|} (|A_i \cap A_j| - \ell)$$

$$= 0$$

as $|A_i \cap A_j| \in L$ and $|A_i \cap A_j| < |A_i|$

$\left(|A_i \cap A_j| \neq |A_i| \text{ because } j < i, \; |A_j| \leq |A_i| \right)$

By the Lemma, $f_1, f_2, \ldots, f_m$ are linearly independent.

Let $\tilde{f}_i(\vec{x})$ be a new polynomial obtained from $f_i(\vec{x})$ by replacing all term x_j^k (for $k \geq 1$)

with the term x_j.

$\Longrightarrow$ For $\forall \vec{y} \in \{0,1\}^n$,

$$f_i(\vec{y}) = \tilde{f}_i(\vec{y}).$$

$\Rightarrow$ The new polynomials $\tilde{f}_1, \tilde{f}_2, \cdots, \tilde{f}_m$ with degree at most $|L|$

are linear independent.

& Each $\tilde{f}_i$ is a linear combination

of the monomials $\prod_{i \in I} x_i$ for $I \subseteq [n]$, $|I| \leq |L|$,

the number of which is $\sum_{k=0}^{|L|} \binom{n}{k}$.

$$\Longrightarrow |\mathcal{F}| = m = \dim\left(\text{span}\{\tilde{f}_1, \cdots, \tilde{f}_m\}\right)$$

$$\leq \dim\left(\left\{\text{such monomials } \prod_{i \in I} x_i\right\}\right)$$

$$= \sum_{k=0}^{|L|} \binom{n}{k}.$$ ☒

<u>Thm 7</u>. Let p be a prime and $L \subseteq \mathbb{F}_p = \{0, 1, \ldots, p-1\}$. Let $\mathcal{F} \subseteq 2^{[n]}$ be a family satisfying that

(1) $|A| \notin L \pmod p$, $\forall A \in \mathcal{F}$,

(2) $|A \cap B| \in L \pmod p$, $\forall A \neq B \in \mathcal{F}$.

Then $|\mathcal{F}| \leq \sum_{k=0}^{|L|} \binom{n}{k}$.

Pf. $\mathcal{F} = \{A_1, \ldots, A_m\}$

$$f_i(\vec{x}) = \prod_{l \in L} (\vec{x} \cdot \vec{1}_{A_i} - l)$$

for $1 \leq i \leq m$. ☒

Rmk. Thm 6 & 7 are the generalizations of Thms 1–3.