Quintic 3-fold

Given a pair Δ, Δ° of polar dual reflexive polytopes we get a pair $X_\Delta, X_{\Delta^\circ}$ of toric varieties, which are Fano.

Anticanonical divisors $Y_\Delta, Y_{\Delta^\circ}$ given Calabi-Yaus, which are candidates for mirror symmetry.

In fact, however, they may be singular, so we need to resolve them: see [CK, §4.1]

Ex Previously, we found dual pair

$$\Delta = \text{span}\{2e_1 - e_2, 2e_2 - e_1, -e_1 - e_2\}$$

$$\Delta^\circ = \text{span}\{e_1, e_2, -e_1 - e_2\}$$

with $X_\Delta \cong \mathbb{P}^2$

Ex Generalizing, we have dual pair

$$\Delta = \mathrm{span}\ \{4e_i - \sum_{j\neq i} e_j\}_{1\leq i\leq 4} \cup \{-\sum_{1\leq i\leq 4} e_i\}$$

$$\Delta^\circ = \mathrm{span}\ \{e_i\}_{1\leq i\leq 4} \cup \{-\sum_{1\leq i\leq 4} e_i\}$$

We have that $\mathrm{Norm}(\Delta) = \mathrm{Cone}(\Delta^\circ)$ is a fan for $\mathbb{P}^4$, so $X_\Delta \cong \mathbb{P}^4$

Recall we constructed divisors in X_Δ from the polytope Δ via a space L of functions f on the torus $\mathbb{C}^{*d} \subset X_\Delta$. The result in this example is the quintics in $\mathbb{P}^4$, as follows. Note these are Calabi-Yau

Take a chart $\mathbb{C}^4 \cong \{x_0 \neq 0\}$ with coordinates $z_1 — z_4$.

The space L then consists of functions f on $\mathbb{C}^{*4}$ where $f(z_1 — z_4) = z_1^{-1} — z_4^{-1} q(z_1 — z_4)$ for q an arbitrary quintic. Then homogenizing we get $F(x_0 — x_4) = x_0 — x_4 q(x_1 x_0^{-1} — x_4 x_0^{-1})$ an arbitrary homogeneous quintic.

The description of the dual X_{Δ° and Y_{Δ° is more complicated. We find (see [CK, §4.2]) that

$$X_{\Delta^\circ} = \mathbb{P}^4 / G \quad \text{finite group}.$$

We have to take a resolution.

However, it can be proved (see work of Batyrev, [CK, Thm 4.1.5]) that the Y_Δ and Y_{Δ° thus obtained satisfy the relation on Hodge numbers predicted by MS.