Toric geometry

We study a class of algebraic varieties determined by combinatorial (and easily-visualized) data. For many problems in algebraic geometry, toric varieties give a class where problem is more easily solved, helping to understand the general case.

Ref Ranganathan, Toric geometry (notes) [R]
(First half.)

We work over $\mathbb{C}$.

Def A toric variety X is an irreducible variety, dim d, with dense open $U \subset X$ where $U \cong \mathbb{C}^{*d}$. Therefore have multiplication $U \times U \to U$. Require that it extends to $X \times U \to X$, group action.

Ex ① $\mathbb{C}^{*d}$ (here $\mathcal{U} = X$)

② $\mathbb{C}^d$ (take $\mathcal{U} = \mathbb{C}^{*d}$)

③ $\mathbb{P}^d$. For this, choose a standard chart $\mathbb{C}^d \subset \mathbb{P}^d$ and take $\mathcal{U} = \mathbb{C}^{*d} \subset \mathbb{C}^d$

④ $\mathbb{P}^d \times \mathbb{P}^e$ (exercise)

Combinatorial data

Def (lattice of one-parameter subgroups of $\mathbb{C}^{*d}$, cocharacter lattice) $N = \mathrm{Hom}(\mathbb{C}^*, \mathbb{C}^{*d}) \cong \mathbb{Z}^d$

Here $\underline{\lambda} \in \mathbb{Z}^d$ gives homomorphism $t \mapsto (t^{\lambda_1}, \_\_, t^{\lambda_d})$.

Def (character lattice) $M = \mathrm{Hom}(\mathbb{C}^{*d}, \mathbb{C}^*) \cong \mathbb{Z}^d$

Here $\underline{\mu} \in \mathbb{Z}^d$ gives $(t_1, \_\_, t_d) \mapsto t_1^{\mu_1} \_\_ t_d^{\mu_d}$

These lattice are dual under pairing $\langle \underline{\lambda}, \underline{\mu} \rangle = \sum \lambda_i \mu_i$

Rem The notation M and N is standard, so it's useful to memorize it.

Question For toric variety X, as we vary $\underline{\lambda} \in N$, what are possible $\lim_{t \to \infty} \underline{\lambda}(t)$ in X ?

one-parameter subgroup of $\mathbb{C}^{*d}$

Answer to this motivates a combinatorial structure called a fan in $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$

We gives examples first.

Ex $X = \mathbb{C}^2 \supset \mathbb{C}^{*2} = \{(t_1, t_2)\}$. $\underline{\lambda} = (\lambda_1, \lambda_2)$

Case 1 $\lambda_1, \lambda_2 > 0$. $t \mapsto (t^{\lambda_1}, t^{\lambda_2}) \to (0, 0)$

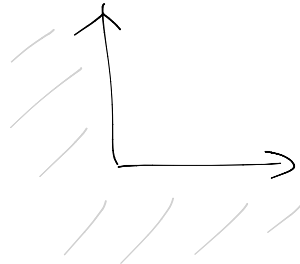
2 $\lambda_1 = 0, \lambda_2 > 0$ $\to (1, 0)$

③ $\lambda_1 > 0,\ \lambda_2 = 0 \qquad \longrightarrow (0, 1)$

④ $\lambda_1, \lambda_2 = 0 \qquad \longrightarrow (1, 1)$

Otherwise limit does not exist

We get picture in $N_{\mathbb{R}}$:



Ex $X = \mathbb{P}^2 \supset \mathbb{C}^2 \supset \mathbb{C}^{*2} = \{(1 : t_1 : t_2)\}$ $\underline{\lambda} = (\lambda_1, \lambda_2)$

Case ① $\lambda_1, \lambda_2 > 0$ $\quad t \longmapsto (1 : t^{\lambda_1} : t^{\lambda_2}) \longrightarrow (1 : 0 : 0)$

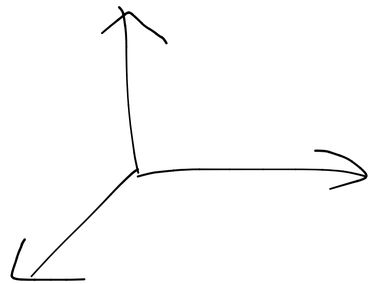
② $\lambda_1 = 0,\ \lambda_2 > 0 \qquad \longrightarrow (1 : 1 : 0)$

③ $\lambda_1 > \lambda_2,\ \lambda_2 < 0 \qquad (1 : t^{\lambda_1} : t^{\lambda_2}) =$

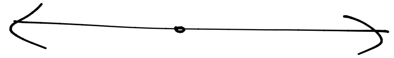
$$(t^{-\lambda_2} : t^{\lambda_1 - \lambda_2} : 1) \longrightarrow (0 : 0 : 1)$$

and so on.

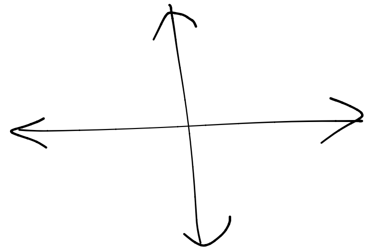
We get picture for $\mathbb{P}^2$:



Ex $\mathbb{P}^1 \supset \mathbb{C} \supset \mathbb{C}^*$ $\quad N_{\mathbb{R}} \cong \mathbb{R}$



Ex $\mathbb{P}^1 \times \mathbb{P}^1$



Later, we'll explain how to get a variety from such pictures.

__Interlude: birational geometry__

Consider algebraic varieties / $\mathbb{C}$.

$\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are __birational__. We have

__Def__ A __rational__ map

$$\begin{tikzcd} X \arrow[r, dashed, "f"] & Y \\ V \arrow[u, phantom, "\cup" rotate=90] \arrow[ur] & \end{tikzcd}$$

is given by a map with V non-empty open.

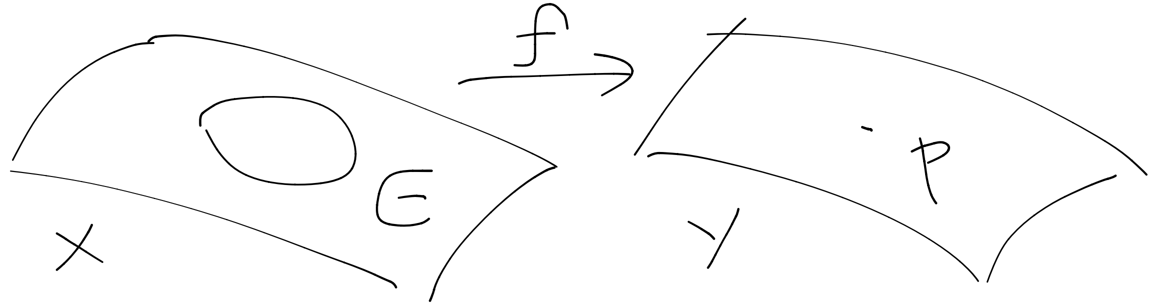
Rem Informally, may say f is a "partially defined" map.

Ex a morphism f gives a rational map ($V = X$)

Def Say f birational if there exists $Y \overset{g}{\dashrightarrow} X$ such that fg, gf are identity (on the open subsets where they can be defined).

Note In this case, write $X \dashleftarrow\dashrightarrow Y$.

Ex Blowup of X in smooth point p. Gives new variety $Y = Bl_p X$, where p is replaced by $E = \mathbb{P}(T_p X) \cong \mathbb{P}^{\dim X - 1}$, and morphism $f : Y \to X$



Now $f|_{X-E}$ is a isomorphism to $Y-p$

So f is a birational map

(we take g to be inverse of iso, defined on $Y-p$)

<u>Ex</u> $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$

We may take rational map

$((x:y),(z:w)) \mapsto (xz : xw : yz)$ Alternatively, if

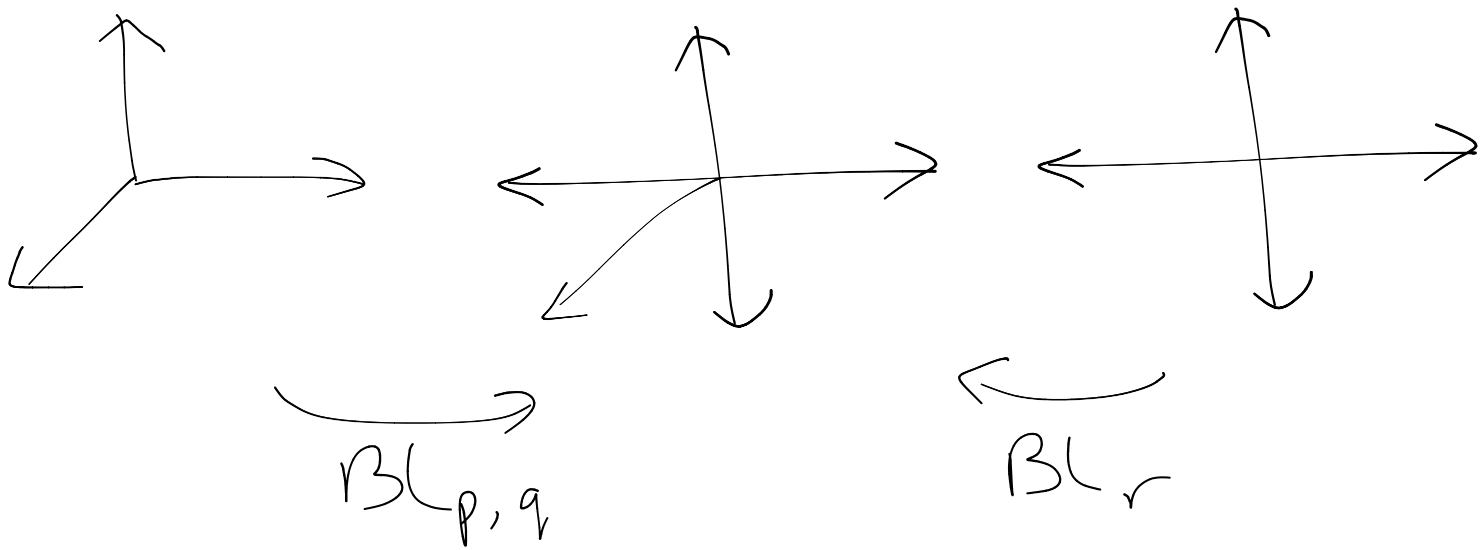
I take $p, q \in \mathbb{P}^2$ distinct, and $r \in \mathbb{P}^1 \times \mathbb{P}^1$ then

$$\mathbb{P}^2 \leftarrow \underbrace{Bl_{p,q}\mathbb{P}^2 \cong Bl_r \mathbb{P}^1 \times \mathbb{P}^1}_{\substack{\text{this in an example of} \\ \text{"del Pezzo surface" } dP_7}} \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$$

These rational maps give the birational map claimed

We don't prove $\cong$ above (it is a good exercise, or see [Vakil, Foundations of Alg Geom, Blowing up]

But we will see that it corresponds to fans as follows, that is, blowup corresponds to adding a new ray. This shows how useful toric diagrams can be!



Calabi-Yau hypersurfaces

For a subvariety $Y \subset X$ we define the normal bundle $N_{Y|X}$ of Y by the short exact sequence

$$0 \to T_Y \to T_X|_Y \to N_{Y|X} \to 0$$

when X and Y are smooth.

If Y is furthermore a divisor (codimension 1) and arises as the zeroes of some section s of a line bundle L on X then $N_{Y|X} \cong L|_Y$.

Ex A point $(a:b) \in \mathbb{P}^1$ is zero of section $bx - ay$ of line bundle $\mathcal{O}(1)$ on $\mathbb{P}^1$

Ex A line $(a:b:0) \in \mathbb{P}^2$ is zeroes of section z of line bundle $\mathcal{O}(1)$ on $\mathbb{P}^2$. Hence for this line $Y \cong \mathbb{P}^1$ have $N_{Y|\mathbb{P}^2} \cong \mathcal{O}_{\mathbb{P}^2}(1)|_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(1)$.

The short exact sequence above implies that

$$\det T_X|_Y \cong \det T_Y \otimes \det N_{Y|X}$$

$$\Longrightarrow \quad \omega_Y \cong \omega_X|_Y \otimes \det N_{Y|X}$$

"adjunction formula"

<u>Ex</u> For Y given as above with $L = \omega_X^\vee$, have

$$\omega_Y \cong \omega_X|_Y \otimes \omega_X^\vee|_Y \quad \text{trivial}$$

Hence for X compact Kähler (for instance, X projective variety) get that Y is Calabi-Yau.

<u>Ex</u> $X = \mathbb{P}^n$, $L = \omega_X^\vee = \mathcal{O}(n+1)$, $\dim Y = n-1$.

<u>Rem</u> Many further examples may be found with X toric.