Tsinghua University
(Pékin)

INTRODUCTION TO DELIGNE-LUSZTIG THEORY

Nov. - Dec. 2022

Group Theory

Let G be a group

- Normal subgroup = subgroup H of G such that $gHg^{-1} = H$, $\forall g \in G$
  $H \triangleleft G$
- Center of G: $Z(G) = \{ z \in G \mid \forall g \in G, zg = gz \}$
  $\triangle$
  G
- Commutator: $[g,h] = g h g^{-1} h^{-1}$
- Derived subgroup / commutator subgroup
  $$D(G) = \langle [g,h] , g,h \in G \rangle \triangleleft G$$

Fact. $G/D(G)$ is the largest abelian quotient of G.

- $D^0(G) = G$, $D^{i+1}(G) = D(D^i(G))$
- G is called perfect (resp. solvable) if $D(G) = G$ (resp. if $\exists n \geq 1$ s.t. $D^n(G) = 1$)

Facts
- A quotient of a perfect group is perfect
- A subgroup or a quotient of a solvable group is solvable.
- A perfect solvable group is trivial.
- A perfect solvable group is trivial.
- If $X \subset G$ and $g \in G$, we set
  $${}^gX = gXg^{-1}$$
- Centraliser: $C_G(X) = \{ g \in G \mid \forall x \in X, gx = xg \}$
- Normalizer:
  $N_G(X) = \{ g \in G \mid {}^gX = X \}$

$gxg^{-1} = x$

Facts. $C_G(X) \triangleleft N_G(X)$; $C_G(G) = Z(G)$

- If G acts on a set S and if $s \in S$, we set
  $\to$ $\mathrm{Stab}_G(s) = \{ g \in G \mid g \cdot s = s \} \subset G$
  (stabilizer)
  $\to$ $\mathrm{Orb}_G(s) = \{ g \cdot s \mid g \in G \} \subset S$

Facts. The map
$$G \longrightarrow \mathrm{Orb}_G(s)$$
$$g \longmapsto g \cdot s$$
induces a bijection $G/\mathrm{Stab}_G(s) \xrightarrow{\sim} \mathrm{Orb}_G(s)$

If G is finite then
$$|G| = |\mathrm{Stab}_G(s)| \cdot |\mathrm{Orb}_G(s)|$$

- $S_n = \{ \text{permutations of } \{1,2,\dots,n\} \} \triangleright A_n$

alternating

What is Deligne - Lusztig theory?

A theory initiated by Deligne and Lusztig (Annals of Math, 1976, ~60 pages) for studying representations of finite reductive groups (e.g. $GL_n(\mathbb{F}_q)$, $Sp_{2n}(\mathbb{F}_q)$, $SO_n(\mathbb{F}_q)$, ..., $E_8(\mathbb{F}_q)$) using geometric methods: Deligne and Lusztig construct varieties on which these groups act, and recover representations on the ℓ-adic cohomology.

Lusztig has obtained:

→ classification of irreducible representations, together with their dimension

→ algorithm for computing character values at semisimple elements

Why studying representations of finite groups?

One of the best way to understand them !

→ Burnside (1904): a group of order $p^a q^b$, with p, q primes, is solvable.

→ Feit - Thompson (1963): a group of odd order solvable

→ Janko (1965) wanted to show that there is no finite simple group G containing an involution g such that $C_G(g) \simeq \mathbb{Z}/2\mathbb{Z} \times A_5$.

He proved that:

- such a group have order 175 560 (he built its character table)
- such a group would have abelian Sylow 2-subgroup
- such a group would have an irreducible rep. of dim. 7 in char. 11.

↳ such a group exists (!!!) J_1

→ classification of finite simple groups (1983 ?, 2004, 2008, ...)

→ Many old unsolved problems (McKay / Alperin / Broué / Brauer ...)

Why studying finite reductive groups?

① Classification of finite simple groups:

- $\mathbb{Z}/p\mathbb{Z}$, p prime
- A_n, $n \geqslant 5$
- 26 sporadic groups
  $M_{11}, M_{12}, \ldots, J_1, \ldots, Co_1, \ldots, M$
  (monster) $\sim 8 \cdot 10^{53}$
- $D(G)/Z(G)$ where G is a finite reductive group
  $PSL_n(\mathbb{F}_q)$, $PSp_{2n}(\mathbb{F}_q), \ldots, E_8(\mathbb{F}_q)$

② Langlands program:

$$\begin{array}{ccc} GL_n(\mathbb{Z}_p) & \hookrightarrow & GL_n(\mathbb{Q}_p) \\ \downarrow & & \\ GL_n(\mathbb{F}_p) & & \end{array}$$

③ Finite reductive groups have an underlying geometric structure: Deligne-Lusztig theory, braid groups, Flag varieties, ...

Why studying D.-L. theory?

- If you are interested in finite reductive groups:
  → in 1976, only the character table of $SL_2(\mathbb{F}_q)$, $GL_n(\mathbb{F}_q)$, $Sp_4(\mathbb{F}_q)$ was known.
  (SL_2: Jordan, GL_n: Green, 1955, Sp_4: Srinivasan 1968)

→ Unsolved problems: cohomology of D.-L. varieties, Broué's conjecture, Ennola duality.

- If you are not:
  → Geometric representation theory
  → Indirect way to understand on ℓ-adic cohomology.
  → Related problems in combinatorics, topology, categorification...

PLAN. 0 Finite fields

1 $SL_2(\mathbb{F}_q)$:

→ Structure

→ character table

. Easy part: Harish-Chandra theory

. Cuspidal part. / Deligne-Lusztig theory for SL_2.

2 Other finite reductive groups

→ Structure

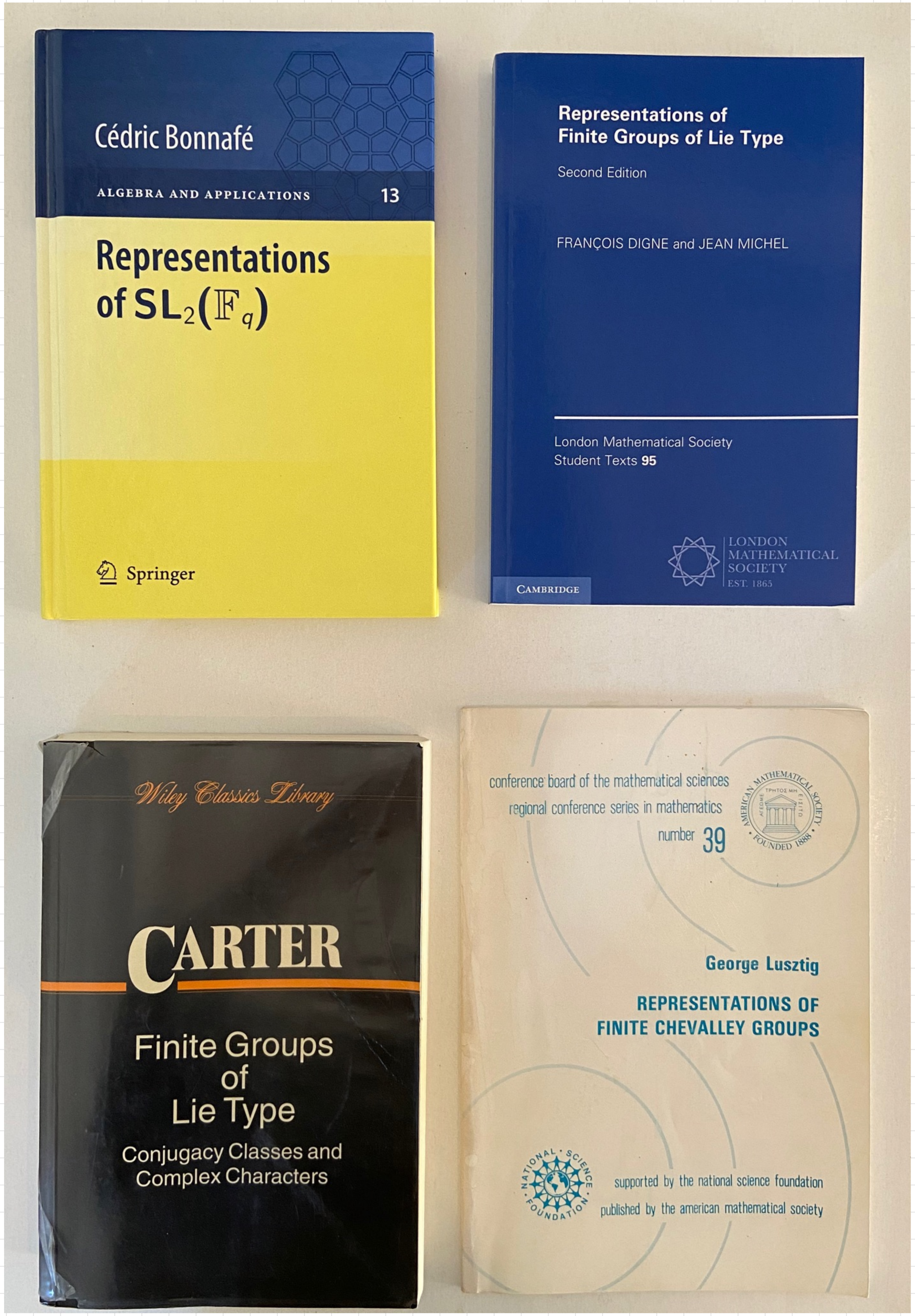
→ Harish-Chandra theory

→ Deligne-Lusztig varieties

Deligne-Lusztig induction

→ Example: $GL_n(\mathbb{F}_q)$

3 Modular representations. (?)



PART 0. Recollection on finite fields.

- p prime number
- $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ "the" finite field with p elements
- $\mathbb{F}$: "an" algebraic closure of $\mathbb{F}_p$.
- q = a power of p
- $\mathbb{F}_q = \{ x \in \mathbb{F} \mid x^q = x \}$

Theorem 0.0.
(a) $\mathbb{F}_q$ is a field with q elements
(b) $\mathbb{F}_q$ is the only subfield of $\mathbb{F}$ with q elements
(c) If K is a finite field with q elements, then $K \simeq \mathbb{F}_q$.

Proof. (a) $(x+y)^p = x^p + y^p$ (Fermat, 1640)
$\rightsquigarrow (x+y)^q = x^q + y^q$

So $\mathbb{F}_q$ is stable under $+, -, \cdot, \frac{1}{\cdot}$: this is a field.

The polynomial $X^q - X \in \mathbb{F}_p[X]$ has at most q roots. They are all simple: $(X^q - X)' = -1$ is prime to $X^q - X$
$\rightsquigarrow \boxed{|\mathbb{F}_q|} = q$

cardinality of a set $|S|$.

(b) Let K be a subfield of $\mathbb{F}$ with q elements, then $K^\times$ is of order $q-1$: so
$\forall x \in K^\times, \quad x^{q-1} = 1$
$\Rightarrow \forall x \in K \quad x^q = x$
$K \subset \mathbb{F}_q$. so $K = \mathbb{F}_q$.

(c) If K is a finite field $\supset \mathbb{F}_p$.
$K/\mathbb{F}_p$ is algebraic
$\rightsquigarrow K \hookrightarrow \mathbb{F}$.
The result follows from (b). ∎

Proposition 0.1.

(a) $X^q - X = \prod_{x \in \mathbb{F}_q} (X - x)$

(b) $\mathbb{F}_q^{+} \underset{gp}{\simeq} (\mathbb{Z}/_p\mathbb{Z})^n$ if $q = p^n$

underlying additive group

(c) $\mathbb{F}_q^{\times} \simeq \mathbb{Z}/(q-1)\mathbb{Z}$

Proof. (c) $\mathbb{F}_q^{\times}$ is an abelian group

$$\underset{gp}{\simeq} \mathbb{Z}/d_1\mathbb{Z} \times \mathbb{Z}/d_2\mathbb{Z} \times \cdots \times \mathbb{Z}/d_r\mathbb{Z}$$

with $d_1 \mid d_2,\ d_2 \mid d_3, \ldots, d_{r-1} \mid d_r$.

In particular, all elements have order d_r.

$$\forall x \in \mathbb{F}_q^{\times},\quad x^{d_r} = 1$$

But $X^{d_r} - 1$ has at most d_r roots.

$\Rightarrow d_r \geq q - 1$.

So $d_r = q - 1$ and $\mathbb{F}_q^{\times}$ is cyclic. ∎

Frobenius automorphism.

$$F : \begin{array}{ccc} \mathbb{F} & \longrightarrow & \mathbb{F} \\ x & \longmapsto & x^q \end{array}$$

automorphism of $\mathbb{F}$

Proposition 0.2. The restriction of F to $\mathbb{F}_{q^n}$ is an automorphism of order n:

$$\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q) = \langle F \rangle \simeq \mathbb{Z}/_n\mathbb{Z}$$

Proof. Exercise. ∎

$\mathbb{F}_{q^n}$ is $\mathbb{F}_q$-vector space of dimension n

(choosing a basis of $\mathbb{F}_{q^n}$ yields an isomorphism

$$\begin{tikzcd}
 & GL_{\mathbb{F}_q}(\mathbb{F}_{q^n}) \arrow[rr, "\sim"] & & GL_n(\mathbb{F}_q) \\
F \arrow[ur, phantom, "\in"] \arrow[urrr, bend left] & \mathbb{F}_{q^n}^{\times} \arrow[u, hook, "d_n"'] \arrow[rr, "\sim"] & & T_n \arrow[u, hook] \arrow[d, equal] \\
 & & & \mathbb{Z}/(q^n-1)\mathbb{Z}
\end{tikzcd}$$

Proposition 0.3. (a)

$$\mathrm{Tr}\, d_n(\xi) = \xi + \xi^q + \dots + \xi^{q^{n-1}}$$
$$\left(= \mathrm{Tr}_{\mathbb{F}_{q^n}/\mathbb{F}_q}(\xi)\right)$$
$$\det d_n(\xi) = \xi \cdot \xi^q \cdots \xi^{q^{n-1}}$$
$$= \xi^{1+q+\dots+q^{n-1}}$$
$$= N_{\mathbb{F}_{q^n}/\mathbb{F}_q}(\xi)$$

(b) $F\, d_n(\xi)\, F^{-1} = d_n(\xi^q) = d_n(\xi)^q$

$\det(F) = (-1)^{n-1}$.

Corollary 0.4. (a) $T_n \simeq \mathbb{Z}/(q^n-1)\mathbb{Z}$

(b) c normalizes T_n:

$$c\, t\, c^{-1} = t^q, \quad \forall t \in T_n.$$

$$\mathbb{Z}/n\mathbb{Z} \ltimes \mathbb{Z}/(q^n-1)\mathbb{Z} \hookrightarrow GL_n(\mathbb{F}_q)$$

Proof (of 0.3).

From Galois theory:

(1) The characteristic polynomial of $d_n(\xi)$ is equal $\prod_{i=0}^{n-1}(X - \xi^{q^i})$

(2) Normal basis theorem: $\exists\, \alpha \in \mathbb{F}_{q^n}^\times$ such that $\alpha, \alpha^q, \dots, \alpha^{q^{n-1}}$ is an $\mathbb{F}_q$-basis of $\mathbb{F}_{q^n}$.

In this basis

$$c = \begin{pmatrix} 0 & & & 0 & 1 \\ 1 & \ddots & & (0) & 0 \\ & \ddots & \ddots & & \\ & (0) & & \ddots & \\ & & & 1 & 0 \end{pmatrix}$$

$\leadsto \det(c) = (-1)^{n-1}$.

$$\boxed{\mu(m) = \begin{cases} 0 & \text{if } r^2 \mid m \text{ for some } r \geq 2 \\ (-1)^{n-1} & \text{if } m = p_1 \cdots p_n,\ p_i \neq p_j \end{cases}}$$

Proposition 0.5. The number of monic irreducible polynomials of degree n over $\mathbb{F}_q$ is equal to

$$\frac{1}{n} \sum_{d \mid n} \mu\left(\frac{n}{d}\right) q^d$$

where μ is the Möbius function