Mirror Symmetry from The Moduli Spaces of Calabi-Yau Manifolds

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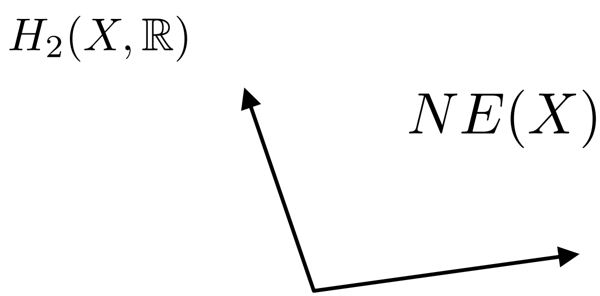
based on works with Hiromichi Takagi

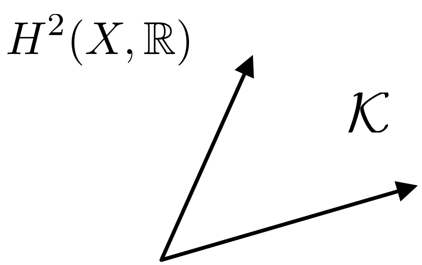
§1. Mirror symmetry of Calabi-Yau manifolds

Definition

X : Calabi-Yau 3-fold $\Leftrightarrow$
- $\cdot$ smooth projective
- $\cdot$ $K_X \simeq \mathcal{O}_X$
- $\cdot$ $H^i(X, \mathcal{O}_X) = 0 \ \ (0 < i < 3)$

A-structure





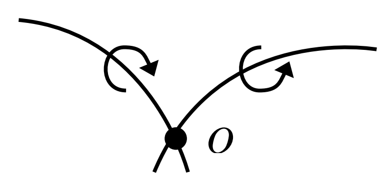
Kähler cone

B-structure

$$\begin{array}{c}\mathfrak{X}\\ \downarrow\\ \mathcal{M}\end{array}$$

Suppose a family of X, then we have a local system $R^3\pi_*\mathbb{C}_{\mathfrak{X}}$

Assume a special boundary point o

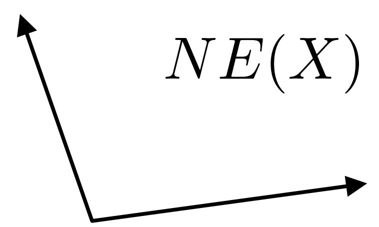


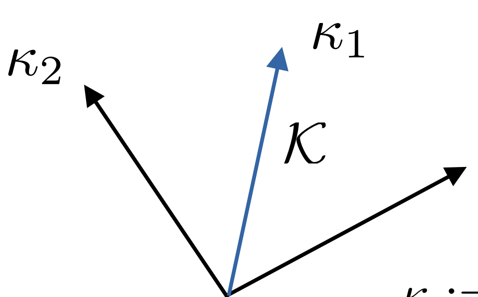
in $\overline{\mathcal{M}}$

Monodromy nilpotent cone $\mathcal{N}$

Mirror symmetry of Calabi-Yau manifolds

A-structure of X





$\kappa := \sum \alpha_i \kappa_i$

$L_\kappa := \kappa \wedge$

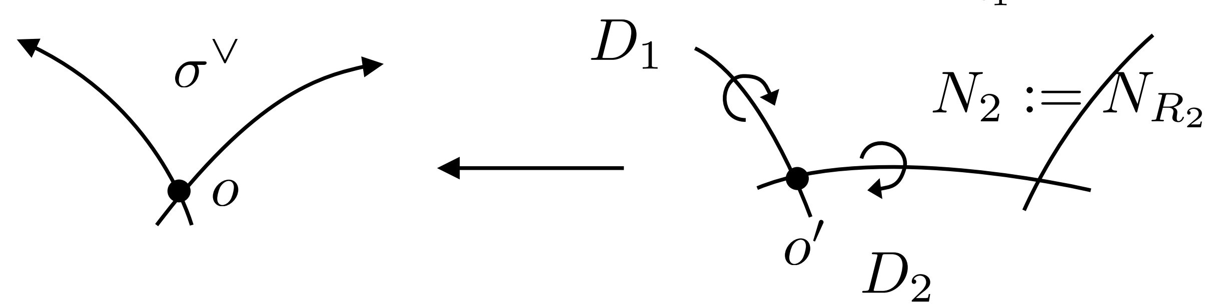
$$H^{even}(X) := H^{0,0} \oplus H^{1,1} \oplus H^{2,2} \oplus H^{3,3)}$$

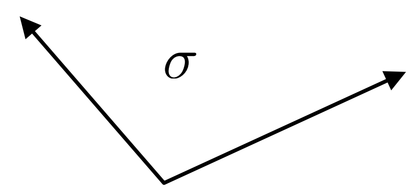
Lefschetz action

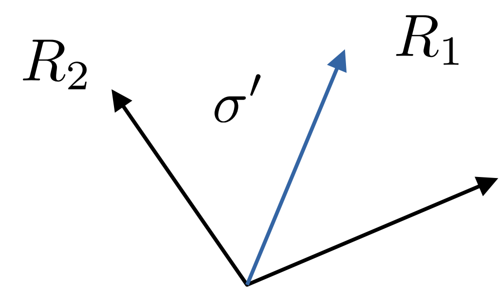
B-structure of $X^\vee$

$N_1 := N_{R_1}$

$N_2 := N_{R_2}$







If $N_1, \cdots, N_r$: nilpotent

$N_\lambda := \sum \lambda_i \log N_i$

$H^3(X^\vee, \mathbb{R})$

$$W^0 \subset W^2 \subset W^4 \subset W^6 = H^3{}_3$$

Definition

X is mirror of $X^\vee$ if

there exists a family $\mathfrak{X}^\vee \to \mathcal{M}$, and its compactification over $\overline{\mathcal{M}}$ and a (normal crossing) boundary point $o \in \overline{\mathcal{M}}$ where we find

$$\begin{array}{ccc} {}^{\exists}\varphi_{\mathbb{R}} : H^{even}(X,\mathbb{R}) & \overset{\sim}{\to} & H^3(X^\vee,\mathbb{R}) \\ \uparrow \text{ch(-)} & & \cup \\ \big(K(X), \chi(-,-)\big) & \overset{\varphi}{\simeq} & \big(H^3(X^\vee,\mathbb{Z}), (\ ,\ )\big) \end{array}$$

s.t.

$$\boxed{L_\kappa = \varphi_{\mathbb{R}}^{-1} \circ N_\lambda \circ \varphi_{\mathbb{R}}}$$

with identifying parameters α_i with λ_i.

'90 (Candelas et al): $\kappa_i * \kappa_j * \kappa_k =$ (Griffiths-Yukawa coupling) + mirror map

'18 (with Takagi) : Movable cone v.s. monodromy nilpotent cone

§2. An interesting example of mirror symmetry

$(A, \mathcal{L})$: abelian surface with $(1,8)$ polarization

$\Phi_{|\mathcal{L}|} : A \hookrightarrow \mathbb{P}^7$ (by theta functions)

Gross-Popescu ($\sim$'00)

$$\mathbb{I}(\mathrm{Im}\Phi_{|\mathcal{L}|}) = I_{w(A)} + \langle 3 \times 3 \text{ minors of Moore matrix } \rangle$$

$$I_w = \left\langle \begin{array}{ll} f_1 = \frac{w_0}{2}(x_0^2 + x_4^2) + w_1(x_1x_7 + x_3x_5) + w_2x_2x_6 & \\ f_2 = \sigma f_1 & \\ f_3 = \sigma^2 f_1 & \\ f_4 = \sigma^3 f_1 & (\sigma : x_i \mapsto x_{i+1}) \end{array} \right\rangle$$

I_w $(w \in \mathbb{P}^2_w)$ defines a special form of $(2,2,2,2)$ complete intersections in $\mathbb{P}^7$.

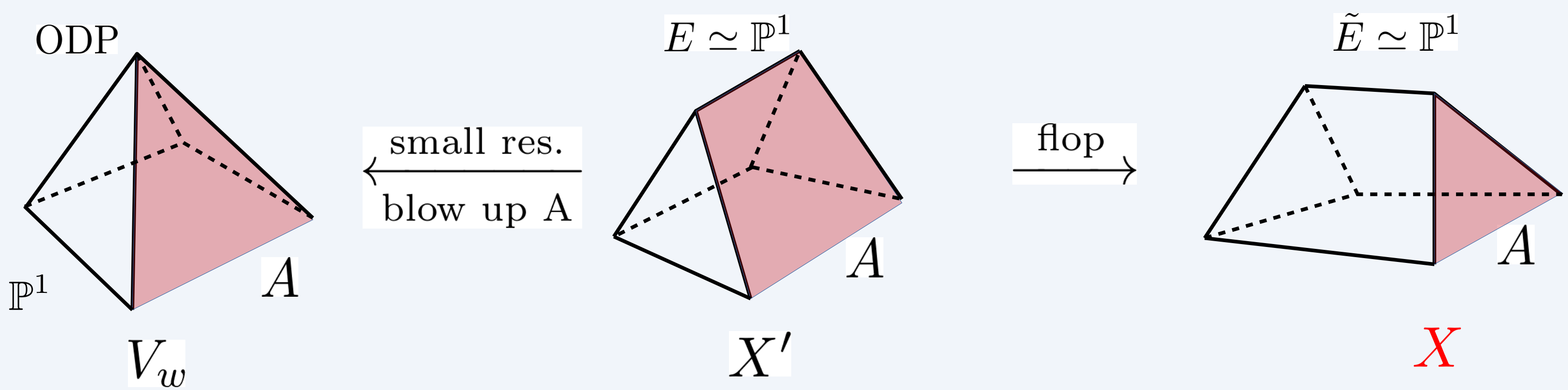
Proposition (Gross-Popescu '01)

$V_w := V(I_w)$ has the following properties:

(1) $\mathcal{H}_8$(Heisenberg group) acts freely on V_w.

(2) V_w is a pencil of $(1,8)$-polarized abelian surfaces with 64 base points.

(3) There are small resolutions as follows: (where we have ODPs)



abelian fibration over $\mathbb{P}^1$

(4) X and X' admit free $\mathbb{Z}_8 \times \mathbb{Z}_8 (= \mathcal{H}_8)$ actions.

- Hodge numbers of X:

$$\begin{matrix} & & & h^{0,0} & & & \\ & & & h^{1,1} & & & \\ h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\ & & & h^{2,2} & & & \\ & & & h^{3,3} & & & \end{matrix} = \begin{matrix} & & & 1 & & & \\ & & & 2 & & & \\ 1 & & 2 & & 2 & & 1 \\ & & & 2 & & & \\ & & & 1 & & & \end{matrix}$$

$\rho = 2$ **and** $\chi(X) = 0$

self-mirror?

Conjecture (Gross,Pavanelli '08)

$$X \xrightarrow{\text{mirror dual}} X/\mathbb{Z}_8$$

$$X/\mathbb{Z}_8 \times \mathbb{Z}_8 \xleftarrow[\text{mirror dual}]{} X/\mathbb{Z}_8$$

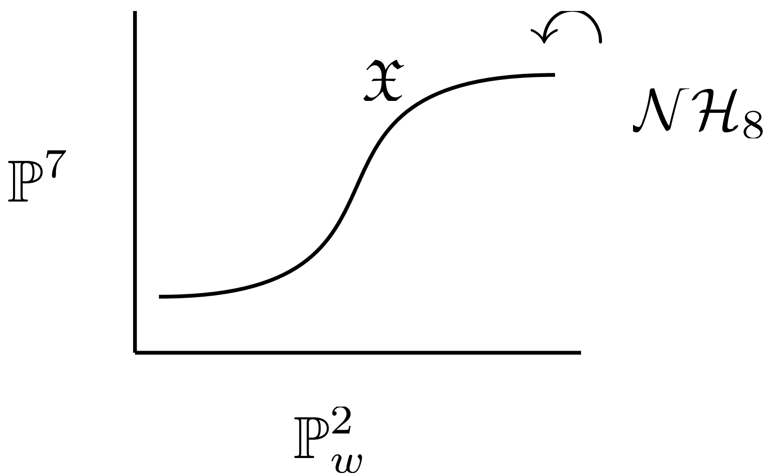
Theorem (Bak '09, Schnell '12)

X and $Y := X/\mathbb{Z}_8 \times \mathbb{Z}_8$ are derived equivalent by fiberwise FM transform

§3. Family of X, $X/\mathbb{Z}_8$, and $X/\mathbb{Z}_8 \times \mathbb{Z}_8$

The ideal $I_w(x) = \langle f_1(w,x), ..., f_4(w,x) \rangle$ defines a family

$$\begin{array}{c} \mathfrak{X} \\ \downarrow \\ \mathbb{P}^2_w \end{array}$$



There is a larger symmetry of I_w

$$I_w(g.x) = I_{g.w}(x) \quad (g \in \mathcal{NH}_8)$$

for $\mathcal{H}_8 \subset \mathcal{NH}_8 \subset GL(8, \mathbb{C})$.

and similarly for $\mathfrak{X}/\mathbb{Z}_8$ and $\mathfrak{X}/\mathbb{Z}_8 \times \mathbb{Z}_8$.

$$\begin{array}{ccc} \mathfrak{X}/\mathbb{Z}_8 & & \mathfrak{X}/\mathbb{Z}_8 \times \mathbb{Z}_8 \\ \downarrow & & \downarrow \\ \mathbb{P}^2_w & & \mathbb{P}^2_w \end{array}$$

$$\left(\begin{array}{c} \text{cf. the mirror quintic} \\ \begin{array}{c} \mathfrak{X}/(\mathbb{Z}_5)^3 \circlearrowleft \mathbb{Z}_5 \\ \downarrow \\ \mathbb{P}^1_\psi \end{array} \end{array} \right)$$

Results (H+Takagi, ’21)

(1) Families

$$\begin{array}{ccc} \mathfrak{X}_{\mathbb{Z}_8} := \mathfrak{X}/\langle\tau\rangle & \text{and} & \mathfrak{X}_{\mathbb{Z}_8\times\mathbb{Z}_8} := \mathfrak{X}/\langle\tau,\sigma\rangle \\ \downarrow & & \downarrow \\ \mathbb{P}^2_w & & \mathbb{P}^2_w \end{array}$$

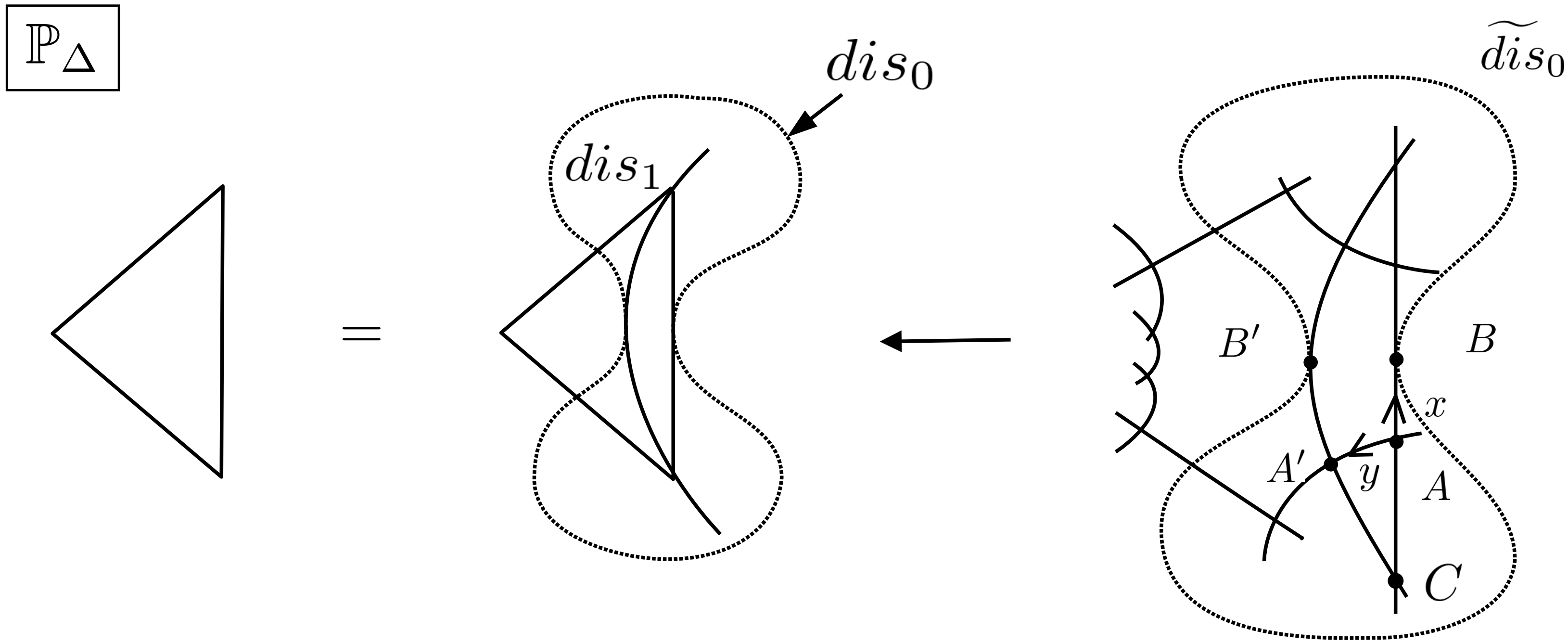
admit ”good” quotients by $\widetilde{G}_0 \subset \mathcal{NH}_8$, and give rise to families

$$\begin{array}{ccc} X/\mathbb{Z}_8 \to \mathfrak{X}_{\mathbb{Z}_8}/\widetilde{G}_0 & & X/\mathbb{Z}_8\times\mathbb{Z}_8 \to \mathfrak{X}_{\mathbb{Z}_8\times\mathbb{Z}_8}/\widetilde{G}_0 \\ \downarrow & & \downarrow \\ \mathbb{P}_\Delta & & \mathbb{P}_\Delta \end{array}$$

over the same $\mathbb{P}_\Delta := \mathbb{P}^2_w/\mathbb{Z}_8$.

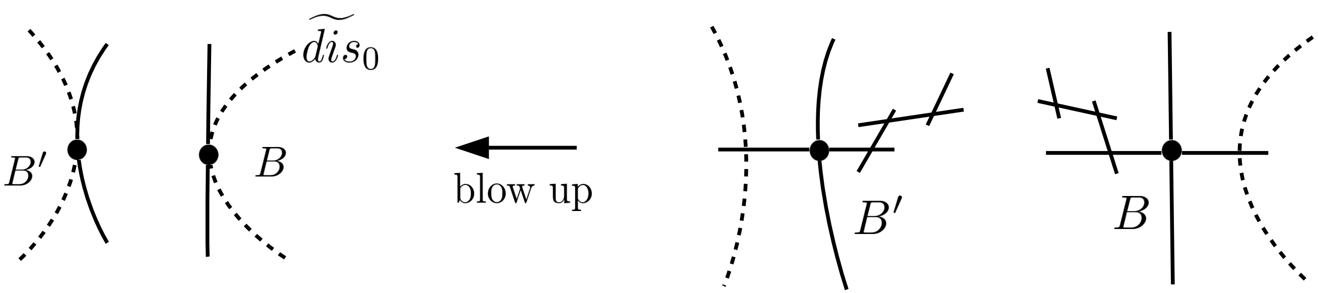
(2) **These two describe the mirror symmetry nicely.**

Global description of the family $\mathfrak{X}_{\mathbb{Z}_8}$ and $\mathfrak{X}_{\mathbb{Z}_8 \times \mathbb{Z}_8}$ over $\mathbb{P}_\Delta$:

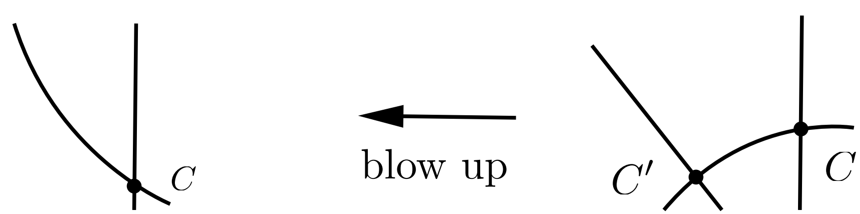


Further blowing-ups:

(1)



(2)



The entire picture:

we can group the boundary points according to integral structures:

One is

$$A \quad A'\,; \qquad B \quad B'$$

$$(X \quad X'; \quad X/\mathbb{Z}_8 \times \mathbb{Z}_8,\; X'/\mathbb{Z}_8 \times \mathbb{Z}_8)$$

$$X/\mathbb{Z}_8 \longrightarrow \mathfrak{X}_{\mathbb{Z}_8}/\widetilde{G}_0 \longrightarrow \mathbb{P}_\Delta$$

The other is

$$C \qquad C'$$

$$(X/\mathbb{Z}_8 \quad X'/\mathbb{Z}_8)$$

$$X/\mathbb{Z}_8 \times \mathbb{Z}_8 \longrightarrow \mathfrak{X}_{\mathbb{Z}_8 \times \mathbb{Z}_8}/\widetilde{G}_0 \longrightarrow \mathbb{P}_\Delta$$

The PF equations (local systems) are the same, but integral structures distinguish the two!

These two families describe the mirror symmetry of

$$X \xrightarrow{\text{mirror dual}} X/\mathbb{Z}_8 \xrightarrow{\text{mirror dual}} X/\mathbb{Z}_8 \times \mathbb{Z}_8$$

§4. Gromow-Witten invariants from the boundary point A

- $\mathrm{Pic}(X)_{/tor} = \mathbb{Z}H_X \oplus \mathbb{Z}A_X$, $H_2(X,\mathbb{Z})_{/tor} = \mathbb{Z}\sigma \oplus \mathbb{Z}\ell$

- $g=0$ BPS numbers from the LCSL A $\rightarrow$ we write $\beta = n\sigma + m\ell$

n\m	0	1	2	3	4	5	6	7	8	
0	0	0	0	0	0	0	0	0	0	
1	64	512	2816	12288	46464	157696	493056	1441792	3989568	$\leftarrow n=1$
2	0	0	4096	98304	1220608	10813440	76775424	464322560	2480783360	$\leftarrow n=2$
3	0	0	2816	195072	6301056	124829696	1772620032	19764707328	183168532288	$\leftarrow n=3$
4	0	0	0	98304	10567680	478740480	13238665216	261369036800	4018366742528	$\leftarrow n=4$
5	0	0	0	12288	6301056	728901120	40797528064	1437499588608	36413468765248	
6	0	0	0	0	1220608	478740480	58763759616	3812602150912	160955539341312	
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

For $n=1$, we have $Z_{0,1}(q) = \dfrac{64}{\bar\eta(q)^8}$

$$Z_{0,n}(q) = P_{0,n}(E_2,E_4,E_6)\left(\frac{64}{\bar\eta(q)^8}\right)^n \qquad \left(\bar\eta(q) := \prod_{n\ge 1}(1-q^n)\ \right)$$

$$P_{0,1}=1,\quad P_{0,2} = \frac{1}{4608}(8\,E_2^2+E_4),$$

$$P_{0,3} = \frac{1}{2654208}(14E_2^4+7E_2^2E_4+E_4^2+2E_2E_6),$$

$$P_{0,4} = \frac{1}{2^{26}3^7}(3008E_2^6+2808E_2^4E_4+1128E_2^2E_4^2+125E_4^3$$
$$+1120E_2^3E_6+528E_2E_4E_6+31E_6^2),$$

$\cdots$.

Conclusion :

I have shown an interesting example which motivates us (or only me?) a global study of the moduli spaces of Calabi-Yau manifolds.

References:

[1] S.H. and H.Takagi "*Mirror symmetry of Calabi-Yau manifolds fibered by (1,8)-polarized abelian surfaces*", Commun. Number Theory Phys. 16 (2022), no. 2, 215–298.

[2] S.H. and H.Takagi, "*Movable vs monodromy nilpotent cones of Calabi-Yau manifolds*", SIGMA Symmetry Integrability Geom. Methods Appl. 14 (2018), Paper No. 039, 37 pp.