2020 - 9 - 27 Kähler geometry

Recall

$c_p(g)$ is called the p-th Chern form

$c_1(g)$ the <u>first chern form</u>

the <u>Ricci form</u>

<u>Def</u> (M, g) Riemannian mfd.

(1) $S(g) (= \mathrm{Scal}(g)) = g^{ij} R_{ij} = R^{j}{}_{j}$ $\quad R_i{}^i$

Scalar curvature.

(2) g is called an Einstein metric

$\underset{\text{def}}{\iff}$ $\exists k \in \mathbb{R}$ s.t. $R_{ij} = k\, g_{ij}$

<u>Lemma</u> If g is an Einstein metric

then $S(g)$ is constant.

(:) $S(g) = g^{ij} R_{ij} = g^{ij}(k\, g_{ij}) = kn$

$n = \dim M$. (:)

Consider the case of compact Kähler mfd.

If g is a Kähler-Einstein metric

then $R_{i\bar{j}} = k\, g_{i\bar{j}}$.

If $k > 0 \Rightarrow c_1(M) > 0$ i.e.

$c_1(M)$ is represented by a positive $(1,1)$-form.

$\sqrt{-1}\, \alpha_{i\bar{j}}\, dz^i \wedge d\bar{z}^j$

$(\alpha_{i\bar{j}})$ positive def Hermitian

(☺ $R_{i\bar{j}} = k\, g_{i\bar{j}}$ $k > 0$
$(g_{i\bar{j}})$ positive def Hermitian)

$K_M = \Lambda^m T'M^*$ canonical line bdl

$K_M^{-1} = \Lambda^m T'M$

$c_1(K_M^{-1}) = c_1(M) > 0$

K_M^{-1} ample. $\Longleftrightarrow$ M : Fano mfd.

If $k = 0 \Longrightarrow c_1(M) = 0$ in $H^2_{DR}(M)$

If $k < 0 \Longrightarrow c_1(M) < 0 \Longleftrightarrow K_M$ is ample.

[Digression Kodaira dimension.]

③

Is converse True ?

Yes for $k<0$. Yau, Aubin 1976

$\boxed{\text{Yes for } k=0 \qquad \text{Yau}}$

No in general for $k>0$. Matsushima

Calabi-Yau mfd
the metric is called the Calabi-Yau metric

K-stability is necessary and sufficient.

We digressed to the history of
(脱線)

K-E problem as follows.

- Matsushima 1956
  reductiveness for KE metrics
- Lichnerowicz 1956
  reductiveness for Kähler metrics
  of constant scalar curvature.

- Futaki 1983 Another obstruction.

- Donaldson - Uhlenbeck - Yau for vector bdles and stability condition. (1980's)

  Yau conjecture for Fano case.

- Tian 1997.
  - K-stability for special degenerations with normal central fiber
  - Ding - Tian's generalized Futaki invariant
  - Mukai - Umemura mfds.

- Donaldson 2002 $L \to M$ ample line bdle
  - Test configuration with possibly non-normal central fiber,
  - Donaldson - Futaki invariant
  - K-stability (polystability) and $\exists$ constant scalar curvature Kähler metric

⑤

- Li-Xu, Odaka, roles of MMP
  Normal central fiber

- Chen-Donaldson-Sun, Tian 2015
  For Fano, $\exists$ KE $\Longleftrightarrow$

- Fujita, Li
  valuative criterion

- Fujita-Odaka
  δ-invariant

- Kewei Zhang
  δ^A invariant

- Siu, Nadel, Tian's work
  on the existence in 1980's
  Tian's α-invariant.