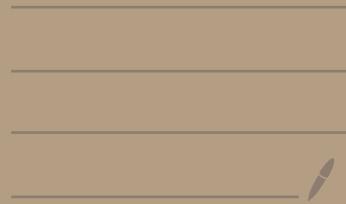


2020 - 9-27

Kähler geometry



①

Recall

$c_p(g)$ is called the p -th Chern form
 $c_1(g)$ the first Chern form.
the Ricci form

Def (M, g) Riemannian mfd.

$$(1) \quad S(g) (= \text{Scal}(g)) = g^{ij} R_{ij} = R^j_j;$$

Scalar curvature.

(2) g is called an Einstein metric

$$\stackrel{\text{def}}{\iff} \exists k \in \mathbb{R} \text{ s.t. } R_{ij} = k g_{ij}$$

Lemma If g is an Einstein metric
 then $S(g)$ is constant.

$$\therefore S(g) = g^{ij} R_{ij} = g^{ij} (k g_{ij}) = k n$$

$n = \dim M.$

∴

Consider the case of compact Kähler mfd.

If g is a Kähler-Einstein metric

$$\text{then } R_{ij} = k g_{ij}.$$

If $k > 0 \Rightarrow c_1(\mu) > 0$ i.e.

(2)

$c_1(\mu)$ is represented by a positive
 $(1,1)$ -form.

$$\sqrt{-1} dz_j \wedge d\bar{z}_i - d\bar{z}_j$$

$(d\bar{z}_j)$ positive def

Hermitian

$$\left(\begin{array}{l} R_{ij} = k g_{ij} \\ (g_{ij}) \text{ positive def Hermitian} \end{array} \right)$$

$\xi_M = \Lambda^m T'M^*$ canonical line bundle

$$\xi_M^{-1} = \Lambda T'M$$

$$c_1(\xi_M^{-1}) = c_1(\mu) > 0$$

ξ_M^{-1} ample. $\Leftrightarrow M$: Fano mfd.

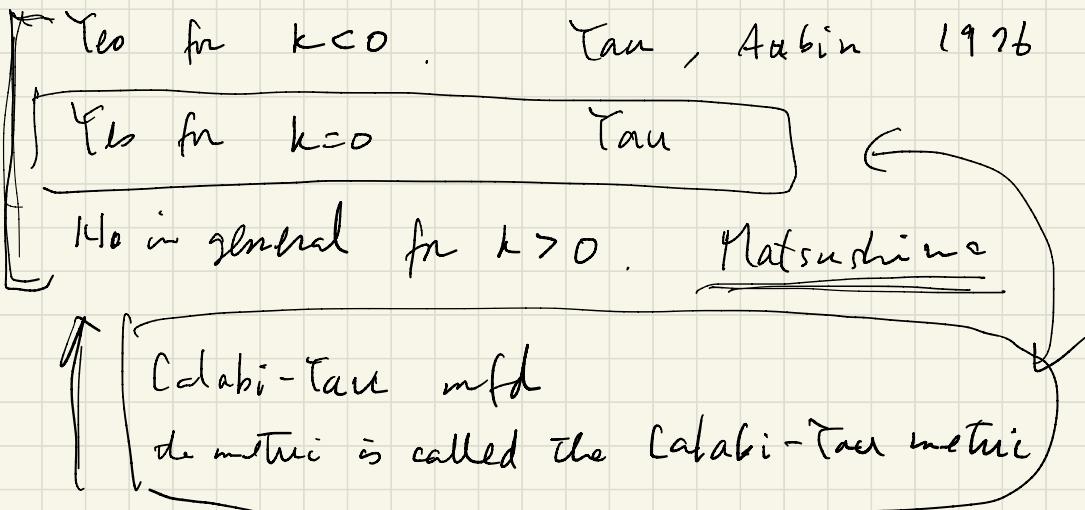
If $k=0 \Rightarrow c_1(\mu)=0$ in $H_{DR}^2(M)$

If $k < 0 \Rightarrow c_1(\mu) < 0 \Leftrightarrow \xi_M$ is ample.

$\left[\text{Dimension Kodaira dimension.} \right]$

(3)

Is converse True ?



K -stability is necessary and sufficient.

We digressed to the history of
(既往史)

$K-E$ problem as follows.

— Matsushima 1956

reductiveness for $K-E$ metrics

— Lichnerowicz 1956
reductiveness for Kähler metrics
of constant scalar curvature.

- Futaki 1983 Another obstruction. (4)
- Donaldson - Uhlenbeck - Taubes for vector
bundles and stability condition.
(1980's)

Tian conjecture for Fano case.

- Tian 1997
 - K-stability for special degenerations with normal central fiber
 - Ding - Tian's generalized Futaki invariant
 - Mukaï - Umemura moduli.
- Donaldson 2002 $L \rightarrow M$ ample line bundle
 - Test configuration with possibly non-normal central fiber,
 - Donaldson - Futaki invariant
 - K-stability (polystability) and constant scalar curvature Kähler metric

(5)

- Li-Xu, Odaka, roles of MNP
Normal central fiber

- Chen-Donaldson-Sun, Tian 2015

F_n Fano, $\exists \text{KE} \iff$

- Fujita, Li
valuative criterion

- Fujita-Odaka
 δ -invariant //
- Kewei Zhang
 δ^A invariant

- Sun, Hadel, Tian's work
on the existence in 1980's
Tian's λ -invariant.