Panorama of Dynamics and Geometry of Moduli Spaces and Applications

Lecture 6. Equidistribution of square-tiled surfaces. Non-correlation of vertical and horizontal foliations

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Solutions of homework assignment problems

$\hat{\text{H}}$omologous saddle connections

Recall that a quadratic differential q on a Riemann surface S defines a canonical (ramified) double cover $p : \hat{S} \to S$ such that $p^*q = \omega^2$ is a square of a holomorphic 1-form ω on $\hat{S}$.

Given an oriented saddle connection γ on S let γ', γ'' be its lifts to the double cover. If $[\gamma'] = -[\gamma'']$ as cycles in

$$H_1(\hat{S}, \{\text{preimages of singularities}\}; \mathbb{Z})$$

we let $[\hat{\gamma}] := [\gamma']$, otherwise we define $[\hat{\gamma}]$ as $[\hat{\gamma}] := [\gamma'] - [\gamma'']$.

Definition The saddle connections γ_1, γ_2 on a flat surface S defined by a quadratic differential q are **$\hat{\text{h}}$omologous** if $[\hat{\gamma}_1] = [\hat{\gamma}_2]$ in

$$H_1(\hat{S}, \{\text{preimages of singularities}\}; \mathbb{Z})$$

under an appropriate choice of orientations of γ_1, γ_2.

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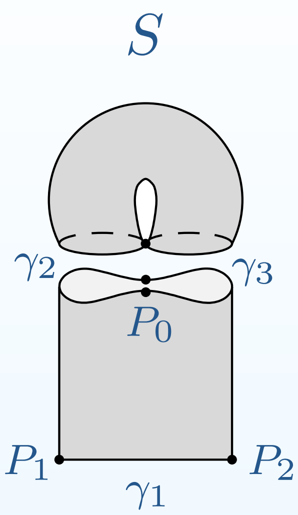
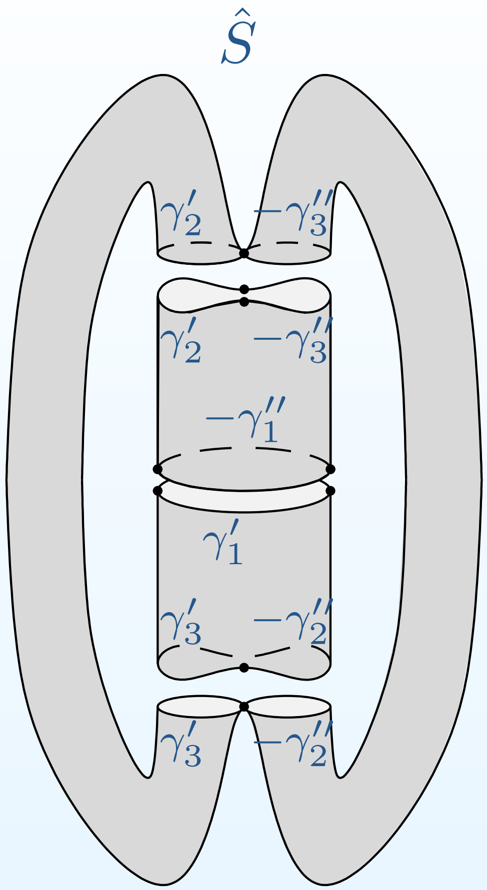
Saddle connection ĥomologous to separatrix loop





Saddle connections $\gamma_1, \gamma_2, \gamma_3$ on the surface S (left picture) are ĥomologous, though γ_1 is a segment joining distinct points P_1, P_2 and γ_2 and γ_3 are closed loops.

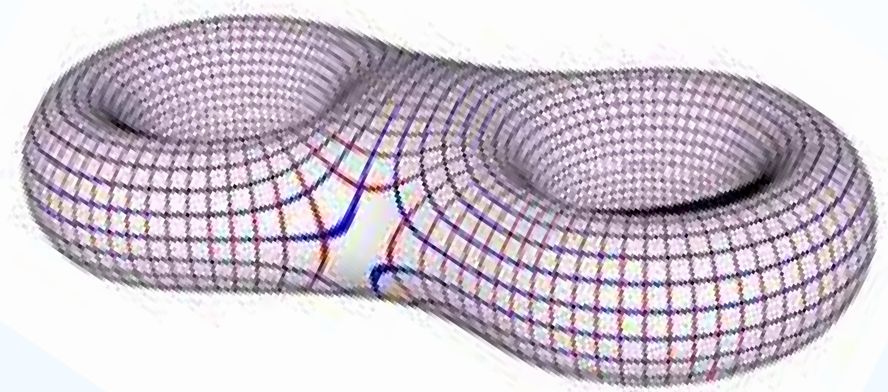
Rigid collections of saddle connections

It follows from the definition that $\hat{h}$omologous saddle connections are parallel on S and that their lengths either coincide or differ by a factor of two.

Theorem (H. Masur, A. Zorich, 2008) *Let S be a flat surface corresponding to a meromorphic quadratic differential q with at most simple poles. A collection $\gamma_1, \dots, \gamma_n$ of saddle connections on S is rigid if and only if all saddle connections $\gamma_1, \dots, \gamma_n$ are $\hat{h}$omologous.*

Theorem (H. Masur, A. Zorich, 2008) *Two saddle connections γ_1, γ_2 on S are $\hat{h}$omologous if and only if they have no interior intersections and one of the connected components of the complement $S \setminus (\gamma_1 \cup \gamma_2)$ has trivial linear holonomy. Moreover, if such a component exists, it is unique.*

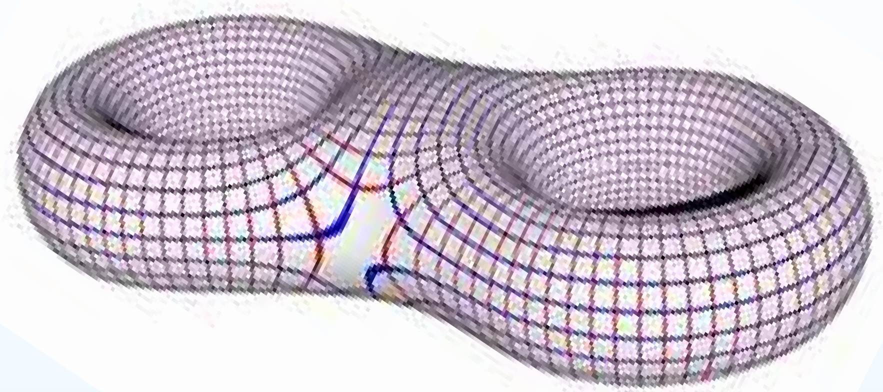
Homework assignment problems


Picture created by Jian Jiang

Questions.

- *To what stratum belongs this square-tiled surface?*
- *Find all realizabe separatrix diagrams for this stratum.*
- *To which of the found diagrams corresponds the red foliation of the square-tiled surface from the picture?*

Which stratum?



Picture created by Jian Jiang

Question.

- *To what stratum belongs this square-tiled surface?*

Answer.

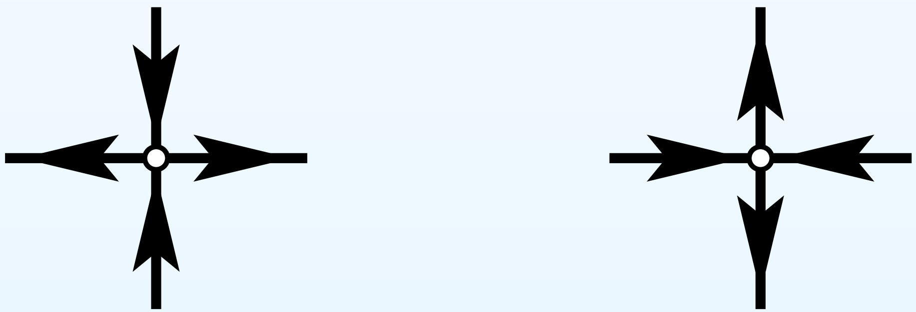
There are two strata in genus two: $\mathcal{H}(2)$ and $\mathcal{H}(1,1)$. The surface in the picture has two symmetric conical singularities, so the ambient stratum is $\mathcal{H}(1,1)$.

One can also honestly count the cone angle at the visible conical singularity. The neighborhood is an octagon composed of four horizontal (blue) sides of the squares and of four vertical (red) sides. Thus, the cone angle is 4π, which excludes stratum $\mathcal{H}(2)$.

Admissible diagrams in $\mathcal{H}(1,1)$

Question.

- *Find all realizabe (admissible) separatrix diagrams for this stratum.*



We have two zeroes. Each has two outgoing and two incoming horizontal separatrices.

Admissible diagrams in $\mathcal{H}(1,1)$

Question.

- *Find all realizabe (admissible) separatrix diagrams for this stratum.*

Let us start with critical graphs (separatrix diagram) having no closed loops. Let us draw one saddle connection and discuss how we can complete it.

Admissible diagrams in $\mathcal{H}(1,1)$

Question.

- *Find all realizabe (admissible) separatrix diagrams for this stratum.*

On the left there is a single outgoing separatrix and on the right — only one incoming. We are forced to join them.

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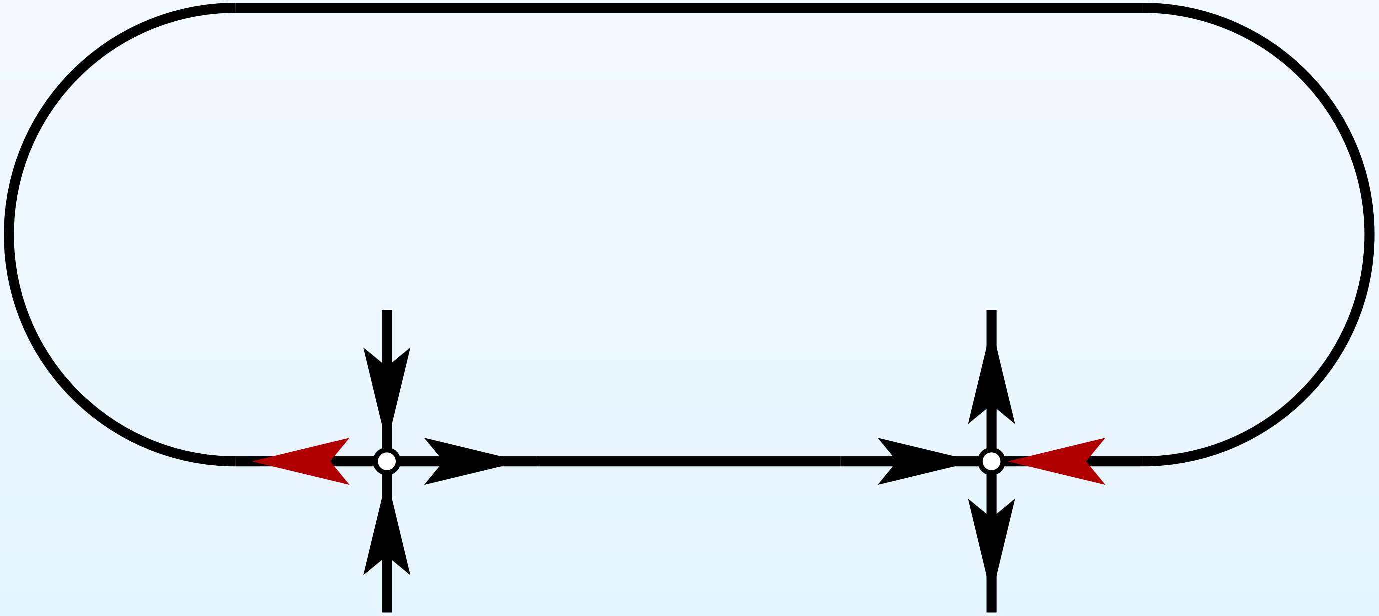
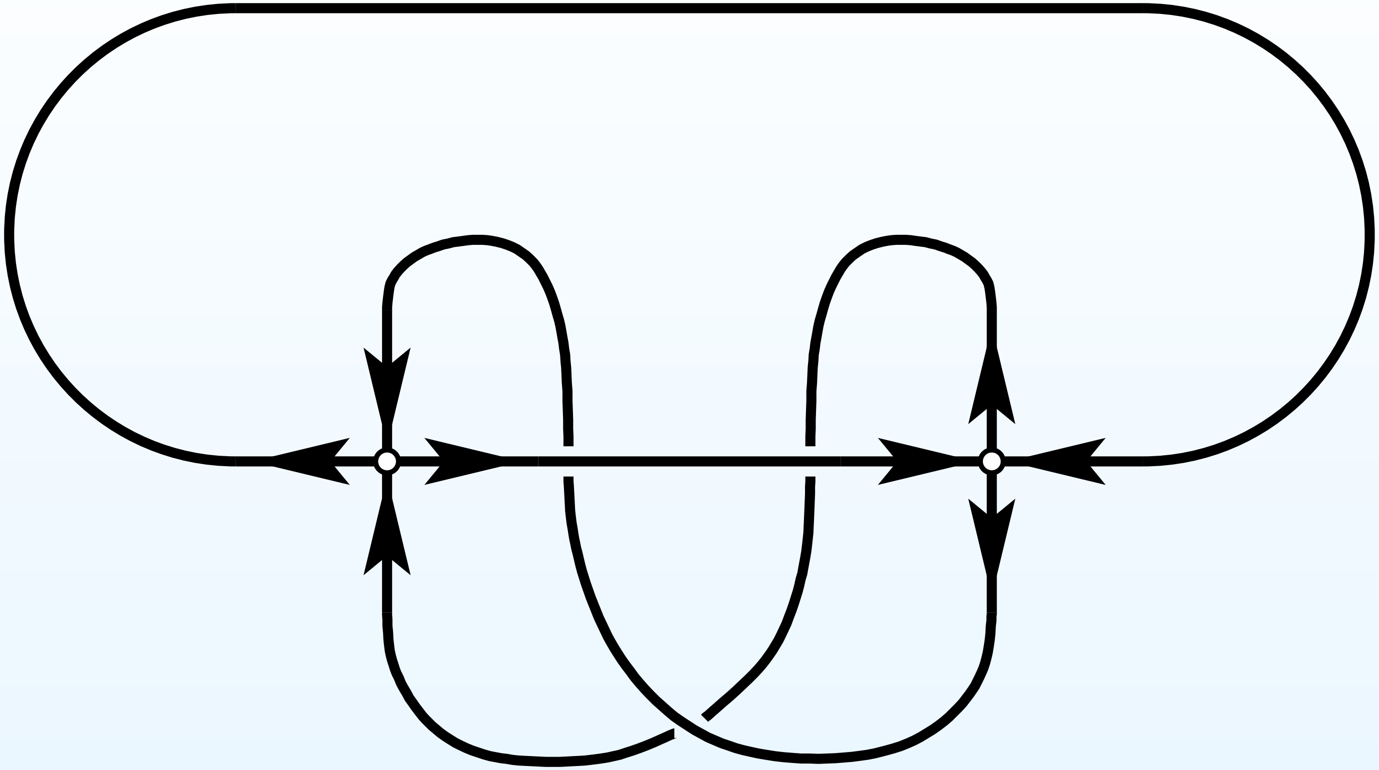
Question.

- *Find all realizabe (admissible) separatrix diagrams for this stratum.*



On the left there is a single outgoing separatrix and on the right — only one incoming. We are forced to join them.

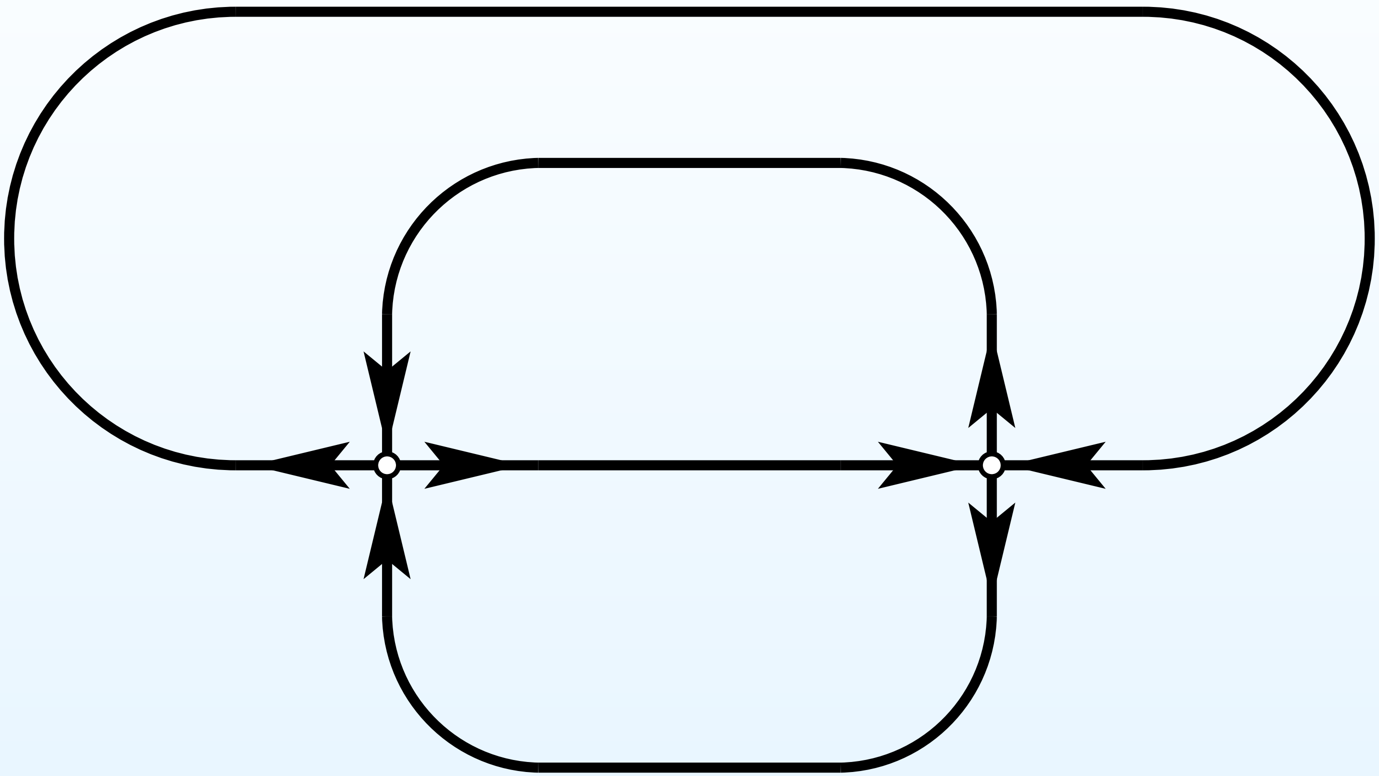
1-cylinder diagram in $\mathcal{H}(1,1)$



This is the first of the two ways of joining the remaining two pairs of separatrix rays.

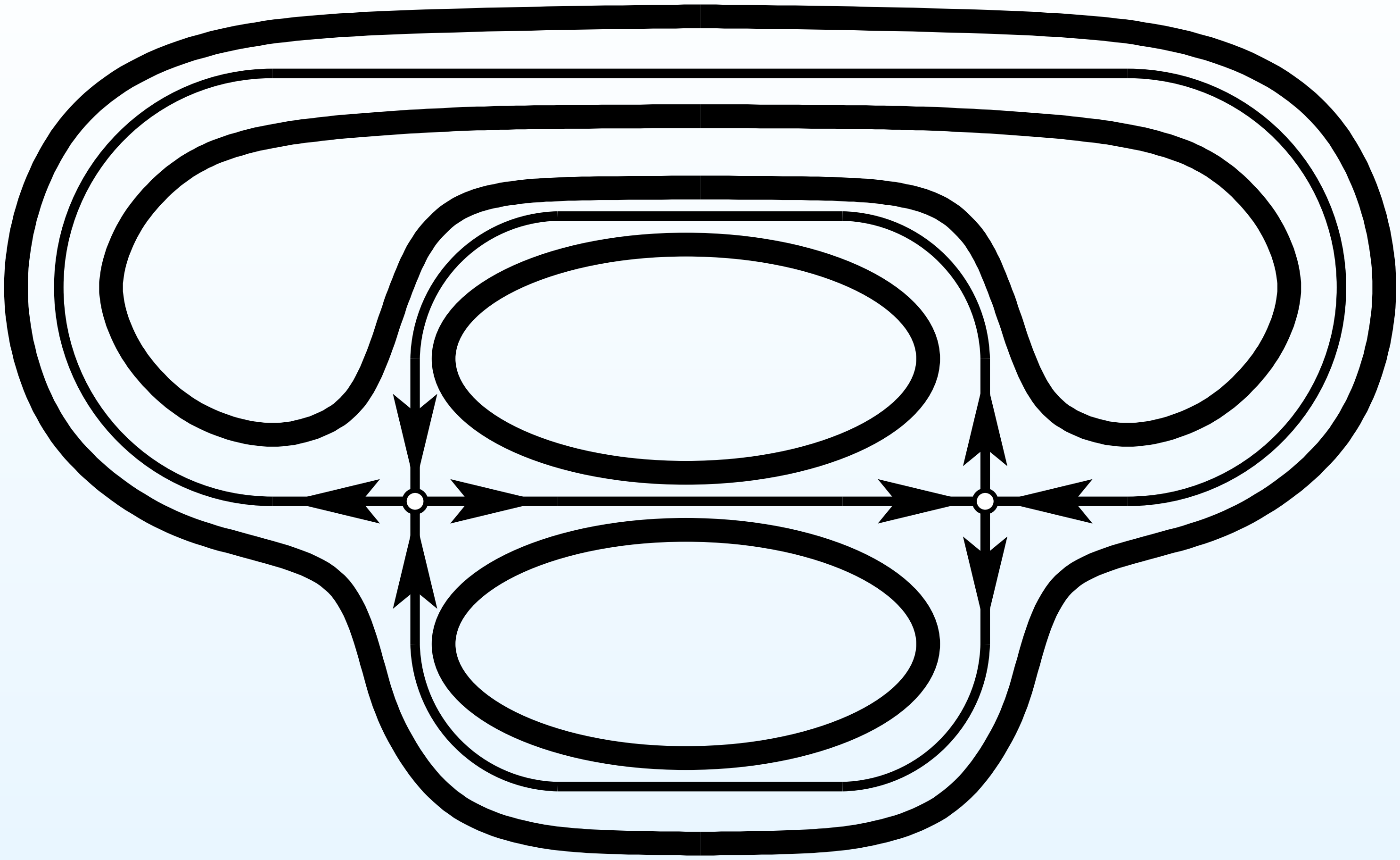
Mandatory Exercise. Check all of the following: The corresponding ribbon graph has two boundary components. Each component follows once each of the four saddle connection, so that the length of each of the two saddle connections is $\ell_1 + \ell_2 + \ell_3 + \ell_4$. There are no realtions on ℓ_i: this diagram is realizable for any choice of the lengths ℓ_i, where $i = 1, \ldots, 4$.

2-cylinder diagram in $\mathcal{H}(1,1)$



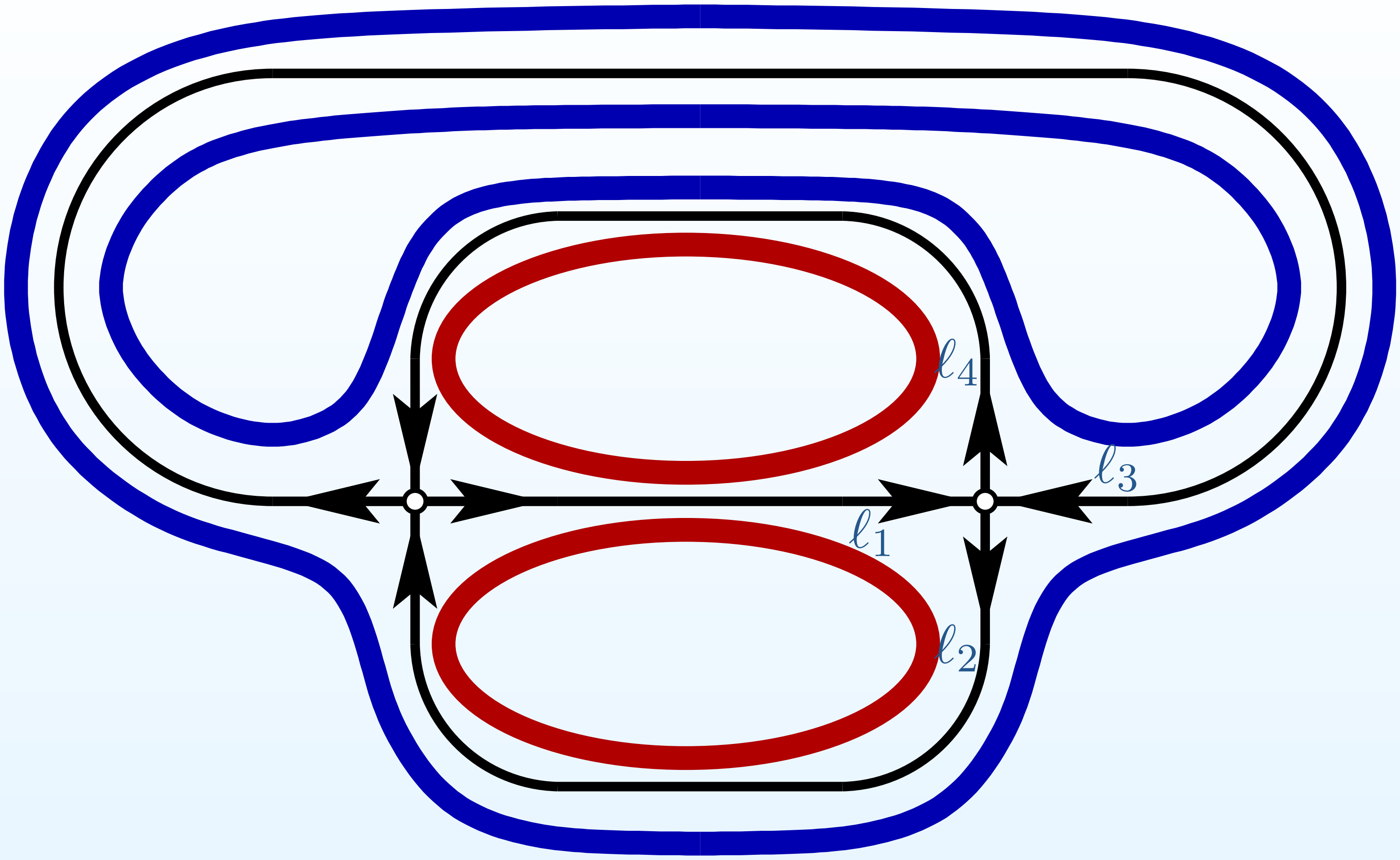
This is the other way to join the remaining two pairs of separatrix rays. Note that every maximal horizontal cylinder has one top and one boundary component. Thus, for every pair of boundary components to which we glue a cylinder, one component has the critical graph on the left and the other component has it on the right.

2-cylinder diagram in $\mathcal{H}(1,1)$



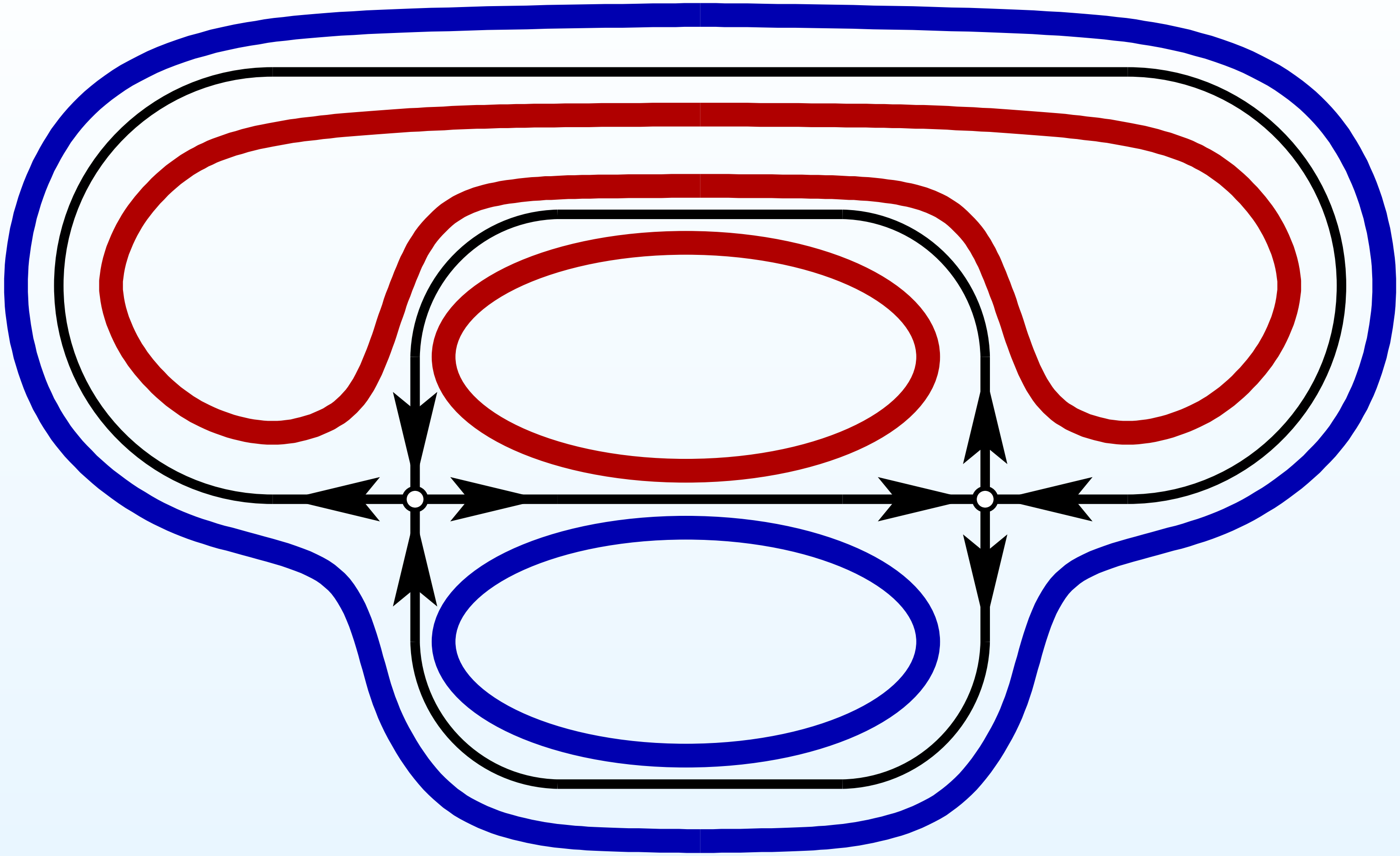
It gives us two ways in which we can organize the four boundary components into two pairs.

2-cylinder diagram in $\mathcal{H}(1,1)$



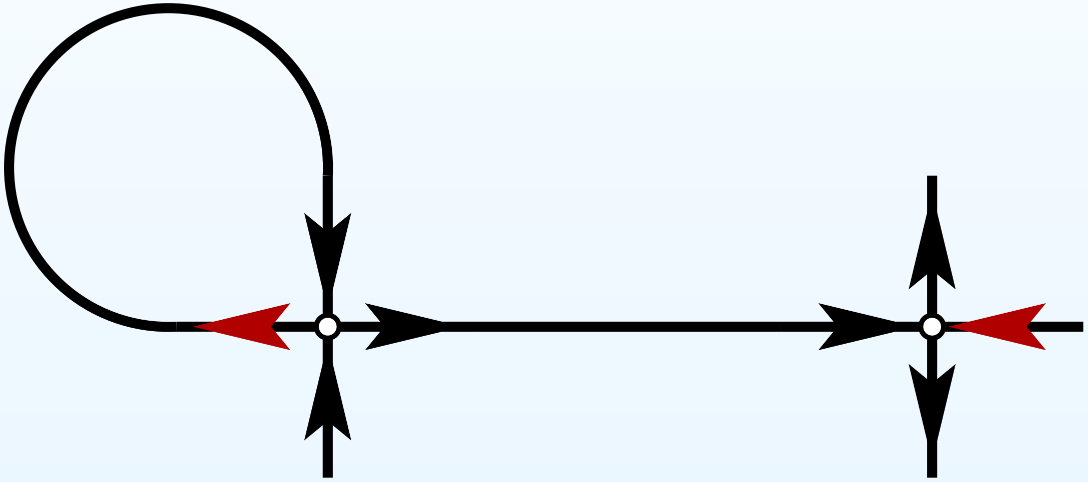
If we choose this way, we see that we have to impose the following conditions on the lengths of saddle connections: $\ell_2 = \ell_4$. Then the red cylinder has the waist curve of length $\ell_1 + \ell_2$ and the blue cylinder has the waist curve of length $\ell_3 + \ell_2$. We get an admissible diagram.

2-cylinder diagram in $\mathcal{H}(1,1)$



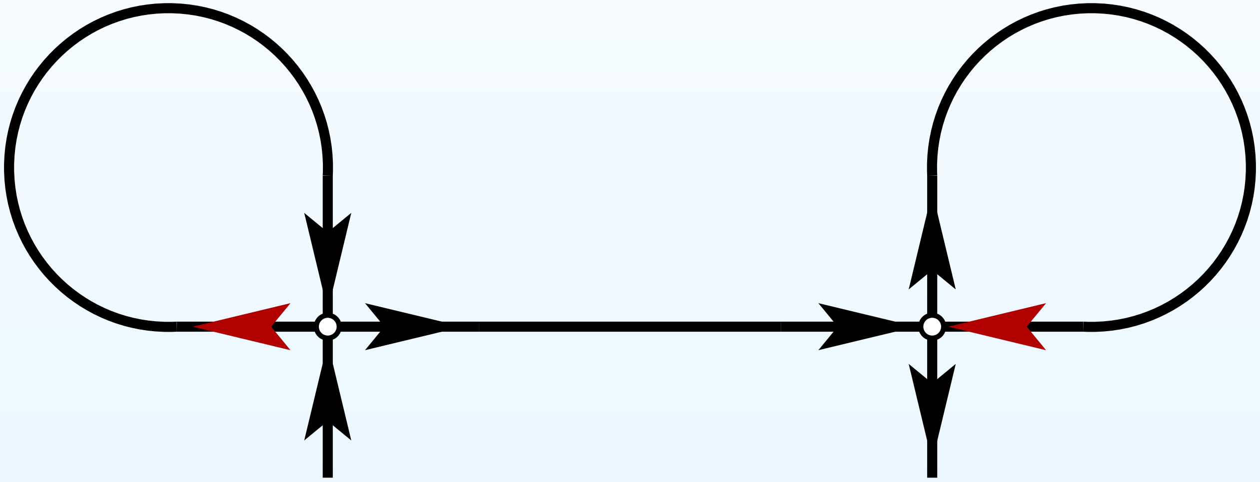
Exercise. Verify that the second way to arrange boundary components into pairs (as in the picture) is symmetric to the first one under interchanging the labels of the two singularities.

Diagrams with two loops in $\mathcal{H}(1,1)$



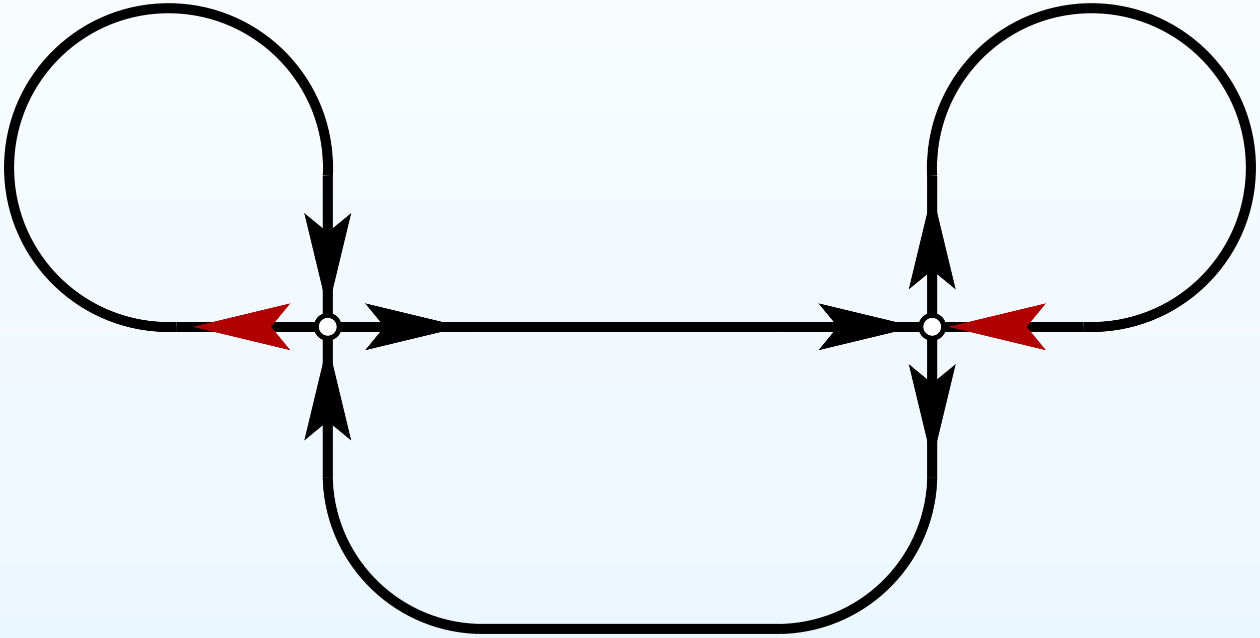
Now we have to consider diagrams having at least one loop. It is clear, that if a diagram has a loop and a saddle connection joining the two zeroes, it has to have another loop at the other zero.

Diagrams with two loops in $\mathcal{H}(1,1)$



There are two choices for the second loop. This is the first possible choice.

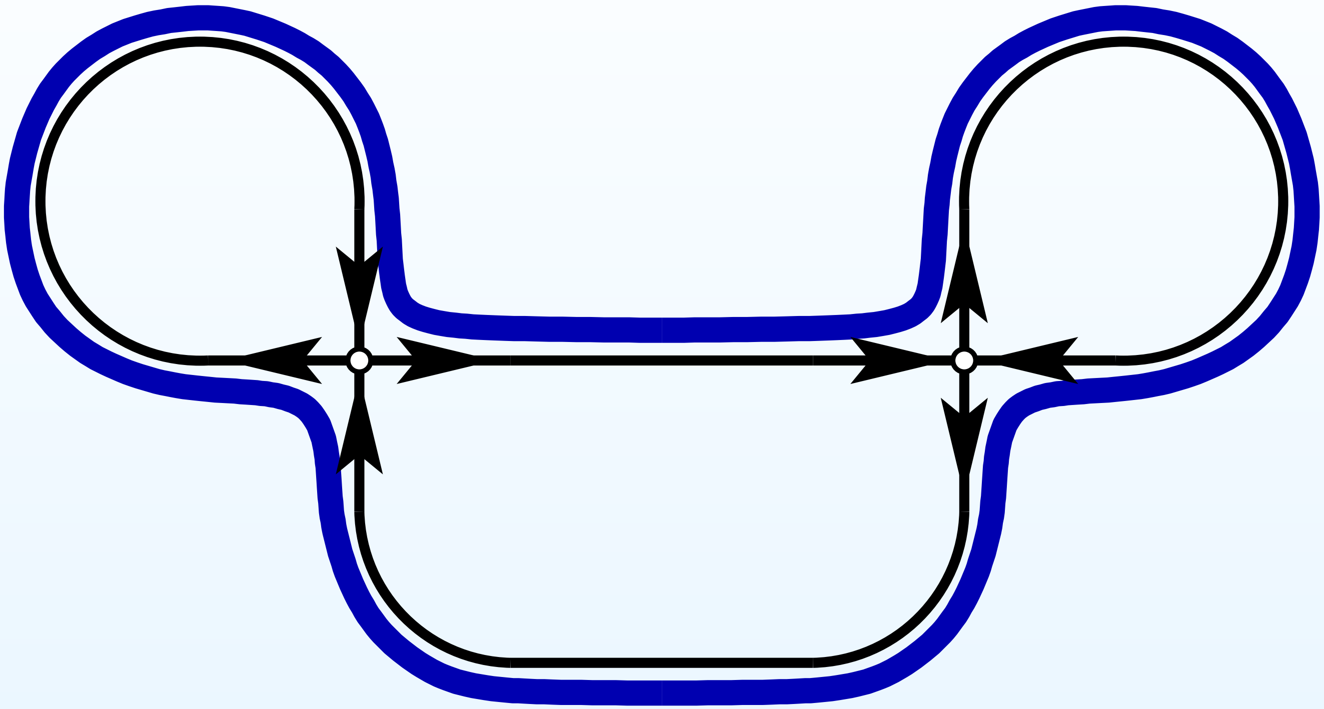
Diagrams with two loops in $\mathcal{H}(1,1)$



There are two choices for the second loop. This is the first possible choice.

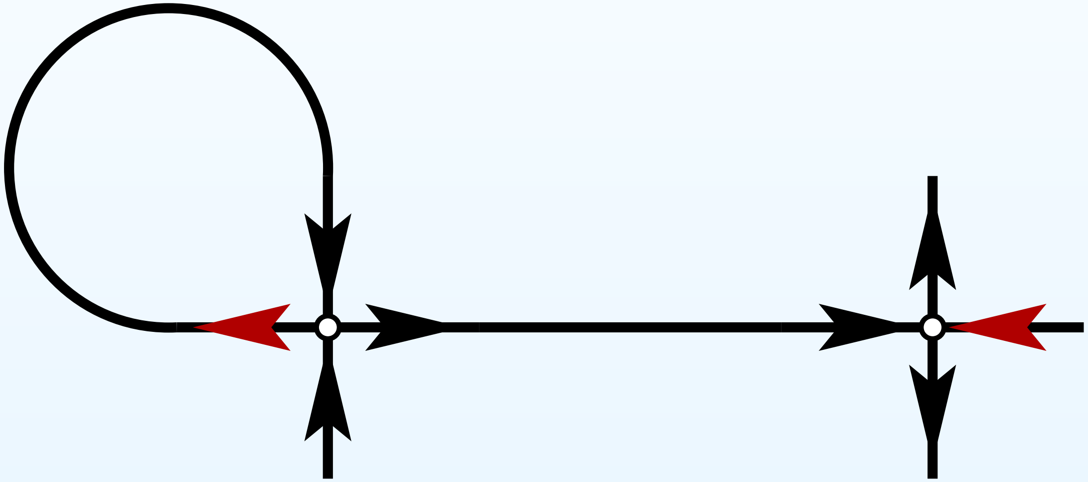
This is the unique way to join the remaining pair of separatrix rays.

Diagrams with two loops in $\mathcal{H}(1,1)$



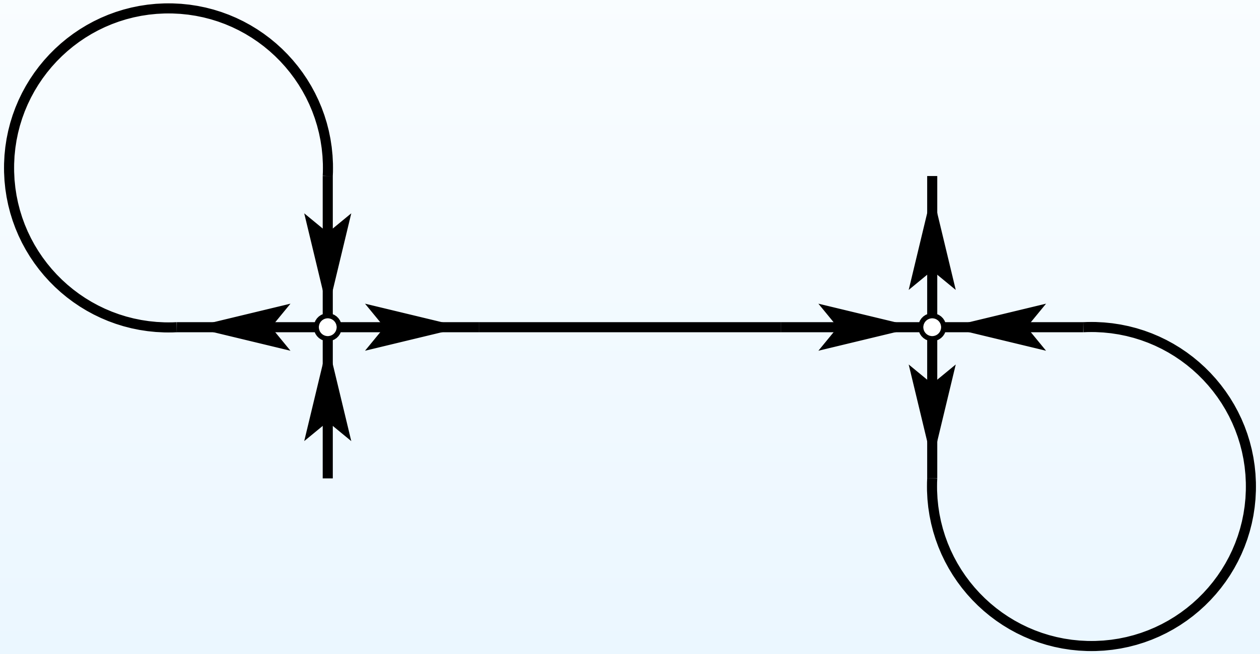
The boundary component of the resulting ribbon graph is longer than any other component for any choice of lengths of saddle connections (edges of the graph). This diagram is not realizable.

Diagrams with two loops in $\mathcal{H}(1,1)$



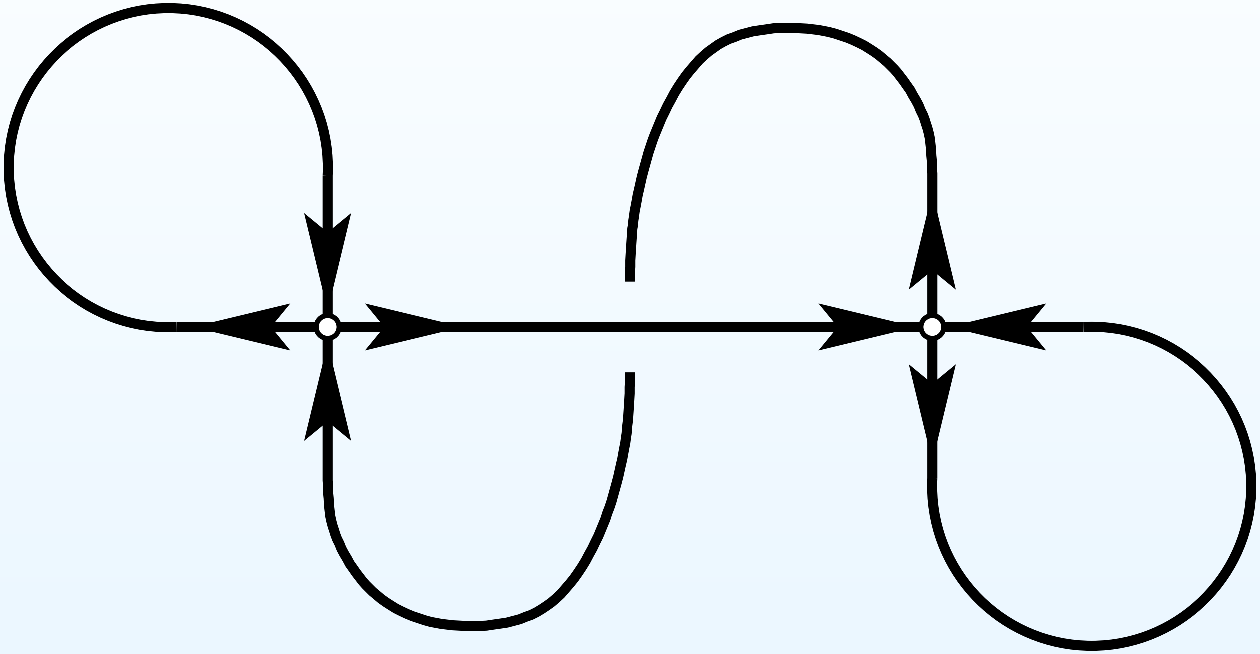
Recall that we are considering diagrams having at least one loop and a saddle connection joining the two zeroes.

Diagrams with two loops in $\mathcal{H}(1,1)$



The second choice for the second loop as in the picture.

Diagrams with two loops in $\mathcal{H}(1,1)$



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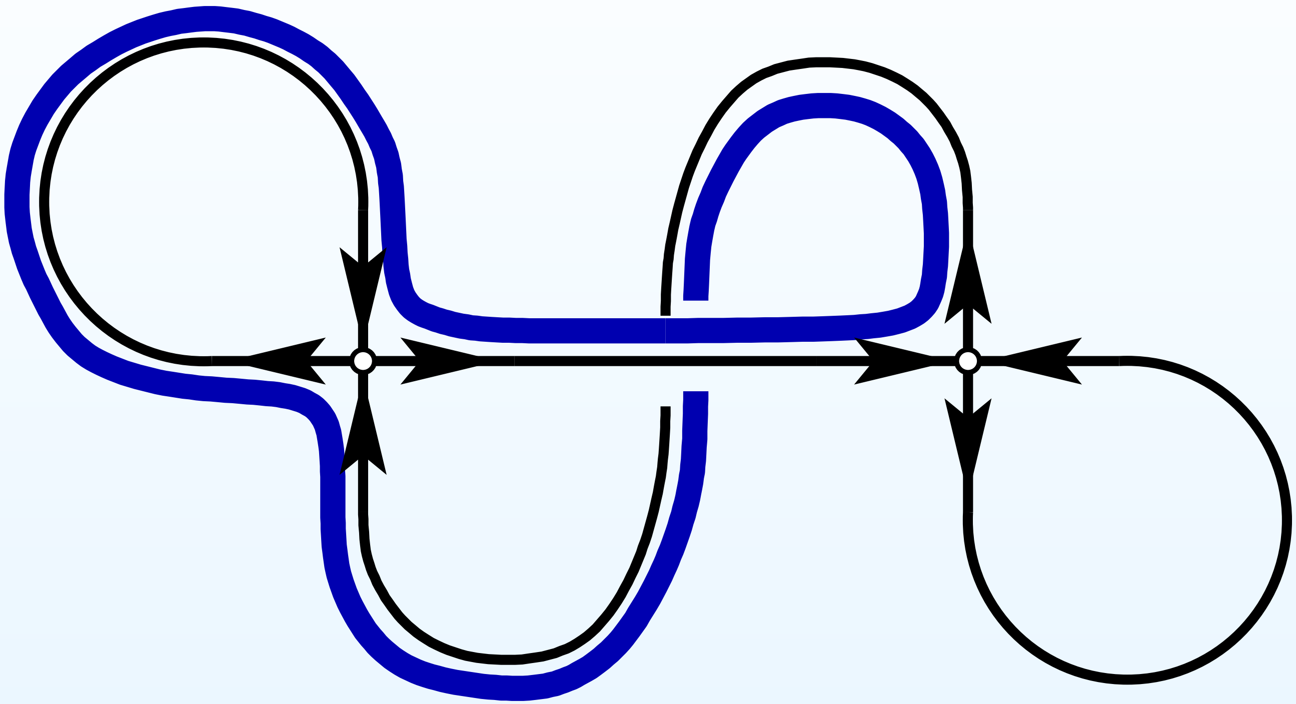
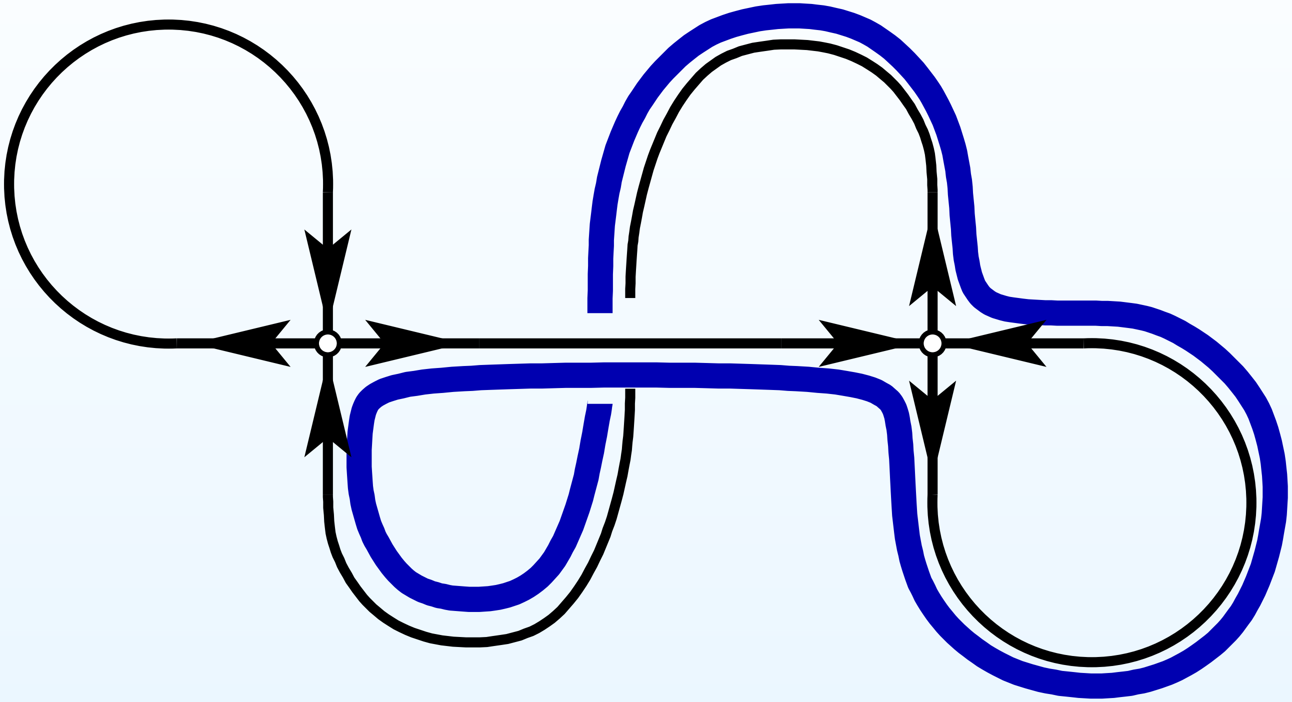
Diagrams with two loops in $\mathcal{H}(1,1)$



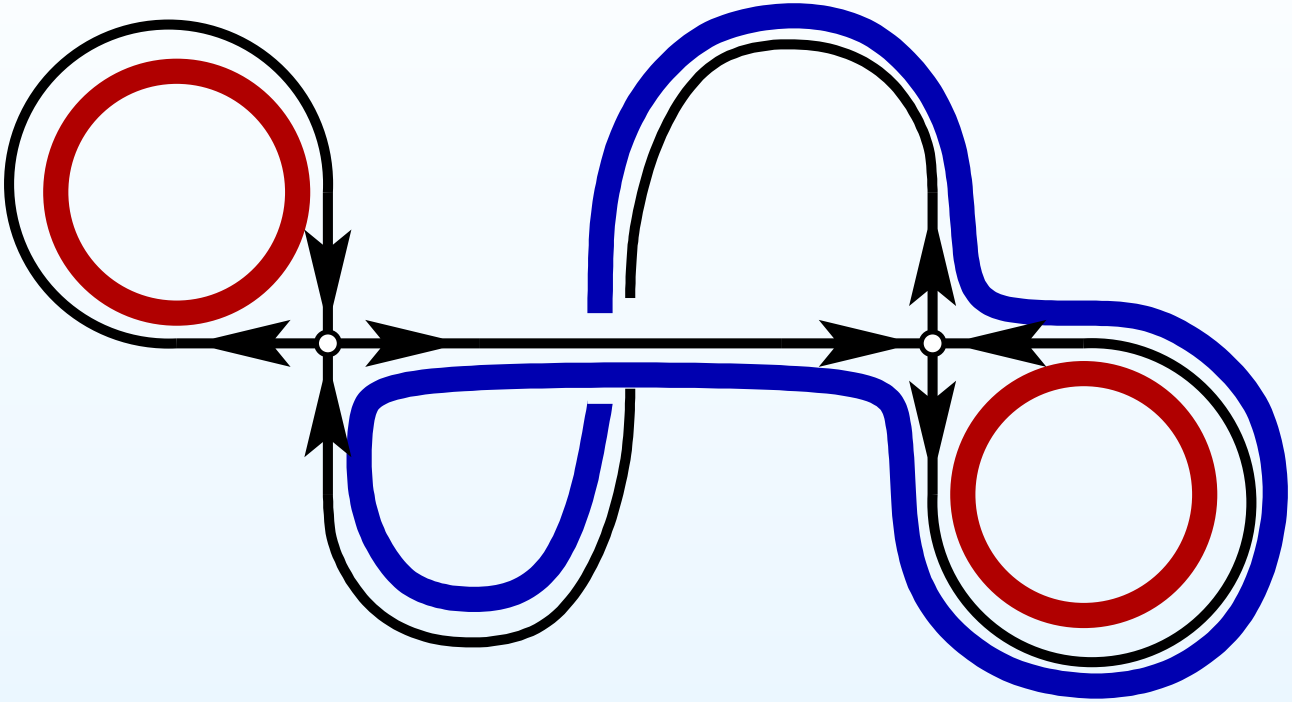
This is one of the four boundary components of the resulting ribbon graph.

Diagrams with two loops in $\mathcal{H}(1,1)$



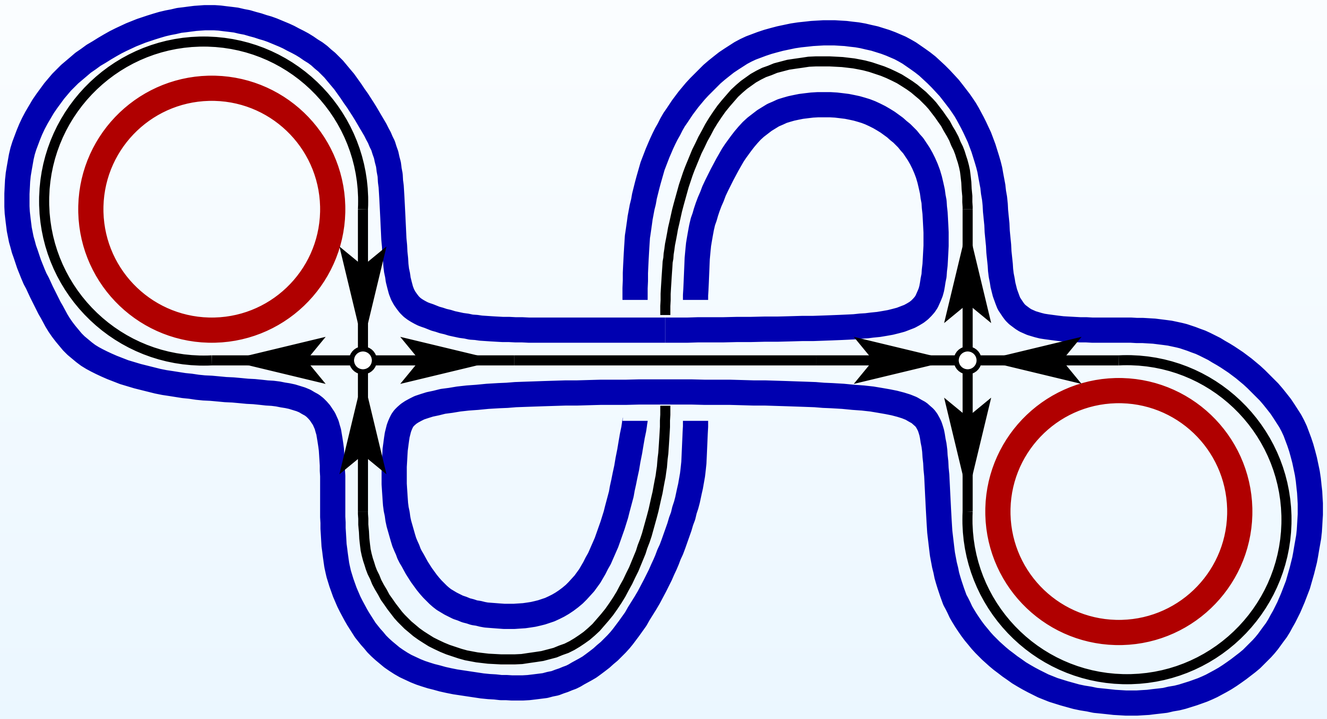
This is one more boundary component.

Diagrams with two loops in $\mathcal{H}(1,1)$



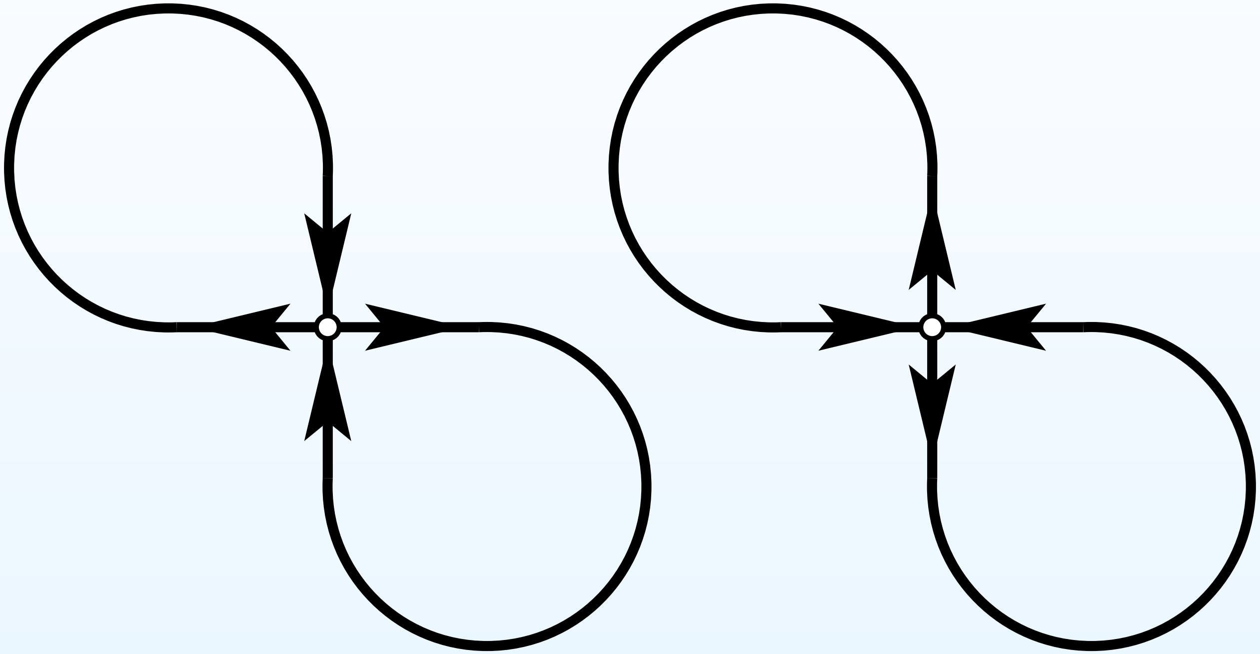
It is really easy to check that the only choice is to paste a cylinder to the pair of red boundary components. This implies a condition that the lengths of the corresponding loops are the same.

Diagrams with two loops in $\mathcal{H}(1,1)$



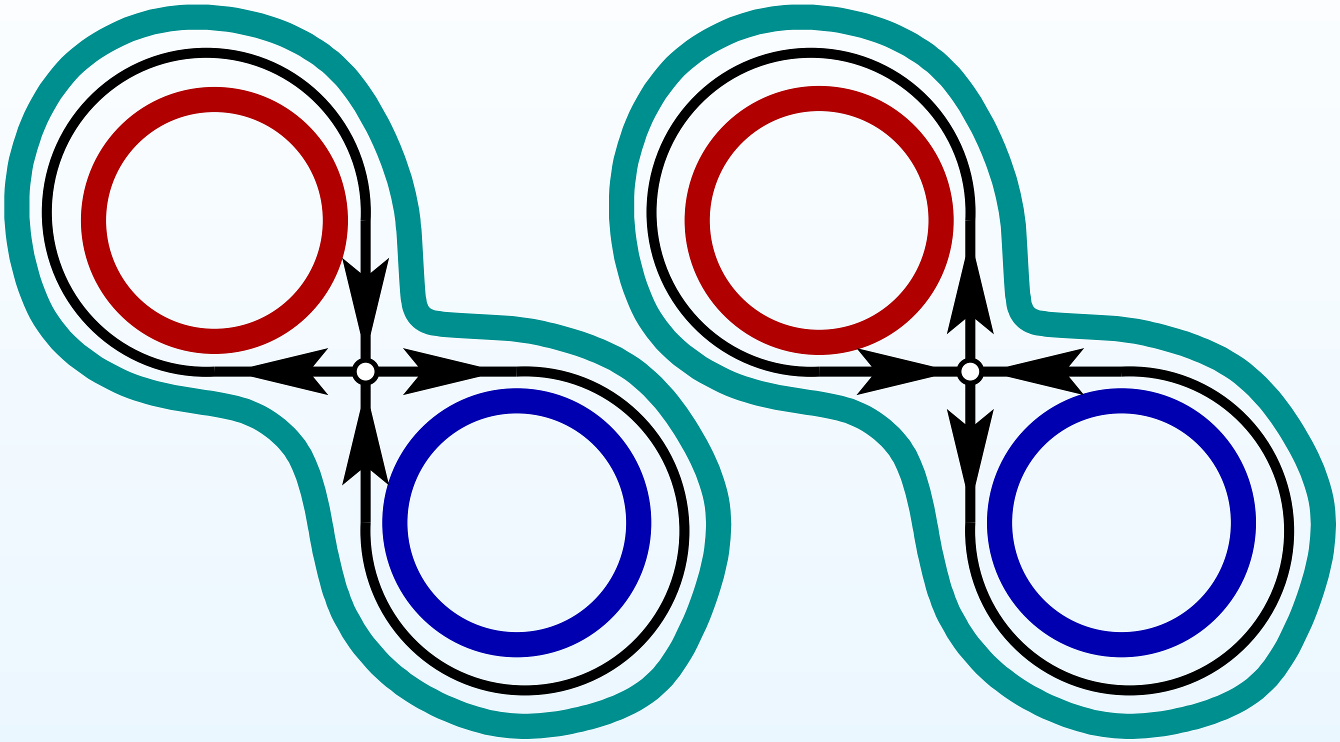
It is really easy to check that the only choice is to paste a cylinder to the pair of red boundary components. This implies a condition that the lengths of the corresponding loops are the same. This automatically implies that the lengths of the blue boundary components are the same. We get one more realizable diagram with two cylinders.

Diagrams with four loops in $\mathcal{H}(1,1)$



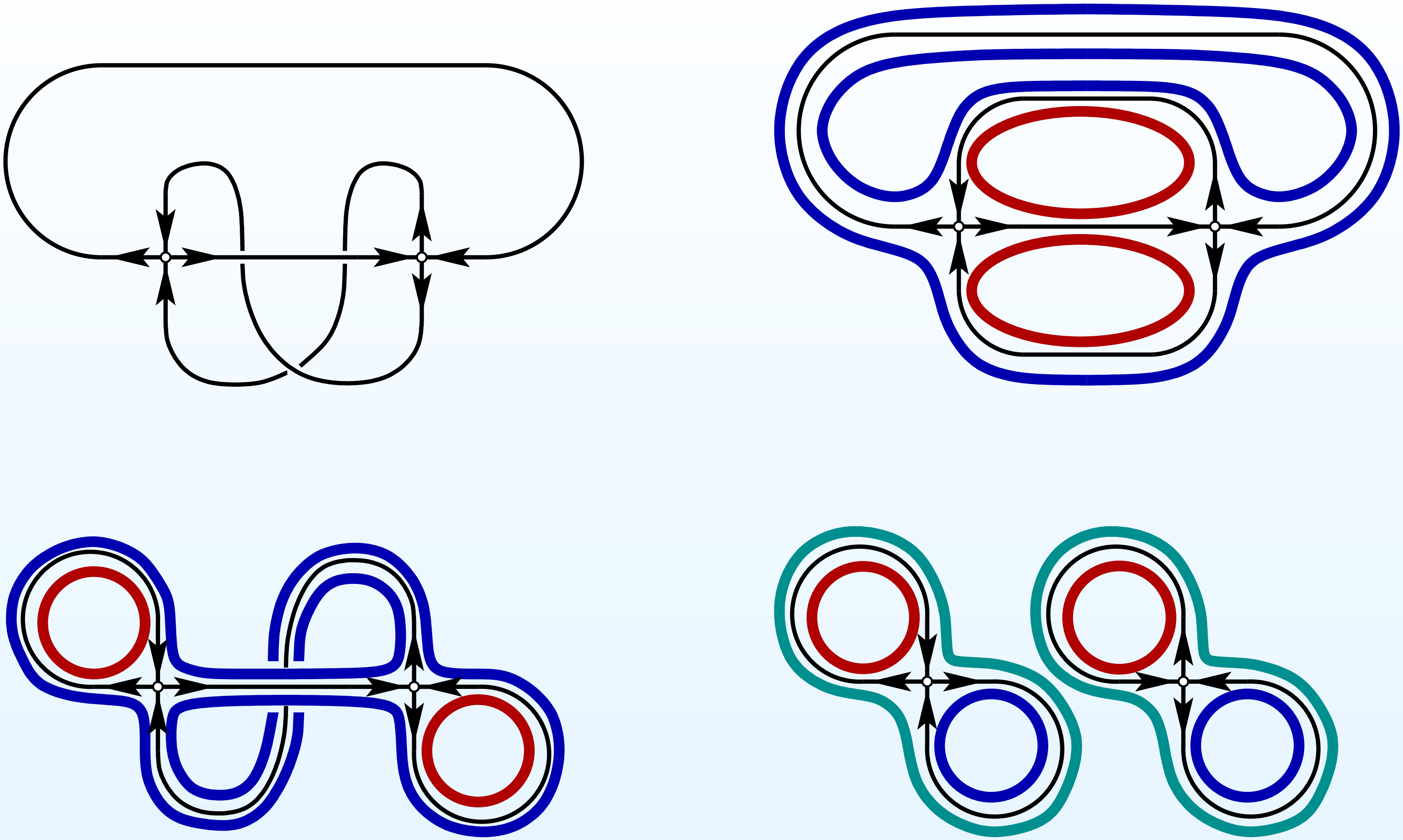
In the remaining case all the edges are loops.

Diagrams with four loops in $\mathcal{H}(1,1)$



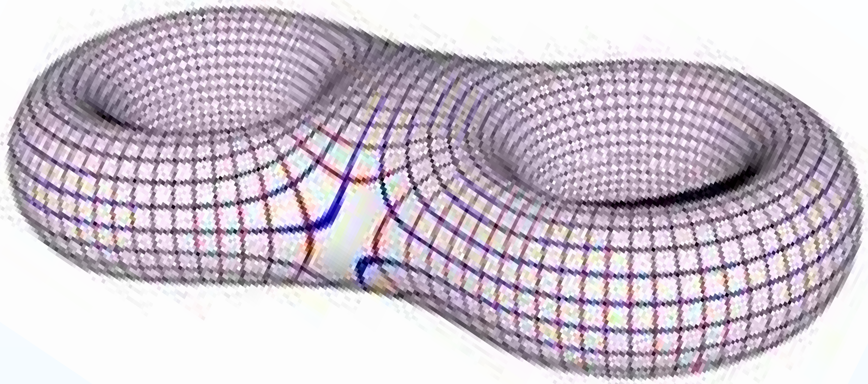
In the remaining case all the edges are loops. There is, clearly only one way to arrange boundary components into pairs. We get the last admissible (realizable) diagram in the stratum $\mathcal{H}(1,1)$.

Admissible diagrams in $\mathcal{H}(1,1)$



These four separatrix diagrams are admissible (realizable) diagrams in the stratum $\mathcal{H}(1,1)$ and there are no other ones (up to interchange of the labelling of the two zeroes).

Which diagram?

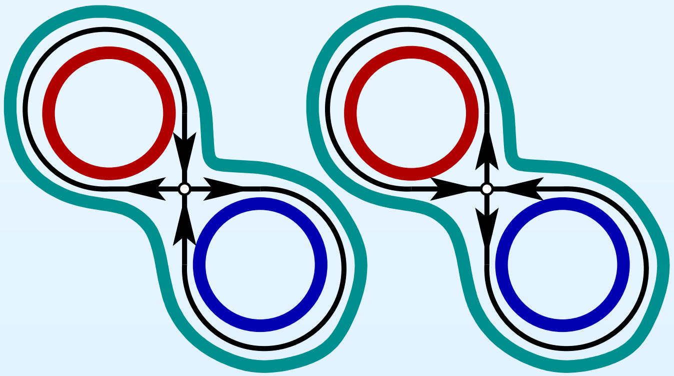

Picture created by Jian Jiang

Question.

● *To which of the found diagrams corresponds the red foliation of the square-tiled surface from the picture?*

Answer.

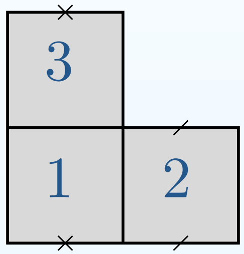
There are, clearly, three distinct cylinders. There only one 3-cylinder diagram in the stratum $\mathcal{H}(1,1)$:



Approach of Eskin and Okounkov

Encoding square-tiled surfaces by pairs of permutations

Consider a square-tiled surface $S \in \mathcal{H}(m_1, \ldots, m_n)$. Enumerate the squares in some way. For the square number j let $\pi_h(j)$ be the number of its neighbor to the right and let $\pi_v(j)$ be the number of the square atop the square number j.



Example. Our favorite L-shaped surface tiled with 3 squares can be encoded by the following two permutations decomposed into cycles:

$$\pi_h = (1,2)(3) \qquad \pi_v = (1,3)(2)$$

Note that there is no canonical enumeration of squares, so the permutations π_h, π_v are defined up to a simultaneous conjugation.

Almost commuting permutations

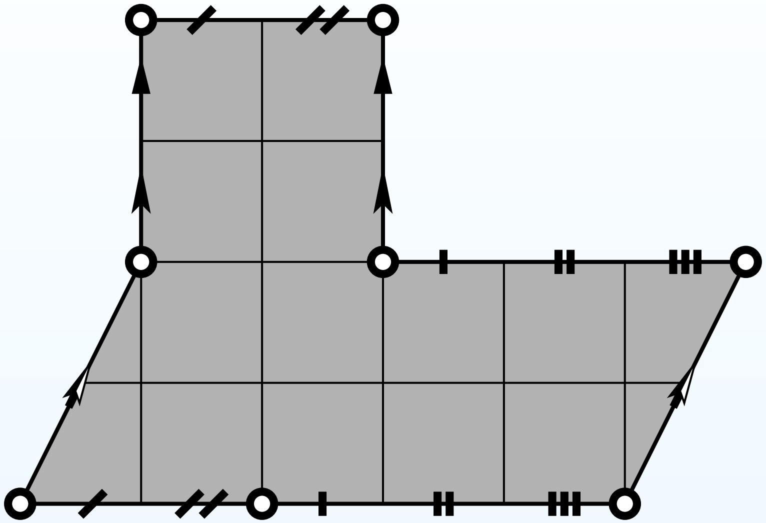
Consider the commutator $\pi' = \pi_h \pi_v \pi_h^{-1} \pi_v^{-1}$ The resulting permutation corresponds to the following path: we start from a square number j, then we move one step right, one step up, one step left, one step down, and we arrive to $\pi'(j)$.

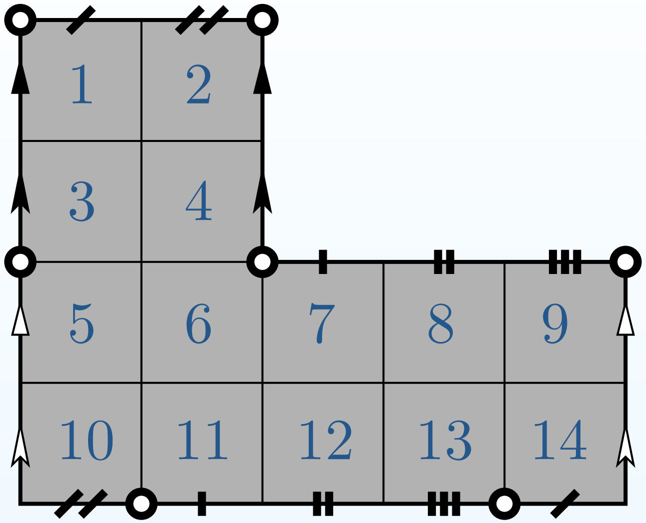
When the total number of squares is large, then for majority of the squares such path brings us back to the initial square; for such squares j we get $\pi'(j) = j$.

For squares having a singularity at the top right corner the path right-up-left-down does not bring us back to the initial square. The commutator $\pi' = \pi_h \pi_v \pi_h^{-1} \pi_v^{-1}$ decomposes into a product of n cycles of lengths $(m_1 + 1), \ldots, (m_n + 1)$ completed with huge number of cycles of length 1.

For example, for any square-tiled surface in $\mathcal{H}(2)$ the commutator is a single 3-cycle completed with plenty of fixed points.

Encoding square-tiled surfaces by pairs of permutations





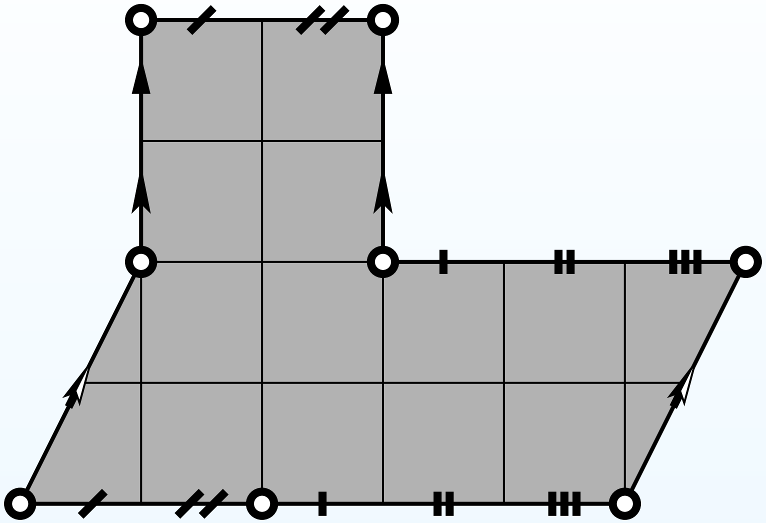
$$\pi_h = (1,2)\,(3,4)\,(5,6,7,8,9)\,(10,11,12,13,14)$$

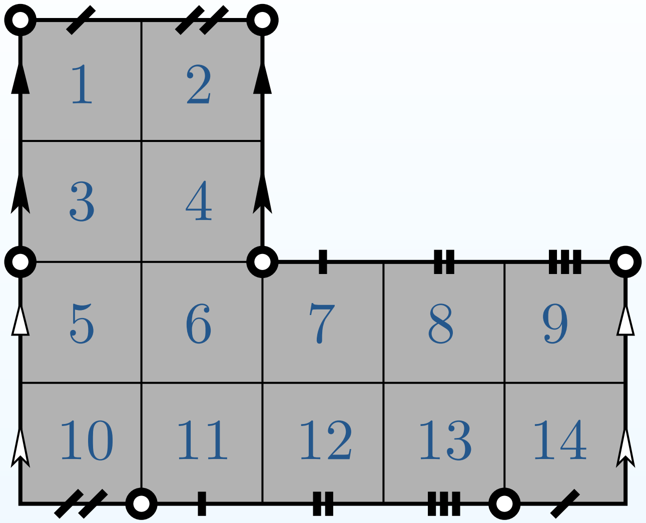
$$\pi_v = (1,14,9,13,8,12,7,11,6,4,2,10,5,3)$$

$$\pi_h\pi_v\pi_h^{-1}\pi_v^{-1} = \mathbf{(2,9,6)}\,(1)\,(3)\,(4)\,(5)\,(7)\,(8)\,(10)\,(11)\,(12)\,(13)\,(14)$$

The commutator $\pi_h\pi_v\pi_h^{-1}\pi_v^{-1}$ decomposes into a single cycle of length 3 completed with cycles of length 1. The cycle of length 3 corresponds to 3 squares, for which the top right corner is located at the conical singularity. There are 4 times more corners of squares at the same singularity, so the cone angle is $3 \cdot 2\pi$, where 3 is the length of the cycle. Our surface lives in $\mathcal{H}(2)$.

Encoding square-tiled surfaces by pairs of permutations





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$$\pi_h\pi_v\pi_h^{-1}\pi_v^{-1} = \mathbf{(2,9,6)}\,(1)\,(3)\,(4)\,(5)\,(7)\,(8)\,(10)\,(11)\,(12)\,(13)\,(14)$$

We conclude that a square-tiled surface $S \in \mathcal{H}(m_1,\dots,m_n)$ tiled with N squares can be encoded by a pair of permutations π_h, π_r (defined up to a common conjugation) such that the commutator $\pi_h\pi_v\pi_h^{-1}\pi_v^{-1}$ decomposes into given number n of cycles of given lengths $(m_1+1),\dots,(m_n+1)$ and π_h, π_r do not have common nontrivial invariant subsets in $1,2,\dots,N$.

Count by A. Eskin, A. Okounkov, R. Pandharipande

Using a version of the description of square-tiled surfaces by pairs of almost commuting permutations and using results of S. Bloch and A. Okounkov, A. Eskin, A. Okounkov and R. Pandharipande proved the following assertion.

Theorem (A. Eskin, A. Okounkov, R. Pandharipande) *For every connected component of every stratum the generating function*

$$\sum_{N=1}^{\infty} q^N \sum_{\substack{N\text{-square-tiled} \\ \text{surfaces } S}} \frac{1}{|Aut(S)|}$$

is a quasimodular form: it is a polynomial in Eisenstein series $G_2(q)$, $G_4(q)$, $G_6(q)$ *of controllable complexity.*

Corollary (A. Eskin, A. Okounkov, R. Pandharipande) *The Masur–Veech volume* $\operatorname{Vol}\mathcal{H}_1^{comp}(m_1,\dots,m_n)$ *of every connected component of every stratum is a rational multiple of* π^{2g}*, where* $2g-2=m_1+\dots+m_n$.


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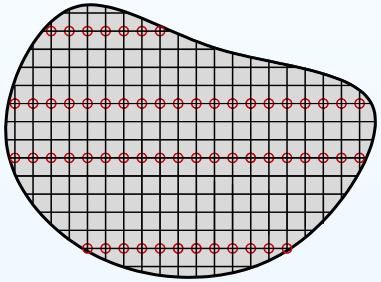
Non-correlation of vertical and horizontal foliations on square-tiled surfaces

Equidistribution and Non-correlation Theorems

Theorem. *The asymptotic proportion $p_k(\mathcal{L})$ of square-tiled surfaces tiled with tiny $\varepsilon \times \varepsilon$-squares and having exactly k maximal horizontal cylinders among all such square-tiled surfaces living inside an open set $B \subset \mathcal{L}$ in a stratum $\mathcal{L}$ of Abelian or quadratic differentials does not depend on B.*



Let $c_k(\mathcal{L})$ be the contribution of horizontally k-cylinder square-tiled surfaces (pillowcase covers) to the Masur–Veech volume of the stratum $\mathcal{L}$, so that $c_1(\mathcal{L}) + c_2(\mathcal{L}) + \dots = \operatorname{Vol} \mathcal{L}$, and $p_k(\mathcal{L}) = c_k(\mathcal{L}) / \operatorname{Vol}(\mathcal{L})$. Let $c_{k,j}(\mathcal{L})$ be the contribution of horizontally k-cylinder and vertically j-cylinder ones.

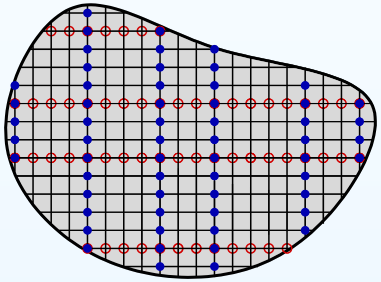
Theorem. *There is no correlation between statistics of the number of horizontal and vertical maximal cylinders:*

$$\frac{c_k(\mathcal{L})}{\operatorname{Vol}(\mathcal{L})} = \frac{c_{kj}(\mathcal{L})}{c_j(\mathcal{L})}\,.$$

This formula is an *asymptotic* formula! Proof: Moore ergodicity theorem.

Equidistribution and Non-correlation Theorems

Theorem. *The asymptotic proportion $p_k(\mathcal{L})$ of square-tiled surfaces tiled with tiny $\varepsilon \times \varepsilon$-squares and having exactly k maximal horizontal cylinders among all such square-tiled surfaces living inside an open set $B \subset \mathcal{L}$ in a stratum $\mathcal{L}$ of Abelian or quadratic differentials does not depend on B.*



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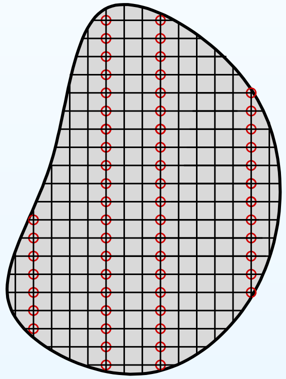
Experimental evaluation of volumes

The Equidistribution Theorem allows to compute approximate values of volumes experimentally. Choose some ball B (or some box) in the stratum. Consider a sufficiently small grid in it and collect statistics of frequency $p_1(B)$ of 1-cylinder square-tiled surfaces (pillow-case covers) in our grid in B.



Now compute the **absolute** contribution $c_1(\mathcal{H})$ of all 1-cylinder square-tiled surfaces to $\operatorname{Vol}\mathcal{H}_1$; it is easier than for k-cylinder ones with $k > 2$. By the Equdistribution Theorem, the volume of the ambient stratum is $\operatorname{Vol}\mathcal{H}_1 = \frac{c_1}{p_1}$.

The statistics $p_1(\mathcal{H})$ can be, actually, collected using interval exchanges, which simplifies the experiment. Approximate values of volumes were extremely useful in debugging numerous normalization factors in rigorous answers in the implementation by E. Goujard of the method of Eskin–Okounkov.

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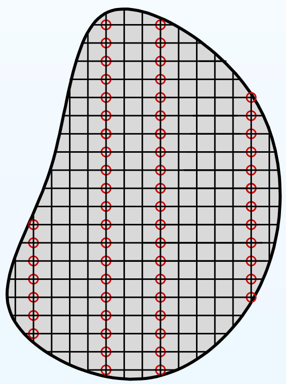

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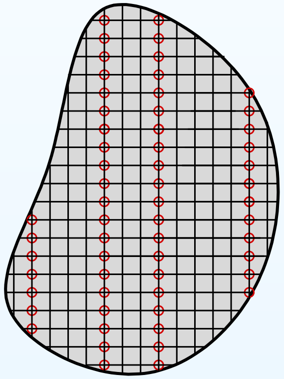

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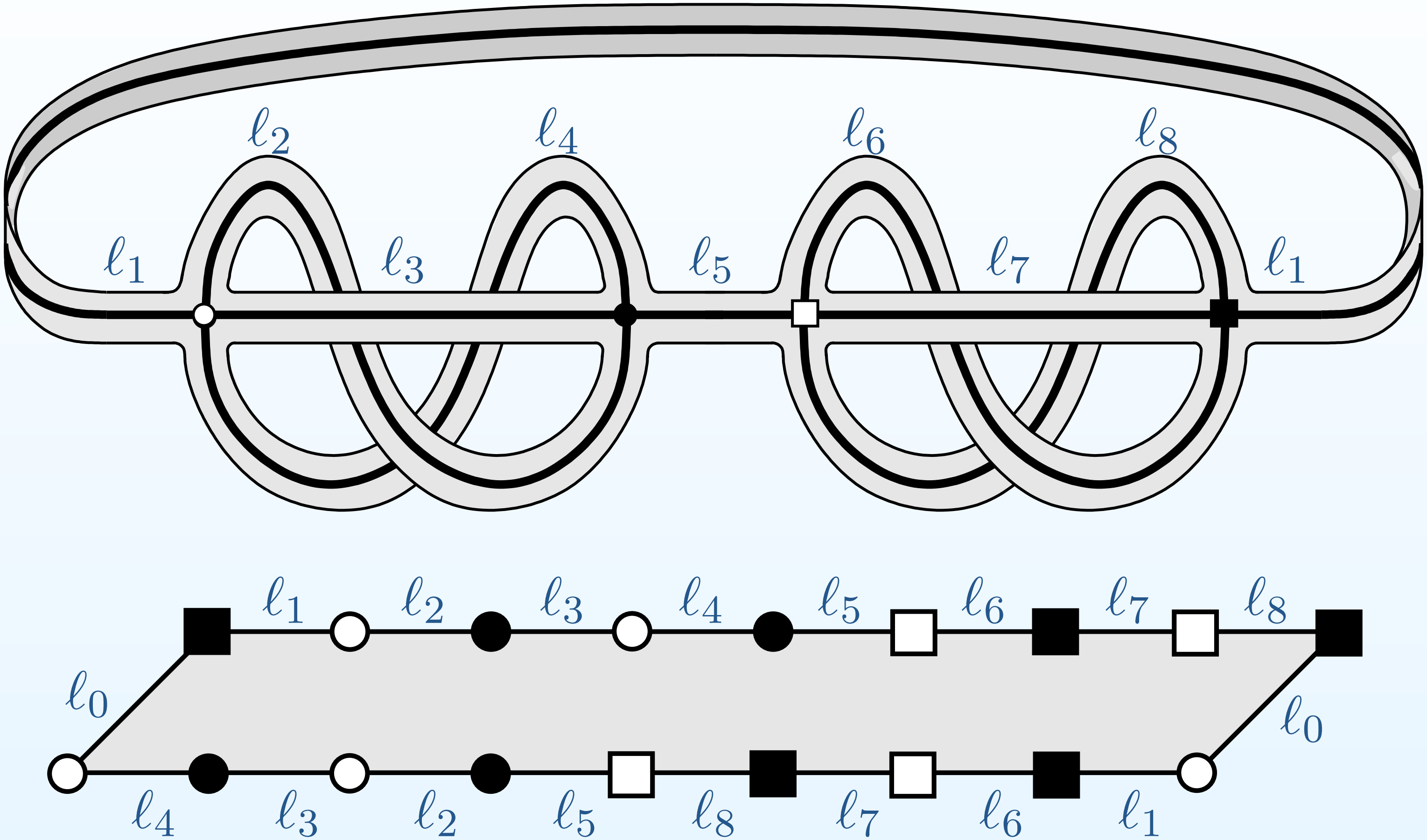
1-cylinder square-tiled surfaces and permutations

1-cylinder surface as a pair of permutations



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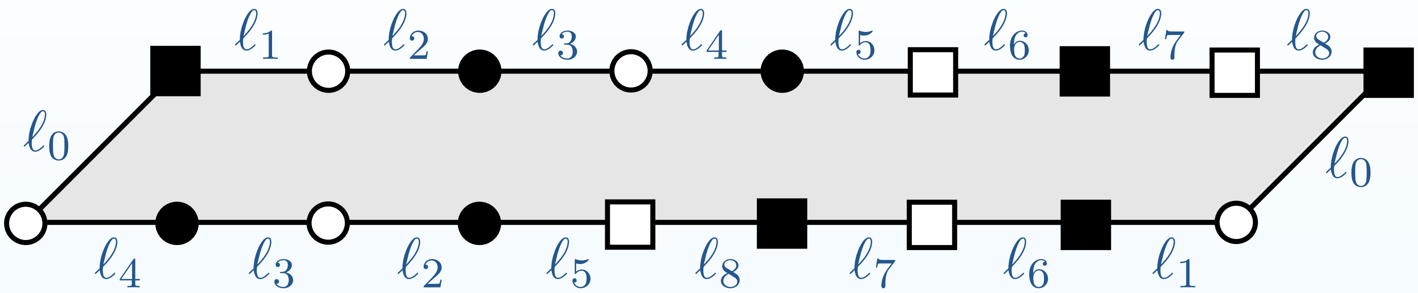

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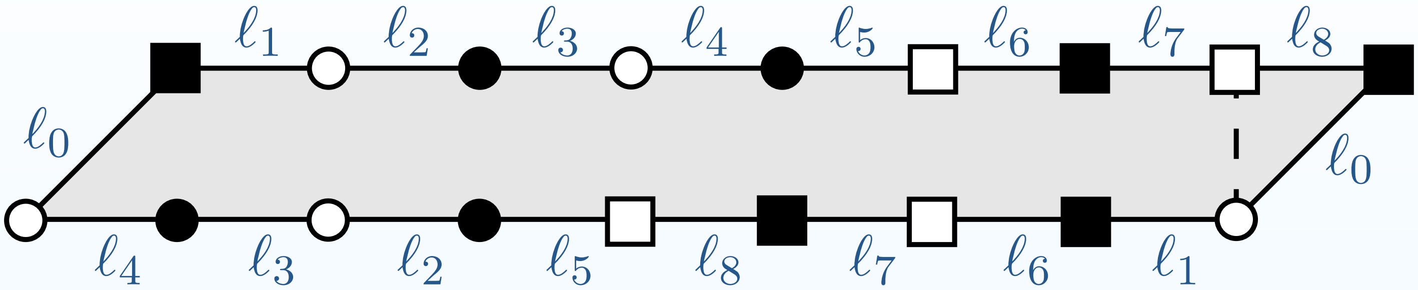

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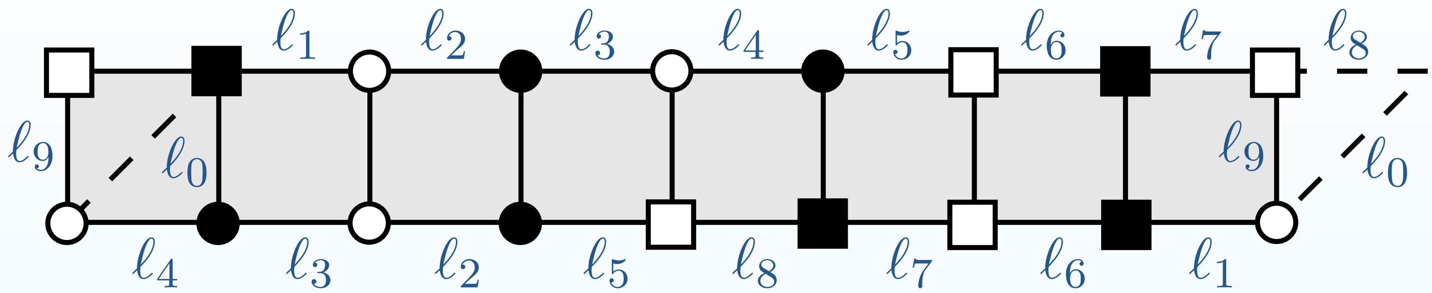

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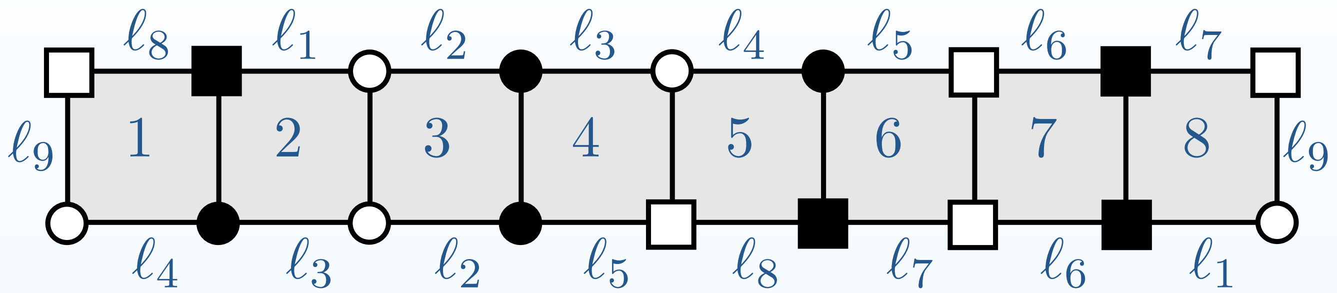

$$\pi_h = (1, 2, 3, 4, 5, 6, 7, 8)$$

$$\pi_v = (1, 5)\,(2, 8, 6, 4)\,(3)\,(7)$$

$$\pi_h \pi_v \pi_h^{-1} \pi_v^{-1} = (1, 7)\,(2, 4)\,(3, 5)\,(6, 8)$$

We see 4 cycles of length 2 which corresponds to $\mathcal{H}(1, 1, 1, 1)$.

Note that by construction, the permutation π_h of a square-tiled surface composed from a single band of squares is a long cycle $\pi_h = (1, \ldots, N)$. Thus, for any π_v, the permutation $\sigma = \pi_v \pi_h^{-1} \pi_v^{-1}$ is also a long cycle. For any pair of long cycles σ, π_h there are exactly N solutions π_v of the equation $\sigma = \pi_v \pi_h^{-1} \pi_v^{-1}$ (if π_v is a solution, $\pi_v (\pi_h)^k$, where $k = 0, 1, \ldots, N-1$, is also a solution).

Frobenius formula

The count of 1-cylinder N-square-tiled surfaces in the stratum $\mathcal{H}(m_1,\dots,m_n)$ is reduced to the count of solutions of the following equation for permutations:

$$(N-\text{cycle})\cdot(N-\text{cycle}) = \text{product of cycles of lengths } m_1+1,\dots,m_n+1,$$

completed with product of cycles of lengths 1.

Frobenius formula expresses this number in terms of characters of the exterior powers of the standard representation $\mathbf{St}_n$ of the symmetric group S_n:

$$\chi_j(g) := \operatorname{tr}(g,\pi_j) \qquad \pi_j := \wedge^j(\mathbf{St}_n) \qquad (0\le j\le n-1)\,.$$

Theorem. *The absolute contribution $c_1(\mathcal{H}(m_1,\dots,m_n))$ of 1-cylinder square-tiled surfaces to the Masur–Veech volume $\operatorname{Vol}\mathcal{H}(m_1,\dots,m_n)$ equals*

$$c_1 = \frac{2}{(d-1)!}\cdot\prod_k \frac{1}{(k+1)^{\mu_k}}\cdot\sum_{j=0}^{d-2} j!\,(n-1-j)!\,\chi_j(\nu)\,.$$

Here $d=\dim\mathcal{H}(m_1,\dots,m_n)$; $\nu\in S_n$ is any permutation with decomposition into cycles of lengths $(m_1+1),\dots,(m_n+1)$; μ_i is the number of zeroes of order i, i.e. the multiplicity of the entry i in the multiset $\{m_1,\dots,m_n\}$.

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For permutations ν representing the principal and the minimal strata the characters $\chi_j(\nu)$ admit easier computation which leads to the following formulae:

$$c_1(\mathcal{H}(1^{2g-2})) = \frac{1}{4g-2} \cdot \frac{4}{2^{2g-2}}\,, \qquad c_1(\mathcal{H}(2g-2)) = \frac{1}{2g} \cdot \frac{4}{2g-1}\,.$$

Contribution of 1-cylinder diagrams.

Theorem. *The contribution c_1 of 1-cylinder square-tiled surfcaes to the volume $\operatorname{Vol}\mathcal{H}_1(m_1,\dots,m_n)$ of any nohyperelliptic stratum of Abelian differentials satisfies*

$$\frac{\zeta(d)}{d+1}\cdot\frac{4}{(m_1+1)\dots(m_n+1)}\le c_1\le\frac{\zeta(d)}{d-\frac{10}{29}}\cdot\frac{4}{(m_1+1)\dots(m_n+1)}\,,$$

where $d=\dim_{\mathbb{C}}\mathcal{H}(m_1,\dots,m_n)$.
(Here we used a result of Zagier.)

Theorem (analog of the Prime Number Theorem). *The relative contribution of 1-cylinder square-tiled surfaces to the volume of the stratum is of the order $1/($dimension of the stratum$)$ when $g\gg 1$:*

$$d\cdot\frac{c_1(\mathcal{H}(m_1,\dots,m_n))}{\operatorname{Vol}(\mathcal{H}_1(m_1,\dots,m_n))}\to 1\ \text{ as }\ g\to+\infty\,,$$

where convergence is uniform for all strata in genus g.

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