Square-tiled surfaces and interval exchanges: geometry, dynamics, combinatorics and applications

Lecture 6. Rauzy–Veech induction as a renormalization.

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"But still, my homeward way has proved too long.
While we were wasting time there, old Poseidon,
it almost seems, stretched and extended space."

J. Brodsky

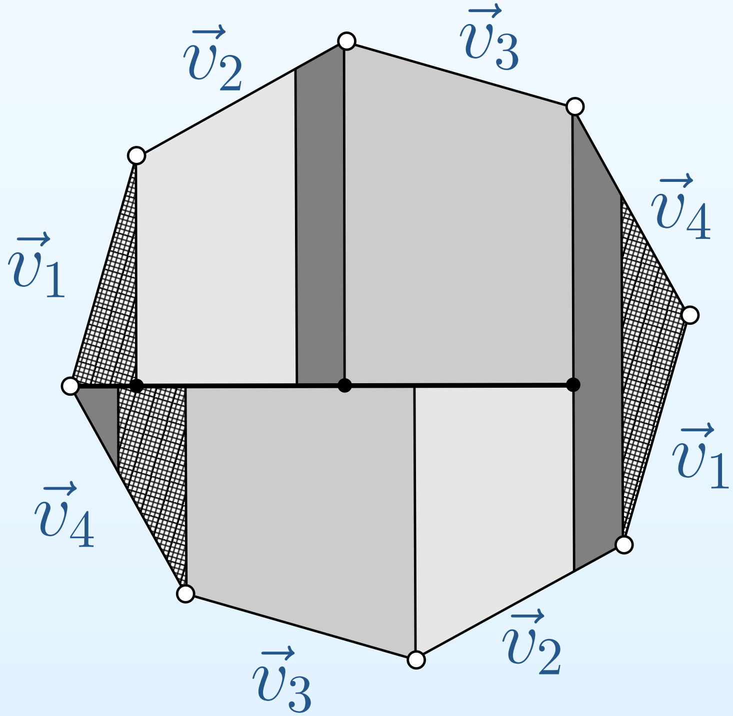
Idea of Renormalization

И все-таки ведущая домой
дорога оказалась слишком длинной,
как будто Посейдон, пока мы там
теряли время, растянул пространство.

И. Бродский

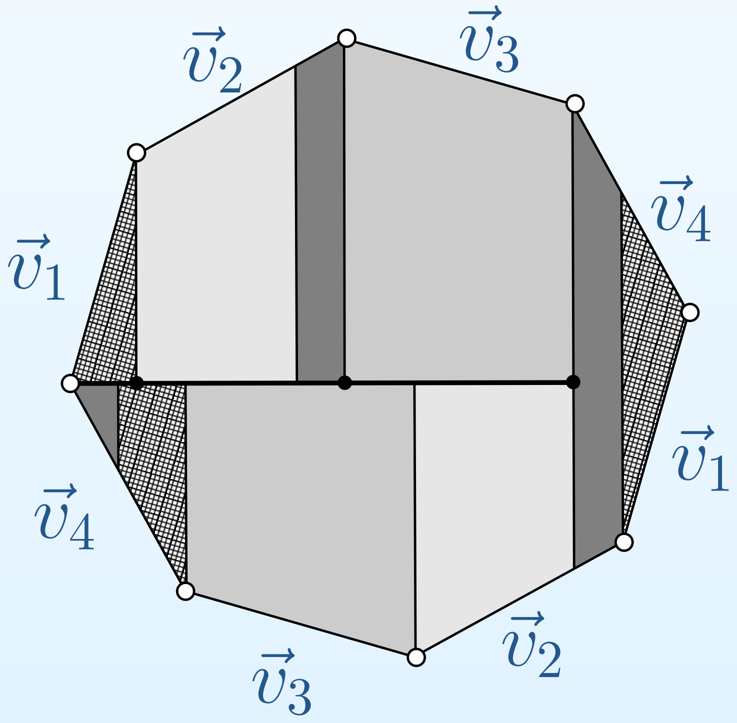
Zippered rectangles

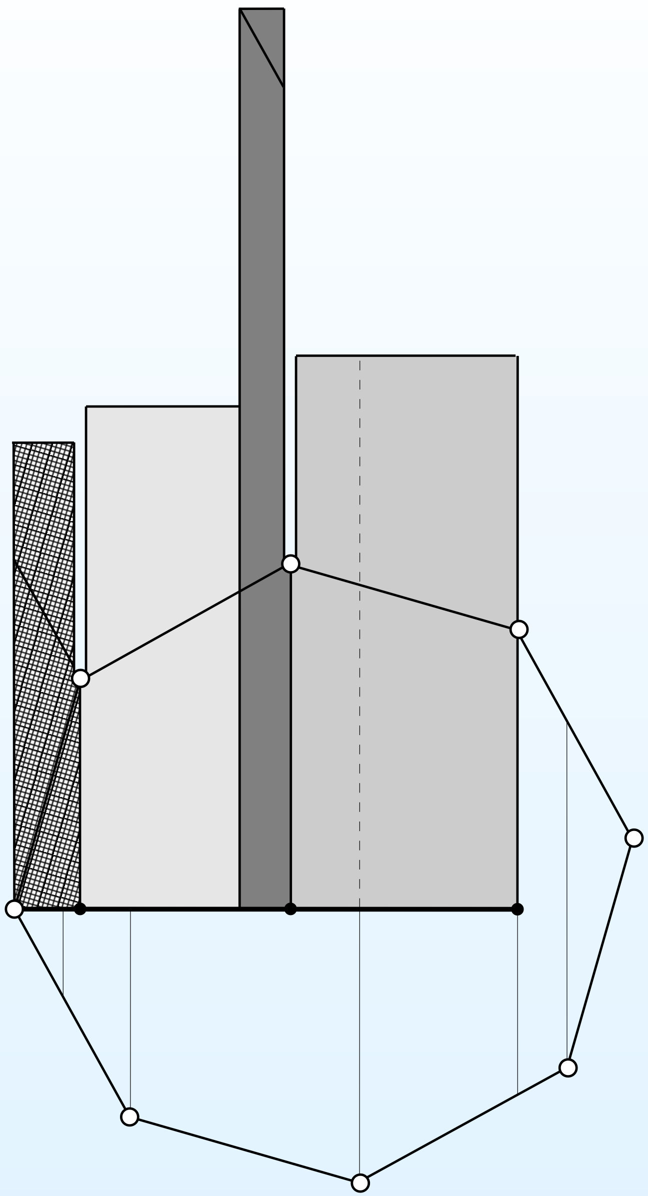
For a general flat surface S the first return map of the vertical flow to a horizontal segment X induces an interval exchange transformation $T : X \to X$.



Zippered rectangles

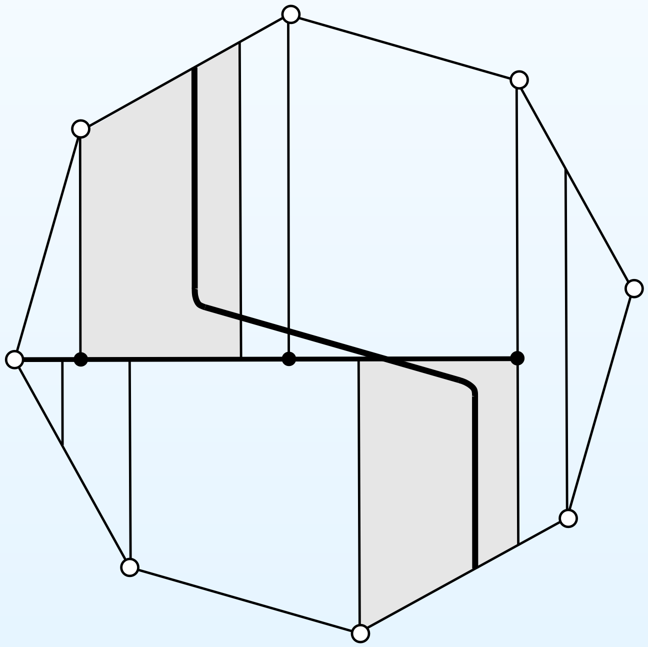
We get a decomposition of S into *zippered rectangles*.





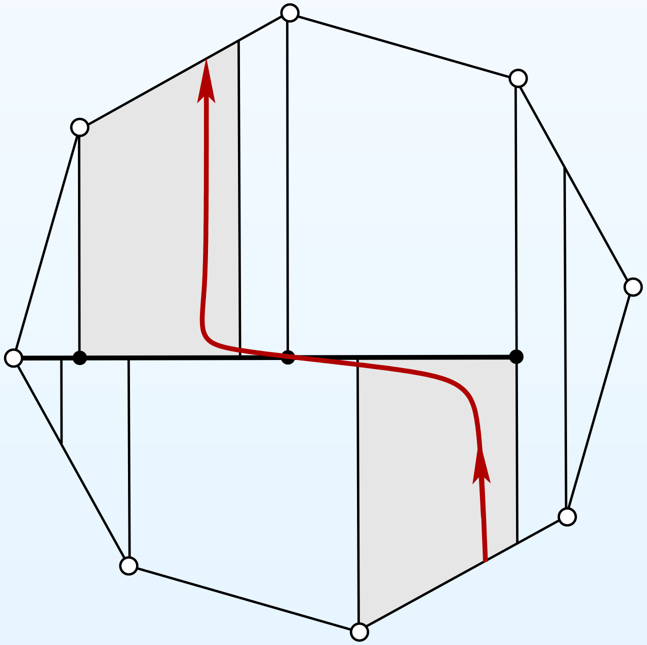
First return cycles

Launch the vertical trajectory from a point $x \in X$. When the trajectory intersects X for the first time join the corresponding point $T(x)$ to the original point x along X to obtain a closed loop $c(x)$. (In the picture this "first return cycle" is smoothed.)



First return cycles

We can deform such a closed path to a path which starts at the left endpoint of the interval, then follows the horizontal interval up to the point x then follows the leaf up to the point $T(x)$ of the first return and then follows the horizontal interval up to the left endpoint of the interval X.



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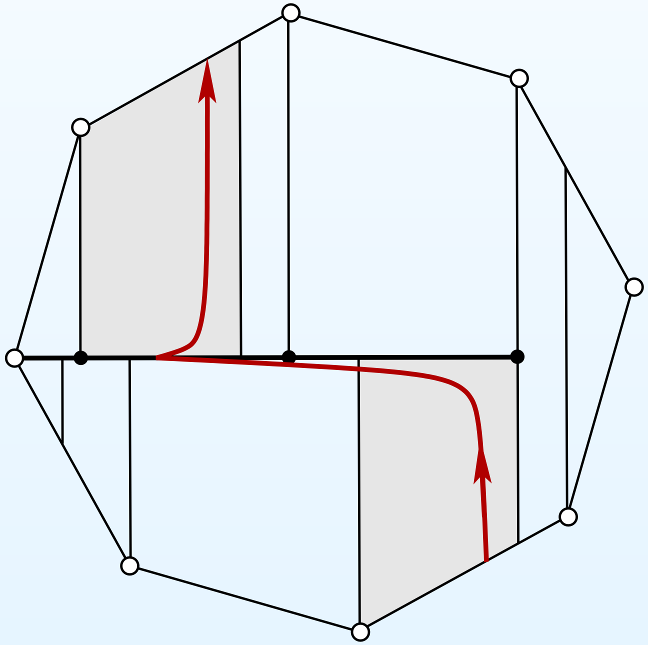

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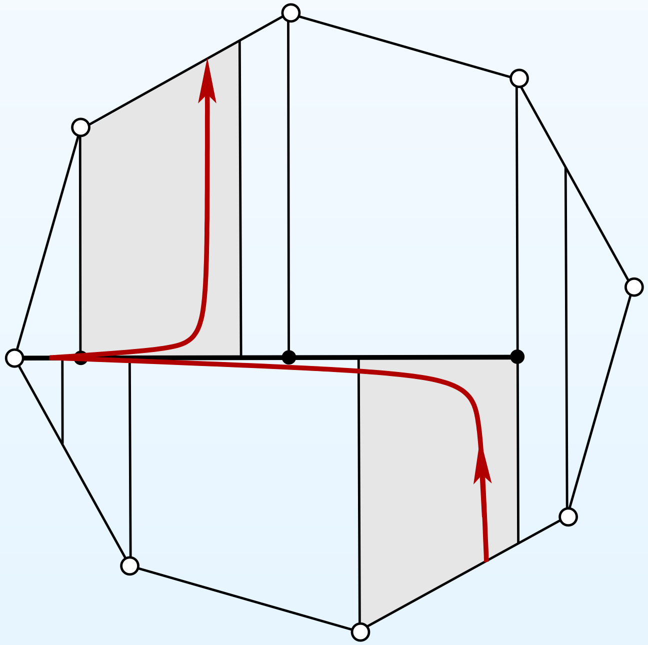

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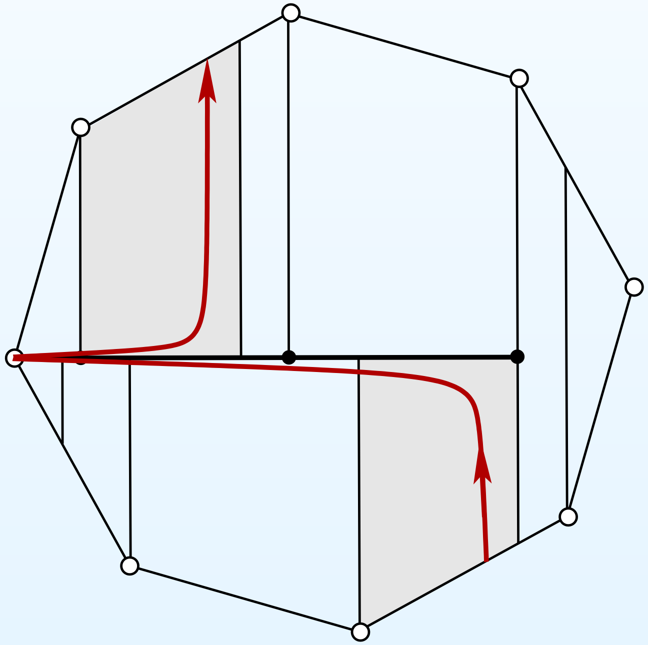
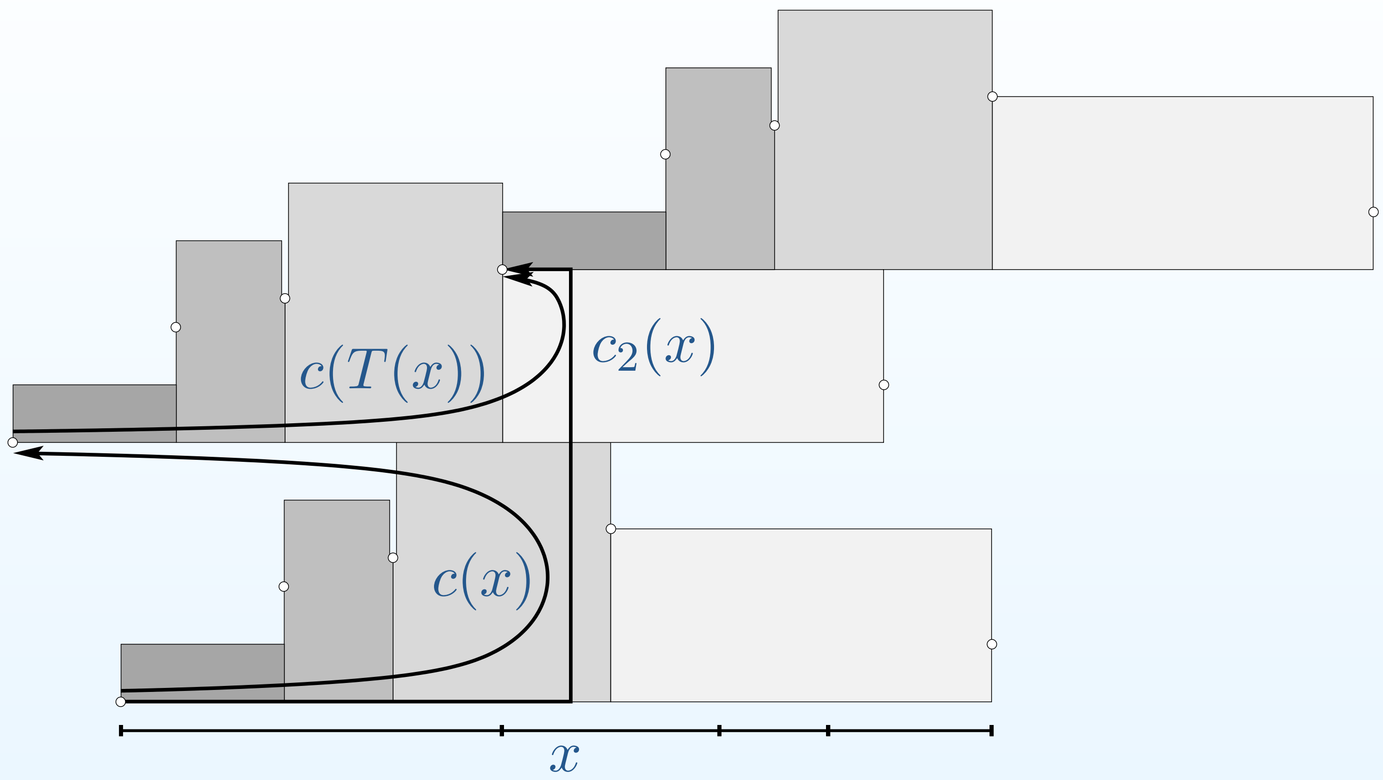


Decomposition of a cycle following a long piece of leaf

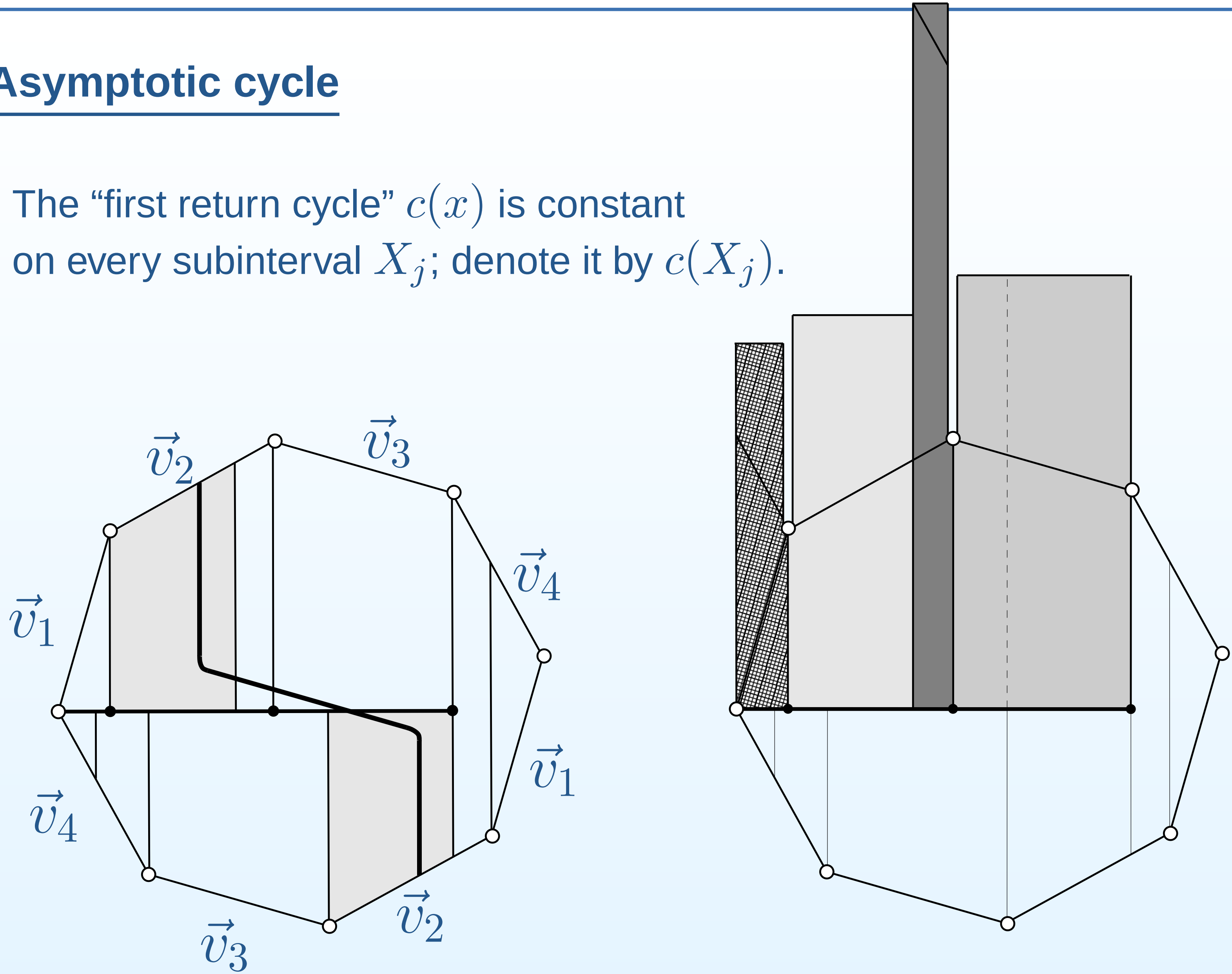


A homology cycle $c_2(x)$ obtained by joining the endpoints of a piece of leaf between a point $x \in X$ and $T(T(x))$ decomposes into the sum of basic cycles $c(x) + c(T(x))$. Similarly, the cycle $c_N(x)$ obtained after N returns of the vertical trajectory to X can be computed as:

$$c_N(x) = c(x) + c(T(x)) + \cdots + c(T^{N-1}(x))\,.$$

Asymptotic cycle

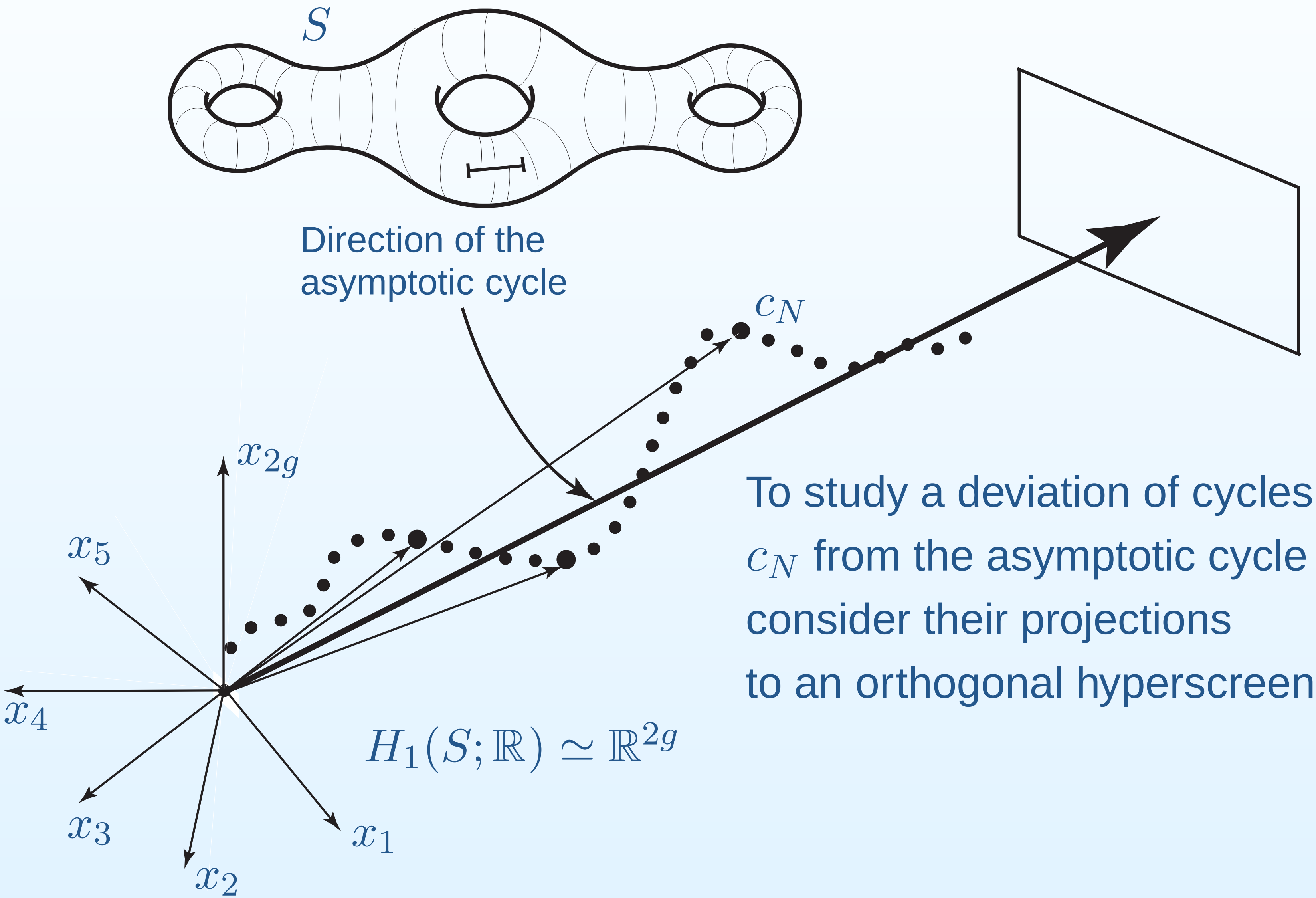
The “first return cycle” $c(x)$ is constant on every subinterval X_j; denote it by $c(X_j)$.



When the interval exchange is ergodic, the asymptotic cycle is computed as

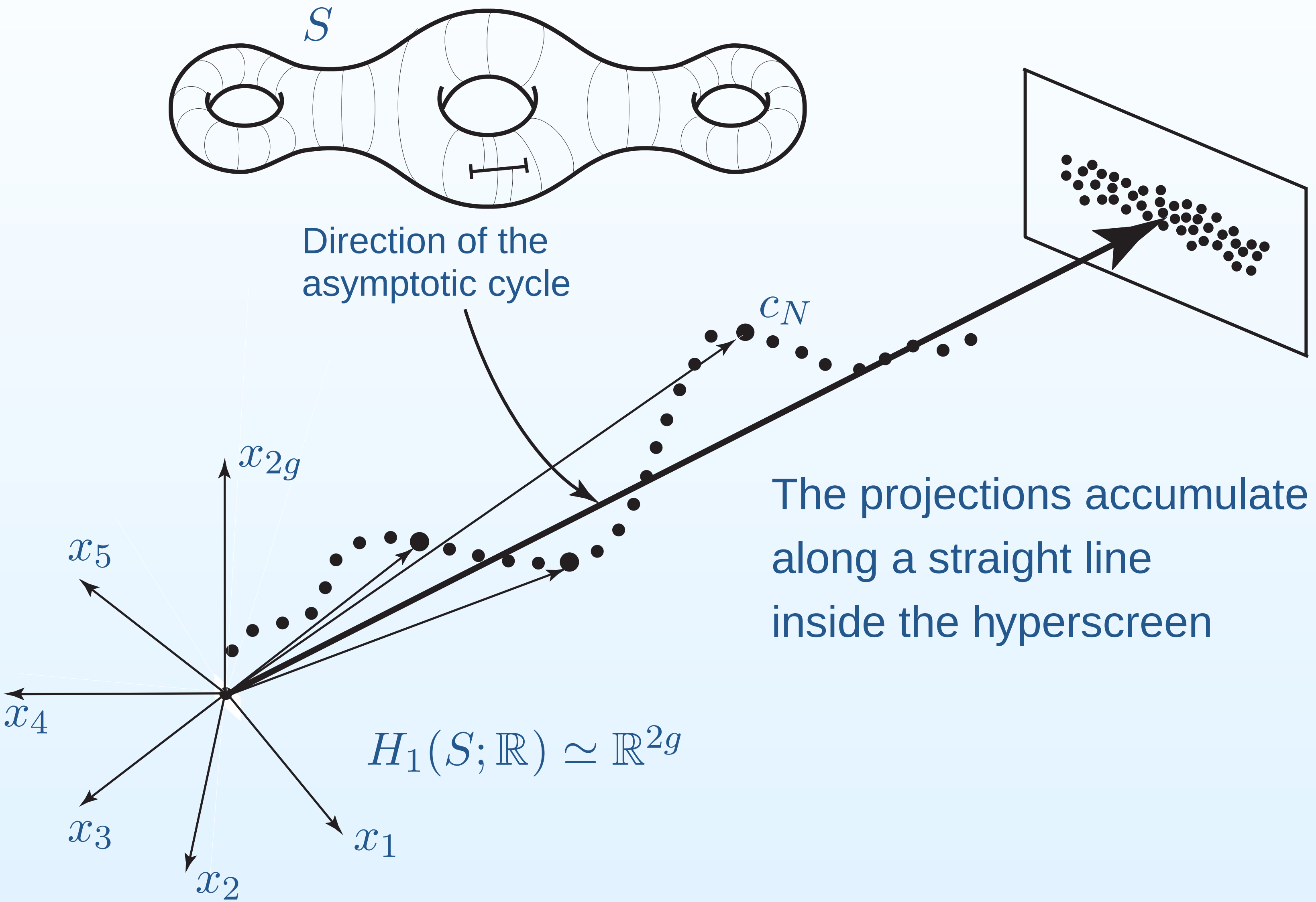
$$\lim_{N\to+\infty} \frac{1}{N} c_N(x) = \frac{|X_1|}{|X|} c(X_1) + \ldots \frac{|X_4|}{|X|} c(X_4)\,.$$

Asymptotic flag: empirical description



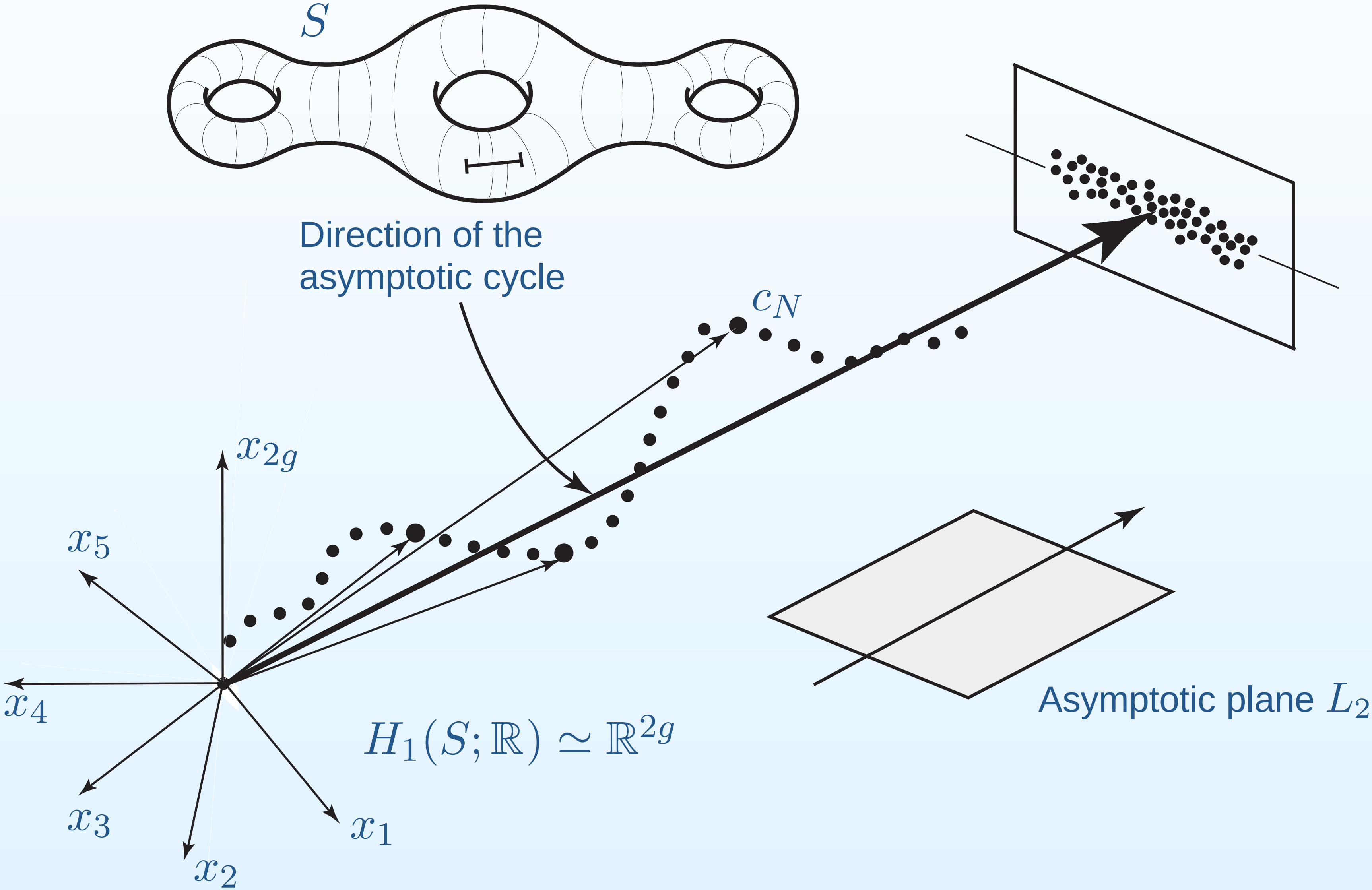
To study a deviation of cycles c_N from the asymptotic cycle consider their projections to an orthogonal hyperscreen

Asymptotic flag: empirical description



The projections accumulate along a straight line inside the hyperscreen

Asymptotic flag: empirical description



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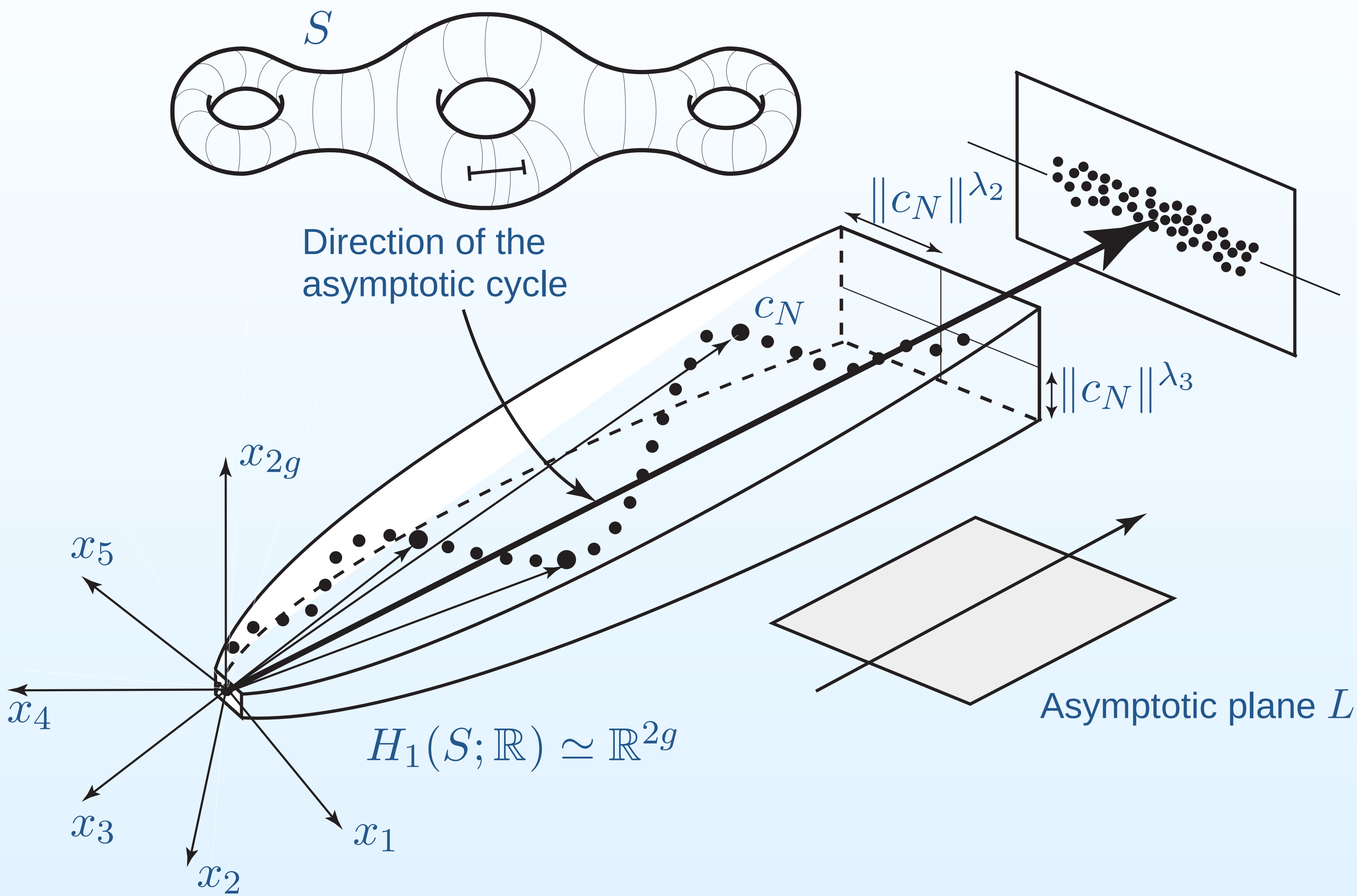


Asymptotic flag

Theorem (A. Zorich, 1999) *For almost any surface S in any stratum $\mathcal{H}_1(d_1,\dots,d_n)$ there exists a flag of subspaces $L_1 \subset L_2 \subset \dots \subset L_g \subset H_1(S;\mathbb{R})$ such that for any $j=1,\dots,g-1$*

$$\limsup_{N\to\infty} \frac{\log \operatorname{dist}(c_N, L_j)}{\log N} = \lambda_{j+1}$$

and

$$\operatorname{dist}(c_N, L_g) \le const,$$

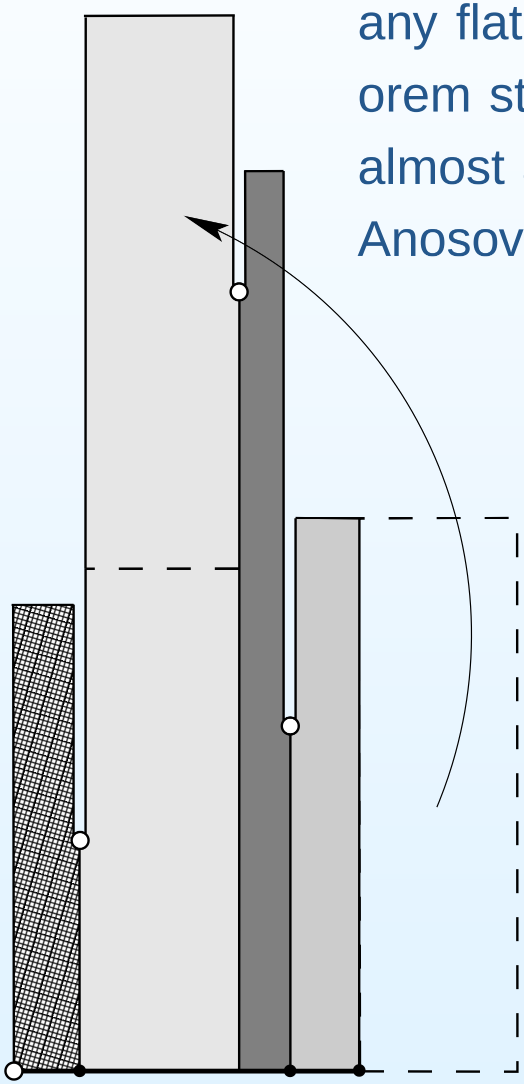
where the constant depends only on S and on the choice of the Euclidean structure in the homology space.

The numbers $1=\lambda_1 > \lambda_2 > \dots > \lambda_g$ are the top g Lyapunov exponents of the Hodge bundle along the Teichmüller geodesic flow on the corresponding connected component of the stratum $\mathcal{H}(d_1,\dots,d_n)$.

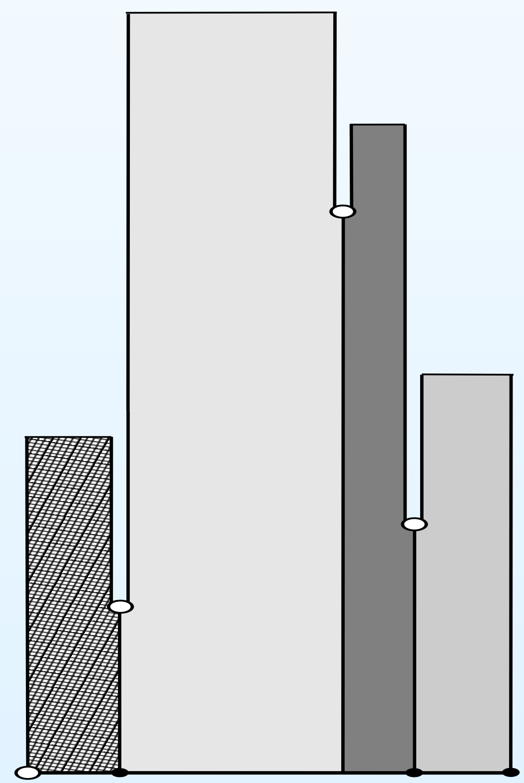
The strict inequalities $\lambda_g > 0$ and $\lambda_2 > \dots > \lambda_g$, and, as a corollary, strict inclusions of the subspaces of the flag, are difficult theorems proved later by Forni (2002) and A. Avila–M. Viana (2007).

Idea of a renormalization

By the theorem of Masur and Veech, the homogeneous expansion-contraction in vertical-horizontal directions regularly brings almost any flat surface, basically, back to itself. Multiplicative ergodic theorem states that, in a sense, there a matrix (one and the same for almost all flat surfaces) which mimics the matrix of a fixed pseudo-Anosov diffeomorphism as if the Teichmüller flow would be periodic.



$$\begin{pmatrix} e^{t_0} & 0 \\ 0 & e^{-t_0} \end{pmatrix}$$



Geometric interpretation of multiplicative ergodic theorem: spectrum of "mean monodromy"

Consider a vector bundle endowed with a flat connection over a manifold X^n. Having a flow on the base we can take a fiber of the vector bundle and transport it along a trajectory of the flow. When the trajectory comes close to the starting point we identify the fibers using the connection and we get a linear transformation $\mathcal{A}(x,1)$ of the fiber; the next time we get a matrix $\mathcal{A}(x,2)$, etc.

The multiplicative ergodic theorem says that when the flow is ergodic a *"matrix of mean monodromy"* along the flow

$$A_{mean} := \lim_{N\to\infty} \left(\mathcal{A}^*(x,N)\cdot\mathcal{A}(x,N)\right)^{\frac{1}{2N}}$$

is well-defined and constant for almost every starting point.

Lyapunov exponents correspond to logarithms of eigenvalues of this "matrix of mean monodromy".

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Hodge bundle and Gauss–Manin connection

Consider a natural vector bundle over the stratum with a fiber $H^1(S;\mathbb{R})$ over a "point" (S,ω), called the *Hodge bundle*. It carries a canonical flat connection called *Gauss—Manin connection*: we have a lattice $H^1(S;\mathbb{Z})$ in each fiber, which tells us how we can locally identify the fibers. Thus, Teichmüller flow on $\mathcal{H}_1(d_1,\dots,d_n)$ defines a multiplicative cocycle acting on fibers of this bundle.

The monodromy matrices of this cocycle are symplectic which implies that the Lyapunov exponents are symmetric:

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_g \geq -\lambda_g \geq \dots \geq -\lambda_2 \geq -\lambda_1$$

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Rauzy–Veech induction

Step of the Rauzy–Veech induction

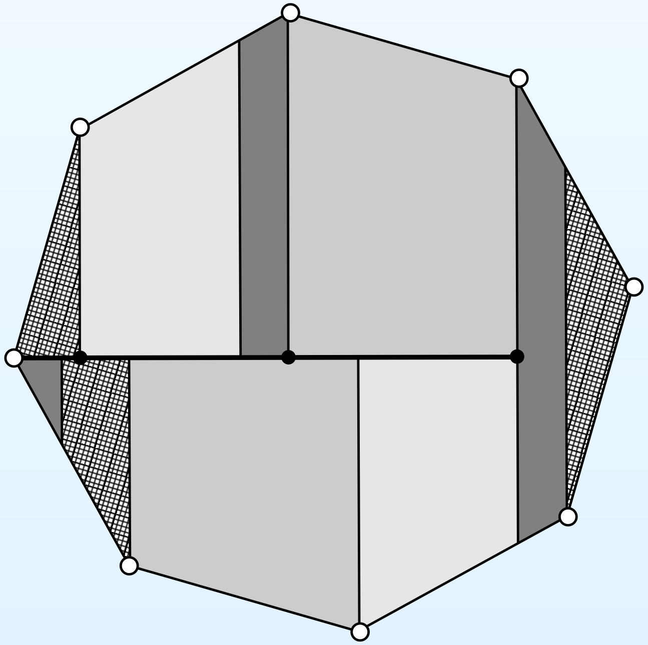
Assume that the vertical foliation on a translation surface S in some stratum $\mathcal{H}(d_1, \ldots, d_n)$ is uniquely ergodic. Launch a horizontal ray $(0x)$ in the positive direction from the singularity P_1. There is a discrete set of points b on the ray such that the first return map of the vertical flow to the subinterval $[0, b[$ is an interval exchange transformation with the minimal possible number of subintervals $m = 2g + n - 1$ under exchange. Choose among such b the smallest one satisfying $|0, b| \geq 1$. Denote by X the resulting horizontal interval of length $|0, b|$. By construction it has his left endpoint at P_1.

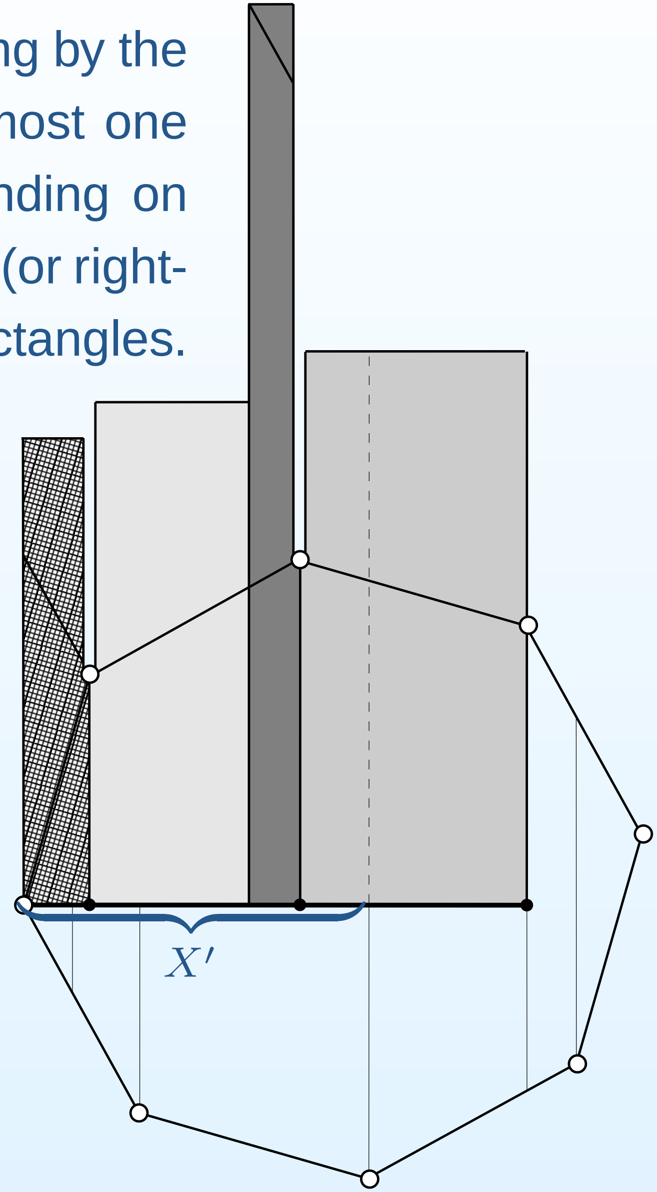
Unwrap the flat surface into m "zippered rectangles" with the base X. Denote by π the permutation such that the rectangles are glued to the bottom of the interval in the order $\pi^{-1}(1), \ldots, \pi^{-1}(m)$.

Compare the width λ_m (length of the horizontal sides) of the rightmost rectangle and the width $\lambda_{\pi^{-1}(m}$ of the rectangle which is pasted to the rightmost position. Chop the smallest of the two lengths from the right of the base X. The first return map of the vertical flow to the resulting subinterval X' is again an interval exchange transformation of m subintervals.

Rauzy–Veech induction geometrically

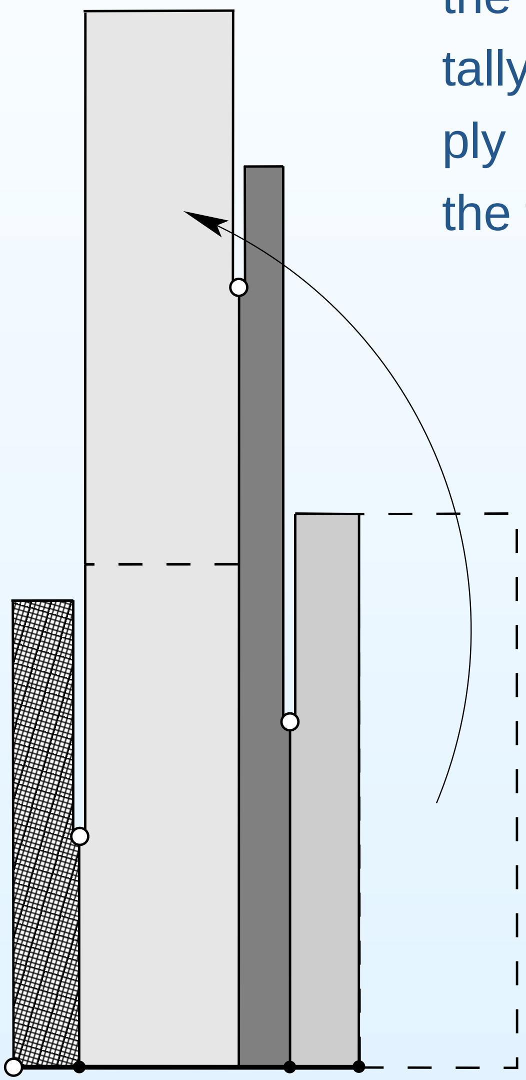
We shorten the base of the zippered rectangles building by the length of the smallest of the two intervals: the rightmost one and the one going to the rightmost position. Depending on which interval is shorter, the entire rightmost rectangle (or rightmost part of it) gets placed atop of one of the other rectangles.





Rauzy–Veech induction geometrically

We can apply the modification as above and then expand the resulting tall and narrow zippered rectangles horizontally and contract them vertically by a factor e^{t_0} or first apply expansion-contraction and them modify the building: the two operations commute.



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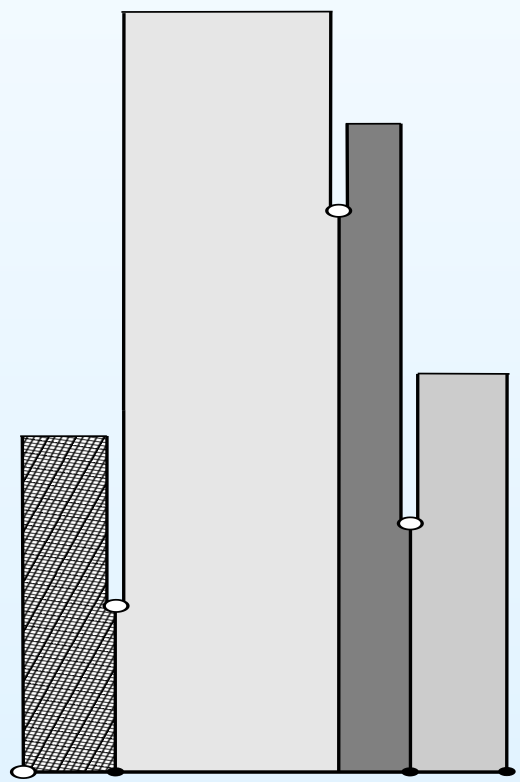

One step of an abstract renormalization

Consider a subinterval $X' \subset X$. Choose it in such way that that the first return map to X' induces an interval exchange transformation $T' : X' \to X'$ of the same number n of subintervals.

New first return cycles $c'(X'_k)$ to the interval X' are expressed in terms of the initial first return cycles $c(X_j)$ by linear relations; the lengths $|X'_k|$ of subintervals of the new partition $X' = X'_1 \sqcup \cdots \sqcup X'_n$ are expressed in terms of the lengths $|X_j|$ of subintervals of the initial partition by dual linear relations:

$$c'(X'_k) = \sum_{j=1}^{n} A_{jk} \cdot c(X_j) \qquad\qquad |X_j| = \sum_{k=1}^{n} A_{jk} \cdot |X'_k|\,,$$

Here a nonnegative integer matrix A_{jk} is completely determined by the initial interval exchange transformation $T : X \to X$ and by the choice of $X' \subset X$.

Time acceleration machine

To construct the cycle c_N representing a long piece of trajectory of the vertical flow we follow the trajectory $x, T(x), \dots, T^{N-1}(x)$ of the corresponding interval exchange transformation and compute the corresponding ergodic sum $c_N(x) = c(x) + \cdots + c(T^{N-1}(x))$.

Passing to a subinterval $X' \subset X$ we can follow the trajectory $x, T'(x), \dots, (T')^{N'-1}(x)$ of the new interval exchange transformation $T' : X' \to X'$. Since X' is shorter than X we cover the initial piece of trajectory of the vertical flow in a smaller number N' of steps.

Passing from T to T' we accelerate the time: that the trajectory $x, T'(x), \dots, (T')^{N'-1}(x)$ follows the trajectory $x, T(x), \dots, T^{N-1}(x)$ but jumps over several iterations of T at a time.

This renormalization is a “time acceleration machine”: instead of getting c_N by following the trajectory $x, ..., T^{N-1}(x)$ of the initial interval exchange transformation for the exponential time $N \sim \exp(const \cdot s)$ we obtain the cycle c_N applying only s steps of the renormalization map.

Strategy

Define an algorithm (generalizing Euclidean algorithm) which associates to an interval exchange transformation $T : X \to X$ some specific subinterval $X' \subset X$ and, hence, a new interval exchange transformation $T' : X' \to X'$ and rescale the length of the interval to one. The algorithm defines a map $\mathcal{T}$ from the space of all interval exchange transformations of a given number n of subintervals to itself.

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Applying recursively this algorithm we construct a sequence of subintervals $X = X^{(0)} \supset X^{(1)} \supset X^{(2)} \supset \dots$ and a sequence of matrices $A = A(T^{(0)}), A(T^{(1)}), \dots$ describing transitions form interval exchange transformation $T^{(r)} : X^{(r)} \to X^{(r)}$ to interval exchange transformation $T^{(r+1)} : X^{(r+1)} \to X^{(r+1)}$.

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Taking a product $A^{(s)} = A(T^{(0)}) \cdot A(T^{(1)}) \cdot \dots \cdot A(T^{(s-1)})$ we can immediately express the "first return cycles" to a microscopic subinterval $X^{(s)}$ in terms of the initial "first return cycles" to X.

Cocycle

We get a matrix-valued function A on the space of interval exchange transformations, and we study the products of matrices A along the orbits $T^{(0)}, T^{(1)}, \ldots, T^{(s-1)}$ of the map on the space of interval exchange transformations. The properties of these products are described by the Oseledets (multiplicative ergodic) theorem, and the cycles c_N have a deviation spectrum governed by the Lyapunov exponents of the cocycle A on the space of interval exchange transformations.

Conceptually, the relation between the the cocycle A over the map $\mathcal{T}$ and the Teichmüller geodesic flow was elaborated in the original paper of W. Veech.

Fundamental domain

Consider the space of flat tori. A point of the fundamental domain provides a canonical parallelogram pattern for the torus. However, any parallelogram in the corresponding lattice represents the same flat torus.

A “canonical” zippered rectangles decomposition of a flat surface also belongs to some fundamental domain defined by a specific choice of a “canonical” horizontal interval X.

Place the left endpoint of X at a conical singularity and choose the length of X in such way that the first return map $T : X \to X$ induced by the vertical flow has minimal possible number $n = 2g + n - 1$ of subintervals.

Among all such horizontal segments X choose the shortest one, which still has length $|X| \geq 1$.

A finite ambiguity corresponds to a finite freedom in the choice of the conical singularity and in the choice of the horizontal ray adjacent to it.

First return of the Teichmüller flow

Morally, the space of zippered rectangles is a ramified covering over the space of flat surfaces. Teichmüller geodesic flow lifts naturally to the space of zippered rectangles. It acts on zippered rectangles by expansion in the horizontal direction and by contraction in the vertical one; i.e. zippered rectangles are modified by a linear transformation

$$\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

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As soon as the Teichmüller geodesic flow brings us out of the fundamental domain, we have to modify the zippered rectangles decomposition to the "canonical one" corresponding to the fundamental domain. The corresponding modification of zippered rectangles (chop an appropriate rectangle on the right, put it atop the corresponding rectangle) corresponds to the *Rauzy—Veech induction*.

Teichmüller geodesic flow versus Rauzy induction

From Teichmüller flow to a discrete map on zippered rectangles

Consider some codimension one subspace Υ in the space of zippered rectangles transversal to the Teichmüller geodesic flow. Choice of Veech: boundary $|X| = 1$ of the fundamental domain.

Teichmüller geodesic flow defines the first return map $\mathcal{S} : \Upsilon \to \Upsilon$ acting as follows. Take a flat surface of unit area decomposed into zippered rectangles Z with the base X of length one. Apply expansion in horizontal direction and contraction in vertical direction. For some $t_0(Z)$ the deformed zippered rectangles can be rearranged to get back to the base of length one; the result is the image of the map $\mathcal{S}$.

Actually, we can first apply the *"cut from right and paste atop"* to the initial zippered rectangles Z and then apply the transformation $\begin{pmatrix} e^{t_0} & 0 \\ 0 & e^{-t_0} \end{pmatrix}$ — the two operations commute.

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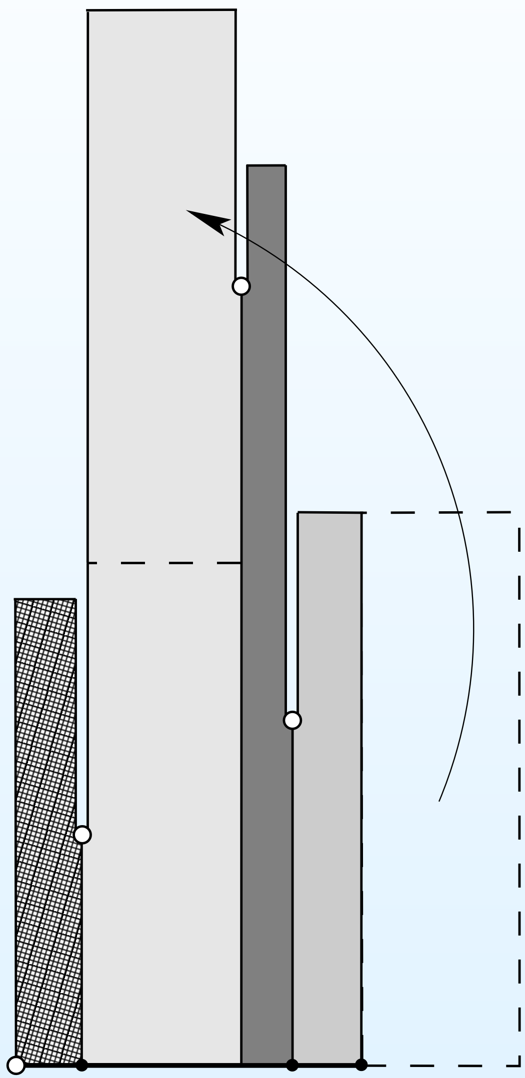
Expand the resulting tall and narrow zippered rectangles horizontally and contract them vertically by a factor e^{t_0}.



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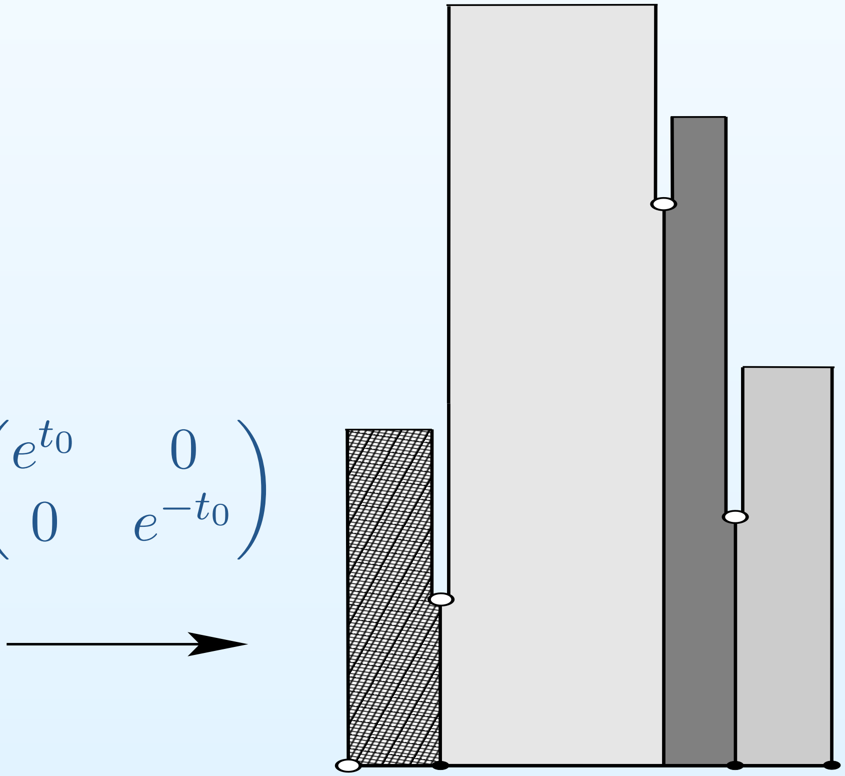

From zippered rectangles to interval exchanges

Zippered rectangles pattern naturally defines an interval exchange transformation — the first return map of the vertical flow to the base X of zippered rectangles.

The map $\mathcal{S}$ of the subspace Υ of zippered rectangles projects to our map $\mathcal{T}$ on the space of interval exchange transformations:

$$\begin{array}{ccc}
\text{Section } \Upsilon \text{ in the space} & & \text{Section } \Upsilon \text{ in the space} \\
\text{of zippered rectangles} & \xrightarrow{\mathcal{S}} & \text{of zippered rectangles} \\
\downarrow & & \downarrow \\
\text{Space of interval} & & \text{Space of interval} \\
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\end{array}$$

The map $\mathcal{S} : \Upsilon \to \Upsilon$ induced by the first return of the Teichmüller flow to the section Υ in the space of zippered rectangles is a suspension of the map $\mathcal{T}$ on the space of interval exchanges.

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The map $\mathcal{S} : \Upsilon \to \Upsilon$ induced by the first return of the Teichmüller flow to the section Υ in the space of zippered rectangles is a suspension of the map $\mathcal{T}$ on the space of interval exchanges.

From zippered rectangles to interval exchanges

Zippered rectangles pattern naturally defines an interval exchange transformation — the first return map of the vertical flow to the base X of zippered rectangles.

The map $\mathcal{S}$ of the subspace Υ of zippered rectangles projects to our map $\mathcal{T}$ on the space of interval exchange transformations:

$$\begin{array}{ccc}
\text{Section } \Upsilon \text{ in the space} & & \text{Section } \Upsilon \text{ in the space} \\
\text{of zippered rectangles} & \xrightarrow{\mathcal{S}} & \text{of zippered rectangles} \\
\downarrow & & \downarrow \\
\text{Space of interval} & & \text{Space of interval} \\
\text{exchange transformations} & \xrightarrow{\mathcal{T}} & \text{exchange transformations}
\end{array}$$

The map $\mathcal{S} : \Upsilon \to \Upsilon$ induced by the first return of the Teichmüller flow to the section Υ in the space of zippered rectangles is a suspension of the map $\mathcal{T}$ on the space of interval exchanges.