The Seiberg-Witten equations (Reference: Manolescu notes 3.6–3.8)

Connection and curvature

E: vector bundle over X

$\Omega^k(X;E) = \Gamma(\Lambda^k T^*X \otimes E) = \{E\text{-valued } k\text{-forms}\}$

Definition: A connection A on E is a linear map

covariant derivative ∇_A $\rightsquigarrow \nabla^A : \Omega^0(X;E) \to \Omega^1(X;E)$ s.t

$\nabla^A(fs) = df \otimes s + f\nabla^A s \quad \forall f: X \to \mathbb{R} \quad s \in \Omega^0(X;E) = \Gamma(E)$

In particular, given vector field $v \in \Gamma(T_*X)$ $s \in \Gamma(E)$

$\nabla^A_v s := (\nabla^A s)\cdot v \in \Gamma(E)$, covariant derivative of s along v.

Given any ∇_A, ∇_B, we have

$(\nabla^A - \nabla^B)(fs) = f\cdot(\nabla^A - \nabla^B)\cdot(s)$

so $\nabla^A - \nabla^B : \Omega^0(X;E) \to \Omega^1(X;E)$ is linear w.r.t. multiplication with $f \in \Omega^0(X)$.

so $((\nabla^A - \nabla^B)s)(x)$ only depends on $s(x)$

$\forall A$

$A = \underset{\text{base}}{A_0} + \underset{\Omega^1(X;\mathrm{End}E)}{\alpha}$

So $s(x) \in E_x \longmapsto ((\nabla^A - \nabla^B)s)(x) \in T^*_xX \otimes E_x$ is an well-defined element in $\Gamma(T^*_xX \otimes \mathrm{End}(E_x))$.

so $\nabla^A - \nabla^B \in \Omega^1(X;\mathrm{End}E) = \Gamma(T^*X \otimes \mathrm{End}E)$

so $\{$connections on $E\}$ is affine over $\Omega^1(X;\mathrm{End}E)$.

E: Hermitian bundle. We say ∇ is a unitary-connection if $\quad d\langle s,t\rangle = \langle \nabla s, t\rangle + \langle s, \nabla t\rangle$

$$\begin{tikzcd} GL(n,\mathbb{C}) \arrow[r, hookrightarrow] & P \arrow[d] \\ \dim_{\mathbb{C}} E & X \end{tikzcd}$$

If ∇^A, ∇^B are unitary connections. then

$$\nabla^A - \nabla^B \in \Omega^1(X, \mathfrak{u}(E)) \subset \Omega^1(X, \mathrm{End}(E))$$

$$\begin{tikzcd} U(n) \arrow[r, hookrightarrow] & P \arrow[d] \\ & X \end{tikzcd}$$

$\mathfrak{u}(E)_x = \{ \ell : E_x \to E_x \mid \ell^* = -\ell \}$

We can extend ∇^A to $\nabla^A : \Omega^k(X;E) \to \Omega^{k+1}(X;E)$ by requiring that

$$\nabla^A(\alpha \otimes s) = d\alpha \otimes s + (-1)^k \alpha \wedge \nabla^A s, \quad \forall \alpha \in \Omega^k(X) \ s \in \Gamma(E).$$

Lemma: $\forall f \in \Omega^0(X)$, $s \in \Omega^k(X;E)$

$$\nabla^A \nabla^A (fs) = f \cdot \nabla^A \nabla^A s.$$

proof: $\nabla^A \nabla^A (fs) = \nabla^A (df \cdot s + f \nabla^A s)$

$$= ddf \cdot s - df \cdot \nabla^A s + df \cdot \nabla^A s + f \nabla^A \nabla^A s$$

$\Omega^0(E) \xrightarrow{\nabla_A} \Omega^1(E) \xrightarrow{\nabla_A} \Omega^2(E) \to \cdots$

$$= f \nabla^A \nabla^A s \qquad \square$$

So $\exists F_A \in \Omega^2(X; \mathrm{End}(E))$ s.t. $\nabla^A \nabla^A s = F_A \wedge s$, $\quad s \in \Omega^k(X,E)$

F_A is called the curvature of A.

Under a local trivilization $E|_U = U \times \mathbb{C}^n$, we have

$\nabla^A s = ds + A \cdot s \qquad A \in \Omega^1(U, \mathrm{End}(E))$

Then $F_A = dA + A \wedge A$.

Chern-Weil formula: $\sum t^k c_k(E) = \det\left(I - \frac{tF_A}{2\pi i}\right)$
formal variable.

$F_A \in \Omega^2(\mathrm{End}(E))$

In particular, $c_1(E) = \left[\frac{i}{2\pi}\mathrm{Tr}(F_A)\right] \in H^2(X;\mathbb{R})$. $\mathrm{Tr}F_A \in \Omega^2(\mathbb{C})$

Connection A on $E \rightsquigarrow$ connection on $\Lambda^k E$ $\quad k = \dim_{\mathbb{C}} E$

(In particular, a connection A^{τ} on $\det(E)$) $\quad$ $\mathrm{Tr}(F_A) = $

If A is unitary, then A^{τ} is unitary. So $F_{A^{\tau}} \in \Omega^2(X; i\mathbb{R})$

Now fix a spin$^{\mathbb{C}}$ structure on X. $\rho: T_*X \to \mathrm{End}(S)$

$s \in \Gamma(S)$

We say a unitary connection A on S is spin$^{\mathbb{C}}$ connection if

$$\nabla_A(\rho(v)\cdot\phi) = \rho(v)\cdot\nabla_A\phi + \rho(\nabla_{LC} v)\phi$$

$\forall\, v \in \Gamma(T_*X)$ $\phi \in \Gamma(S)$. $\quad \nabla_{LC} \in \Omega^1(X; T_*X)$

Altenative definition: $\mathrm{Spin}^{\mathbb{C}} \longrightarrow SO(n)$

connection on $S \longmapsto$ connection on T_*X

We say A is spin$^{\mathbb{C}}$ if $A \longmapsto$ Levi-Civita

If A, B are both spin$^{\mathbb{C}}$ connections $\nabla := \nabla^A - \nabla^B$

Then $\nabla(\rho(v)\cdot s) = \rho(v)\cdot \nabla s$ $\quad \nabla \in \Omega^1(X; \mathrm{End}(S,\rho))$

On the other hand, $\nabla \in \Omega^1(X; u(S))$

so $\nabla \in \Omega^1(X; i\mathbb{R})$ $\qquad \mathrm{End}(S,\rho) \cap u(S) = i\cdot\mathbb{R}$

so {spin$^{\mathbb{C}}$-connections} is an affine space over $\Omega^1(X; i\mathbb{R})$.

$$\rho : \underset{\substack{|2| \\ T_*X}}{T^*X} \to \mathrm{End}(S)$$

Given a spin$^{\mathbb{C}}$ connection A, we define the Dirac operator $D_A : \Gamma(S) \xrightarrow{\nabla_A} \Gamma(T^*X \otimes S) \xrightarrow{\rho} \Gamma(S)$

$$\alpha \otimes \phi \longmapsto \rho(\alpha)\cdot\phi$$

$S = S^+ \oplus S^-$, so $\underline{\underline{D_A}} = \begin{bmatrix} 0 & D_A^+ \\ D_A^- & 0 \end{bmatrix}$. $\quad D_A^+ : \Gamma(S^+) \to \Gamma(S^-)$

Alternative definition: Take orthonormal basis $\{e_1, e_2, \cdots, e_n\}$ of $T_{\underline{\underline{x}}}X$.

Then $(D\phi)(x) = \sum_{k=1}^{n} \rho(e_i)\cdot(\nabla^A_{e_i}\phi)(x)$

Example: $X = \underline{\underline{\mathbb{R}^n}}$, standard metric, trivial spin$^{\mathbb{C}}$ connection.

Then $\not{D}\phi = \sum \rho(e_i)\cdot \dfrac{\partial\phi}{\partial x_i}$. If we set $\rho(e_i) = A_i$

Then $\not{D}\phi = \sum A_i \cdot \dfrac{\partial \phi}{\partial x_i}$ with $A_i^2 = -1$ $\quad A_i \cdot A_j = -A_j A_i$ $\;(i \neq j)$

$D_A : \Gamma(S) \to \Gamma(S)$ $\qquad$ $\Gamma(S^+) \underset{D_A^-}{\overset{D_A^+}{\rightleftarrows}} \Gamma(S^-)$

Properties of Dirac operator

- If $A' = A + \alpha$ $\quad \alpha \in \Omega^1(X; i\mathbb{R})$

 Then $\not{D}_{A'}\phi = \not{D}_A\phi + \rho(\alpha)\cdot\phi$

- $\not{D}$ is formally self-adjoint. i.e.

 $\langle \not{D}\phi, \psi \rangle_{L^2} = \langle \phi, \not{D}\psi \rangle_{L^2}$ $\quad \forall \phi, \psi \in \Gamma(S)$

- Weitzenbock formula. $\qquad \rho : \underline{\underline{T^*X}} \to \mathrm{End}(S)$

$\not{D}_A^2 : \Gamma(S) \to \Gamma(S)$ Dirac Laplacian $\quad \Lambda^k T^*X \to \mathrm{End}(S)$

$\Omega^0(X;S) \xrightarrow{\nabla_A} \Omega^1(X;S)$ has a formal-adjoint

$\Omega^1(X;S) \xrightarrow{\nabla_A^*} \Omega^0(X,S)$ s.t.

$\langle \nabla_A s, \alpha \rangle_{L^2} = \langle s, \nabla_A^* \alpha \rangle$ $\quad \forall s \in \Omega^0(X;S)$
$\qquad \alpha \in \Omega^1(X;S)$.

$\nabla_A^* \nabla_A : \Gamma(S) \to \Gamma(S)$ Laplacian. $\qquad$ Scalar curvature

Formula: $\not{D}_A^2\phi = (\nabla_A)^*\nabla_A\phi + \rho(F_{A^c}^+)\cdot\phi + \frac{s}{4}\phi$

$F_{A^c}^+ = \frac{F_{A^c} + * F_{A^c}}{2}$ $\qquad$ A^c: connection on $\det(S)$

The Seiberg-Witten equations

Recall $\rho: \Lambda^* T^*X \to \mathrm{End}(S)$ restricts to

$$\rho: \Lambda^2_+ T^*X \xrightarrow{\cong} su(S^+)$$

Here $su(S^+)_x = \{ L: S^+_x \to S^+_x \text{ s.t. } L^* = -L, \mathrm{tr}(L) = 0 \}$

$$\rho: \Omega^2_+(T^*X; i\mathbb{R}) \xrightarrow{\cong} \{ S \in \mathrm{End}(S^+) \mid S^* = S, \mathrm{tr}(S) = 0 \}$$

Given $\phi \in \Gamma(S^+)$, we have

$\phi^*\phi \in \mathrm{End}(S^+)$ $\quad \psi \longrightarrow \langle \psi, \phi \rangle \cdot \phi$

$(\phi^*\phi)_0 : \psi \longmapsto \langle \psi, \phi \rangle \cdot \phi - \frac{|\phi|^2}{2} \psi$

$(\phi^*\phi)_0$ satisfies : 1) $((\phi^*\phi)_0)^* = (\phi^*\phi)_0$ 2) $\mathrm{tr}(\phi^*\phi)_0 = 0$

So $\rho^{-1}(\phi^*\phi)_0 \in \Omega^2_+(X; i\mathbb{R})$

fix : Riemanian metric
spin$^{\mathbb{C}}$ str.

Seiberg-Witten equations (A, ϕ) $\qquad \phi \in \Gamma(S^+)$

$$\begin{cases} F^+_{A^{\mathbb{C}}} - \rho^{-1}(\phi^*\phi)_0 = 0 \\ \underbrace{\not{D}^+_A \phi = 0}_{} \end{cases}$$

$\leftarrow$ curvature eq. Solve : A, ϕ

$\uparrow \Omega^2_+(X; i\mathbb{R})$

$\leftarrow$ Dirac eq.

A : spin$^{\mathbb{C}}$ connection $\quad A^{\mathbb{C}}$: induced connection on $\det(S^+)$

$$F^+_{A^{\mathbb{C}}} = \frac{F_{A^{\mathbb{C}}} + *F_{A^{\mathbb{C}}}}{2} \in \Omega^2_+(X; i\mathbb{R})$$

$\int \langle \not{D}^+_A \phi, \phi \rangle$

$+ \int |F_A|^2 + \cdots$

Gauge symmetry

$$\begin{tikzcd} \boxed{S^+} \arrow[r, "\nabla_A"] \arrow[d, "u"] & \Omega^1(S^+) \arrow[d, "u"] \\ S^+ \arrow[r, "\nabla_{A-u^{-1}du}"] & \Omega^1(S^+) \end{tikzcd}$$

$\mathrm{End}(S,\rho)$

The gauge group $\mathcal{G} := \Gamma(X, \mathrm{End}(S^+, \underline{\rho}))$
$$= C^\infty(X, U(1))$$

$\mathcal{G}$-acts on $\{(A,\phi)\}$ by $u\cdot(A,\phi) = (A - u^{-1}du, u\cdot\phi)$

$u^{-1}du = u^*(id\theta)$

$$\not{D}^+_{A-u^{-1}du}(u\phi) = u\cdot \not{D}^+_A\phi$$

$$(u\cdot A)^t = A^t - 2u^{-1}du \Rightarrow F_{A^t} = F_A$$

So $\{(A,\phi) \mid F^+_{A^t} = \rho^{-1}(\phi^*\phi)_0,\ \not{D}^+_A\phi = 0\}$ has an action of $\mathcal{G}$.

The Seiberg-Witten moduli space

$$\mathcal{M}_{SW} := \{(A,\phi) \text{ solutions}\}/\mathcal{G}$$

$\underline{u\cdot A} = \underline{A} - u^{-1}du$

$d^*(u^{-1}du) = 0$

To get rid of this large symmetry group, we can pick a base spin^c connection A_0.

$A = A_0 + \alpha \quad \alpha \in \Omega^1(X; i\mathbb{R})$

$d^*\alpha = 0$

We say A is in Coulomb gauge with respect to A_0 if

$$d^*(A - A_0) = 0$$

Coulomb gauge fixing eq.

$$\mathcal{M}_{SW} = \{(A,\phi) \mid F^+_A = \rho^{-1}(\phi^*\phi)_0,\ \not{D}^+_A\phi = 0,\ \underline{d^*(A-A_0) = 0}\}/\mathcal{G}_h$$

Here $\mathcal{G}_h = \{\text{harmonic map } X \to S^1\} \cong H^1(X;\mathbb{Z}) \times S^1$.

Given A $\exists\, u$ s.t. $d^*(A - u^{-1}du - A_0) = 0$

$\mathcal{M}_{SW} = \{\text{solutions of S.W. eqs}\}/\mathcal{G}$

Theorem: $\mathcal{M}_{SW}$ is compact.

proof: Assume (A, ϕ) is a solution

Then $\not{D}_A^+ \phi = 0 \Rightarrow \not{D}_A^- \not{D}_A^+ \phi = 0$

$\Rightarrow \nabla_A^* \nabla_A \phi + \frac{s}{4}\phi + \frac{1}{2}\rho(F_A^+)\phi = 0$

$\Rightarrow \langle \nabla_A^* \nabla_A \phi, \phi \rangle + \frac{s}{4}|\phi|^2 + \frac{1}{2}\langle \rho(F_{A^c}^+)\phi, \phi \rangle = 0$

$\frac{1}{2}\langle \rho(F_{A^c}^+)\phi, \phi \rangle = \frac{1}{2}\langle (\phi\phi^*)_0 \phi, \phi \rangle$

$= \frac{1}{2}\langle \frac{1}{2}|\phi|^2\phi, \phi \rangle = \frac{1}{4}|\phi|^4$

$\langle \nabla_A^* \nabla_A \phi, \phi \rangle + \frac{s}{4}|\phi|^2 + \frac{1}{2}|\phi|^4 = 0.$

$\Delta |\phi|^2 = d^* d \langle \phi, \phi \rangle$

$= d^*(\langle \nabla_A \phi, \phi \rangle + \langle \phi, \nabla_A \phi \rangle)$

$= 2\langle \nabla_A^* \nabla_A \phi, \phi \rangle - 2\langle \nabla_A \phi, \nabla_A \phi \rangle$

$\leq 2\langle \nabla_A^* \nabla_A \phi, \phi \rangle$

so $\Delta |\phi|^2 \leq -\frac{s}{2}|\phi|^2 - |\phi|^4$

suppose $|\phi|$ achieves it's max at x_0

Then

$$0 \le \Delta|\phi|^2(x_0) \le -\frac{S}{2}|\phi(x_0)|^2 - |\phi(x_0)|^4$$

so $|\phi(x_0)| = 0$ or $\le \sqrt{-\frac{S}{2}}$

so $C^0(|\phi|) \le C$

Now suppose we have a sequence $\{(A_i, \phi_i)\}$ of solutions. After gauge transformation, we may assume $A_i = A_0 + \alpha$ $\quad d^*\alpha_i = 0$

A_0: base connection

(If $b_1(X) > 0$, also assume $\Omega^1(x) \longrightarrow \mathcal{H}^1$
$\{\alpha_i\} \longmapsto$ bounded

so
$$\begin{cases} 2d^+\alpha_i = -F^+_{A_0} + \rho^{-1}(\phi_i^*\phi_i)_0 \\ \not{D}_{A_0}\phi_i = -\rho(\alpha_i)\cdot\phi_i \\ d^*\alpha_i = 0 \end{cases}$$

LHS: elliptic operator, we can do a bootstraping argument to show that $C^k(\alpha_i, \phi_i)$ bounded for any k. So after passing to subsequence (α_i, ϕ_i) converges in C^∞ topology.