2020 · 10 · 19

finite graph $G=(V,E)$

$\sigma \in \{+1,-1\}^V$

Hamiltonian $H(\sigma) = -\sum_{x\sim y} \sigma_x \sigma_y$

$H(\sigma) = -(\#\text{agree} - \#\text{disagree})$

$= 2\#\text{disagree}(\sigma) - \#E.$

Ising model : $\beta>0$ inverse temperature

$$\mu_{\beta,G}[\sigma] = \frac{1}{Z}\exp(-\beta H(\sigma))$$

$$Z = \sum_{\sigma} e^{-\beta H(\sigma)}.$$

$\partial G \subset V$ boundary set

Fix some boundary conditions $b \in \{+1,-1\}^{\partial G}$.

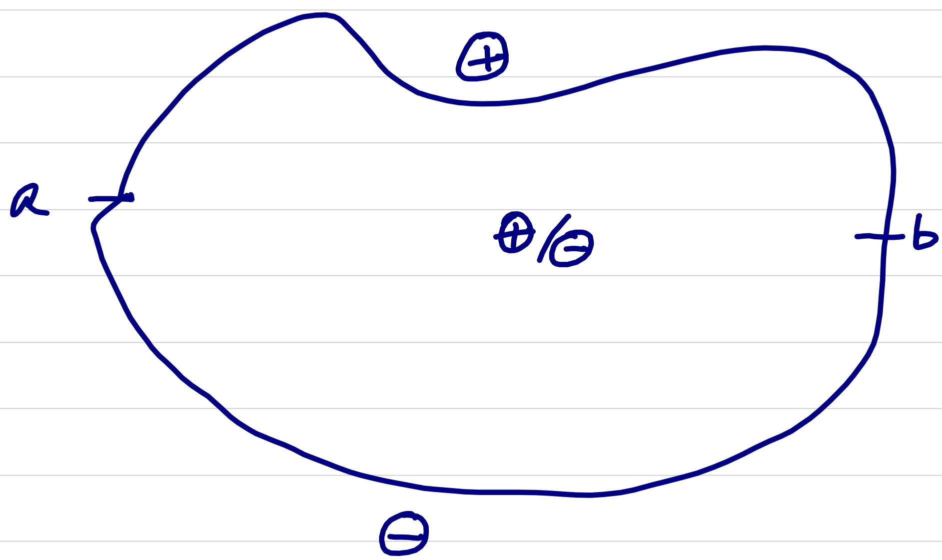
$$\mu^b_{\beta,G}[\sigma] = \frac{1}{Z}\exp(-\beta H(\sigma))$$

$\sigma \in \{+1,-1\}^G$ s.t $\sigma = b$ on ∂G.

free boundary conditions. $\mu^{f}_{\beta,G}$.

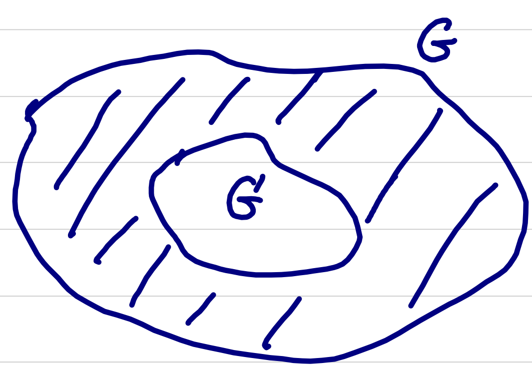
$\mu^{\oplus}_{\beta,G}$, $\mu^{\ominus}_{\beta,G}$.

Dobrushin b.c.



b.c. induced by the config. outside G.

Domain Markov property



$$\mu^{b}_{\beta,G}[\,\cdot \mid \sigma_x = \psi_x,\ x \in G \setminus G'\,]$$

$$= \mu^{\psi}_{\beta,G'}$$

FKG inequality.

Fix a finite graph G, boundary conditions b, $\beta > 0$.

partial order $\sigma \le \sigma'$ iff $\sigma_x \le \sigma'_x$, $\forall x \in V$.

increasing event A: $\mathbb{1}_A(\sigma)$. increasing.

$$\sigma \in A, \ \sigma \le \sigma' \Rightarrow \sigma' \in A.$$

for two increasing events A, B,

$$\mu^b_{\beta,G}[A \cap B] \ge \mu^b_{\beta,G}[A] \, \mu^b_{\beta,G}[B].$$

Pf: Holley's inequality.

Cor. (comparison between boundary conditions).

$b_1 \le b_2$. $\quad \mu^{b_1}_{\beta,G} \le_{st} \mu^{b_2}_{\beta,G}$

$\forall$ increasing event A, $\quad \mu^{b_1}_{\beta,G}[A] \le \mu^{b_2}_{\beta,G}[A]$.

Prop. $\beta > 0$. $\exists$ two (possibly equal) ∞-volume measures $\mu_\beta^{\oplus}$, $\mu_\beta^{\ominus}$:

for any event A depending on finitely many vertices

$$\lim_n \mu_{\beta,\Lambda_n}^{\oplus}[A] = \mu_\beta^{\oplus}[A]$$

$$\lim_n \mu_{\beta,\Lambda_n}^{\ominus}[A] = \mu_\beta^{\ominus}[A].$$

Pf: increasing event A finitely measurable.

$$\{\mu_{\beta,\Lambda_n}^{\oplus}[A]\}_n \downarrow. \quad \mu_\beta^{\oplus}[A].$$

A finitely measurable

$$\mu_{\beta,\Lambda_n}^{\oplus}[A] \to \mu_\beta^{\oplus}[A].$$

$\{\mu_\beta^{\oplus}[A] : A \text{ finitely measurable}\}$

can be extended to a proba. measure.

Prop. $\mu_\beta^{\oplus}$, $\mu_\beta^{\ominus}$ translation-invariant, ergodic.

For $\mu_\beta^{\oplus}$, $\mu_\beta^{\ominus}$: either no ∞-cluster a.s.

or $\exists!$ ∞-cluster a.s

Thm. [Phase transition]

$$\mu_{\beta,\Lambda_n}^{\oplus}[\sigma_0] > 0$$

$$n\to\infty. \qquad \downarrow$$

$$\mu_\beta^{\oplus}[\sigma_0]$$

$\exists\, \beta_c \in (0, \infty)$,
$$\begin{cases} \mu_\beta^{\oplus}[\sigma_0] = 0, & \beta < \beta_c \quad \Big| \quad T > T_c \\ \mu_\beta^{\oplus}[\sigma_0] > 0, & \beta > \beta_c. \quad \Big| \quad T < T_c \end{cases}$$

$$\beta_c = \frac{1}{2}\log(1+\sqrt{2}).$$

Edwards-Sokal coupling.

Fix a finite graph G and $p \in (0,1)$

1st. ω: random-cluster model with $(p,2)$ free-b.c.

$$\phi[\omega] \propto \left(\frac{p}{1-p}\right)^{o(\omega)} 2^{\underline{k(\omega)}}$$

2nd. assign to each cluster of ω a spin $\oplus/\ominus$ with

proba. $\frac{1}{2}$, indept. $\rightarrow$ σ.

conclusion: $\sigma \sim$ Ising model on G with free-b.c.

$$\beta = \frac{1}{2}\log\frac{1}{1-p} > 0.$$

Pf: P: 1st. $\omega \sim RCM(p,2)$

2nd. assign to each cluster of ω a spin. σ

$$P[(\omega,\sigma)] = \frac{1}{Z^f_{p,2,G}} \left(\frac{p}{1-p}\right)^{o(\omega)} 2^{k(\omega)} \left(\frac{1}{2}\right)^{k(\omega)}$$

$$= \frac{1}{Z^f_{p,2,G}} \left(\frac{p}{1-p}\right)^{o(\omega)}.$$

Goal: marginal law of σ under P.

Q: 1st. $\sigma \sim$ Ising free-b.c. $\beta > 0$.

2nd. $\omega \in \{0,1\}^E$:

$e \in E$, $e = (x,y)$, $\sigma_x \neq \sigma_y$, $\omega(e) = 0$.

$\sigma_x = \sigma_y$, $\omega(e) = 1$ with p

$\omega(e) = 0$ with $1-p$.

$$Q[(\omega,\sigma)] = \frac{1}{Z^f_{\beta,G}} e^{-2\beta \# \mathrm{disagree}(\sigma)} \cdot p^{o(\omega)} \cdot (1-p)^{c(\omega) - \# \mathrm{disagree}}$$

$H(\sigma) = 2 \# \mathrm{disagree}(\sigma) - \# E$

$$= \frac{1}{Z^f_{\beta,G}} \left(\frac{e^{-2\beta}}{1-p}\right)^{\# \mathrm{disagree}(\sigma)} p^{o(\omega)} (1-p)^{c(\omega)}$$

$$\propto \left(\frac{e^{-2\beta}}{1-p}\right)^{\# \mathrm{disagree}(\sigma)} \left(\frac{p}{1-p}\right)^{o(\omega)}.$$

$e^{-2\beta} = 1-p$

$Q[(\omega,\sigma)] \propto \left(\frac{p}{1-p}\right)^{o(\omega)}$.

$P = Q$.

marginal law of σ under P = marginal law of σ under Q

= Ising.

$$\mu^{f}_{\beta,G}[\sigma_x\sigma_y]$$

$$= P[\sigma_x\sigma_y]$$

$$= P[\ \mathbb{1}\{x \overset{w}{\leftrightarrow} y\} + \mathbb{1}\{x \overset{w}{\not\leftrightarrow} y\}\cdot 0\ ]$$

$$= \phi^{0}_{p,2,G}[x \overset{w}{\longleftrightarrow} y].$$

$$\mu^{\oplus}_{\beta,G}[\sigma_x]$$

$$= P[\sigma_x]$$

$$= P[\ \mathbb{1}\{x \leftrightarrow \partial G\} + \mathbb{1}\{x \not\leftrightarrow \partial G\}\cdot 0\ ]$$

$$= P[\ x \overset{w}{\leftrightarrow} \partial G]$$

$$= \phi^{1}_{p,2,G}[x \leftrightarrow \partial G].$$

$$\mu^{\oplus}_{\beta,\Lambda_n}[\sigma_0] = \phi^1_{p,2,\Lambda_n}[0 \leftrightarrow \partial\Lambda_n]$$

$$\to \phi^1_{p,2}[0 \leftrightarrow \infty] \begin{cases} = 0, & p < p_c \\ > 0, & p > p_c. \end{cases}$$

$$\beta = \frac{1}{2}\log\frac{1}{1-p}.$$

$$\beta_c = \frac{1}{2}\log\frac{1}{1-p_c}$$

$$\mu^{\oplus}_{\beta}[\sigma_0] = 0, \quad \beta < \beta_c$$

$$> 0, \quad \beta > \beta_c.$$

Recall. $p_c = \frac{\sqrt{q}}{1+\sqrt{q}} = \frac{\sqrt{2}}{1+\sqrt{2}}$

$$\beta_c = \frac{1}{2}\log(1+\sqrt{2}).$$

Fix a finite graph $G=(V,E)$

$\mathcal{E}_G$: even subgraphs of G.

$\omega = (V, E(\omega))$. $v \in V$, degree of v in ω is even.

$A \subset V$, $\mathcal{E}_G(A)$: subgraphs $\omega \in G$

- $v \in V \setminus A$, degree of v in ω is even
- $v \in A$, degree of v in ω is odd.

Note : • $A = \phi$. $\mathcal{E}_G(\phi) = \mathcal{E}_G$.

- $\# A$. odd, $\mathcal{E}_G(A)$ is empty.

High temperature expansion

$$Z^f_{\beta,G} = \sum_\sigma e^{-\beta H(\sigma)}$$

$$= 2^{\#V} \cosh(\beta)^{\#E} \sum_{\omega\in\mathcal{E}_G} \tanh(\beta)^{o(\omega)}.$$

Pf: $$e^{+\beta\underbrace{\sigma_x\sigma_y}_{\pm 1}} = \cosh(\beta) + \sigma_x\sigma_y \sinh(\beta)$$

recall: $\cosh(\beta) = \dfrac{e^\beta + e^{-\beta}}{2}$, $\sinh(\beta) = \dfrac{e^\beta - e^{-\beta}}{2}$

$$e^{+\beta\sigma_x\sigma_y} = \cosh(\beta)\,(1 + \sigma_x\sigma_y \tanh(\beta)).$$

$$Z^f_{\beta,G} = \sum_\sigma e^{+\beta\sum_{x\sim y}\sigma_x\sigma_y}$$

$$= \sum_\sigma \prod_{(x,y)\in E} e^{\beta\sigma_x\sigma_y}$$

$$= \sum_\sigma \prod_{(x,y)\in E} \cosh(\beta)\,(1+\sigma_x\sigma_y\tanh(\beta))$$

$$= \cosh(\beta)^{\#E} \sum_\sigma \prod_{(x,y)\in E} (1+\sigma_x\sigma_y\tanh(\beta))$$

$$= \cosh(\beta)^{\#E} \sum_\sigma \sum_{\omega\subset E} \tanh(\beta)^{o(\omega)} \prod_{(x,y)\in\omega} \sigma_x\sigma_y$$

$$Z^{f}_{\beta,G} = \cosh(\beta)^{\#E} \sum_{\sigma} \sum_{\omega \subset E} \tanh(\beta)^{o(\omega)} \prod_{(x,y)\in\omega} \sigma_x \sigma_y .$$

$$= \cosh(\beta)^{\#E} \sum_{\omega \subset E} \tanh(\beta)^{o(\omega)} \boxed{\sum_{\sigma} \prod_{(x,y)\in\omega} \sigma_x \sigma_y}$$

$$\underbrace{\sum_{\sigma} \prod_{(x,y)\in\omega} \sigma_x \sigma_y}_{1} = \begin{cases} 0 , & \exists v \text{ s.t. degree } v \text{ in } \omega \text{ is odd} \\ 2^{\#V} , & \omega \in \mathcal{E}_G \end{cases}$$

$$Z^{f}_{\beta,G} = 2^{\#V} \cosh(\beta)^{\#E} \sum_{\omega \in \mathcal{E}_G} \tanh(\beta)^{o(\omega)}$$

$\sigma \in \{+1, -1\}^V$.

define $\omega[\sigma] \in \{0, 1\}^{E(G^*)}$

$\forall\, e = (x, y) \in E$, $\quad e^*$

$$\omega[\sigma](e^*) = \begin{cases} 1, & \text{if } \sigma_x \neq \sigma_y \\ 0, & \text{else.} \end{cases}$$

Ising: $\quad \mathbb{P}[\sigma] \propto e^{-\beta H(\sigma)}$

$$-\beta H(\sigma) = \beta \sum_{(x,y)\in E} \sigma_x \sigma_y$$

$$= -2\,\#\text{disagree} + \#E.$$

$$= -2\, o(\omega)$$

$$\mathbb{P}[\omega] \propto e^{-2\beta\, o(\omega)}.$$

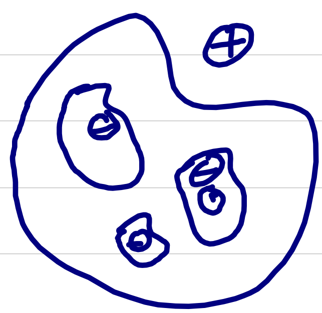
$$Z^{\oplus}_{\beta,G} = \sum e^{-\beta H(\sigma)}$$

$$= e^{\beta \# E} \sum_{\omega \in \mathcal{E}_{G^*}} e^{-2\beta o(\omega)}$$

$$-\beta H(\sigma) = \beta \sum_{(x,y)\in E} \sigma_x \sigma_y$$

$$= \beta \# E - 2\beta \# \text{disagree}(\sigma)$$

$$= \beta \# E - 2\beta\, o(\omega)$$



$$Z^{\oplus}_{\beta,G} = \sum_{\sigma} e^{-\beta H(\sigma)}$$

$$= e^{\beta \# E} \sum_{\omega \in \mathcal{E}_{G^*}} e^{-2\beta o(\omega)}$$

Krammers – Wannier duality

$$\tanh(\beta^*) = e^{-2\beta} .$$

$$Z^{f}_{\beta^*, G^*} = 2^{\# V(G^*)} \cdot \cosh(\beta)^{\# E} \sum_{\omega \in \mathcal{E}_{G^*}} \tanh(\beta^*)^{o(\omega)}$$

$$Z^{\oplus}_{\beta, G} = e^{\beta \# E} \sum_{\omega \in \mathcal{E}_{G^*}} e^{-2\beta\, o(\omega)}$$

$$2^{\# V(G^*)} \cdot \cosh(\beta^*)^{\# E} Z^{\oplus}_{\beta, G} = e^{\beta \# E} Z^{f}_{\beta^*, G^*} .$$

$G = \Lambda_n$, $\log$. $\frac{1}{\# E(\Lambda_n)}$.

$$\frac{\# V(\Lambda_n)}{\# E(\Lambda_n)} \log 2 + \cosh(\beta^*) + \frac{1}{\# E(\Lambda_n)} \log Z^{\oplus}_{\beta, \Lambda_n}$$

$$= \beta + \frac{1}{\# E(\Lambda_n)} \log Z^{f}_{\beta^*, \Lambda_n^*}$$

$n \to \infty$ $\qquad \tanh(\beta^*) = e^{-2\beta}$

$$\log 2 + \cosh(\beta^*) + f(\beta) = \beta + f(\beta^*).$$

Physicists: f has only one point at which f is not analytic. β_c.

if $\beta_c^* \neq \beta_c$, f will have two non-analytic points

contradiction!

$$\beta_c^* = \beta_c.$$

$$\tanh(\beta) = e^{-2\beta}$$

has only one solution $\beta_c = \frac{1}{2}\log(1+\sqrt{2})$.