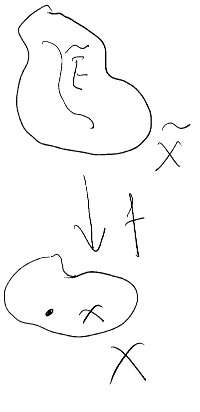
Last time: We have seen discrepancy of sum exceptional divisor over X.

Any divisor can viewed as a valuation of the field $K(X) = \mathbb{C}(X)$

Notation: ① X normal projective. A divisor E over X is following data $(\widetilde{X}, \widetilde{E})$ such that $\widetilde{X}$ is proj normal, birational to X with the birational morphism $f: \widetilde{X} \longrightarrow X$ and $\widetilde{E}$ is a prime divisor in $\widetilde{X}$

② the center of E in X is $f(\widetilde{E})$. It is always a closed irreducible subvariety of X.

Remark ① This definition of E does not depend on the model $(\widetilde{X}, \widetilde{E})$ chosen, but just the the valuation associated to E.



② $0 \le \dim(\operatorname{center}(E)) \le \dim X - 1$

For example, if $f: \widetilde{X} \to X$ is blowup of a smooth point $x \in X$, and with exceptional divisor $\widetilde{E}$, then $E = (\widetilde{X}, \widetilde{E})$ is a divisor over X. And $\operatorname{center}(E) = \{x\}$

Notation: A divisor E over X is called exceptional if $\dim(\text{center}(E)) \leq \dim X - 2$

Remark: There are much more valuation on $K(X)$ than the valuations given by divisors.

[KM98, 2.3]

<u>Pairs</u>: Def: A pair (X, Δ) consists of a normal quasi-proj var X, and a $\mathbb{Q}$-Weil divisors Δ. And the adjoint canonical divisor is $K_X + \Delta$

Δ is called the boundary of the pair (X, Δ)

Remark: In the litterature, Δ is often supposed to be ≥ 0 and Δ containing negative coeff is called a sub-pair.

<u>Reason for pairs</u>: ① X singular, normal, q-proj var.

$r: \tilde{X} \to X$ resolution, we write $K_{\tilde{X}} \equiv r^* K_X + F$.

Then $(\tilde{X}, \Delta = -F)$ is pair with adjoint canonical divisor

$$K_{\tilde{X}} + \Delta = K_{\tilde{X}} - F \equiv r^* K_X$$

numerical prop of $K_X \longleftrightarrow$ numerical prop of $K_{\tilde{X}} + \Delta$

② <u>Important</u>: Adjunction formula: $H \subseteq X$ smooth prime divisor in smooth proj var X.

Then $K_H = i^*(K_X + H)$, where $i: H \hookrightarrow X$.

Here $(X, \Delta = H)$ is a pair

discrepancy for pairs.

Def: (X,Δ) a pair, we can resolve the singularities $r: \tilde{X} \to X$ so that $r_*^{-1}\Delta \cup \operatorname{excep}(r)$ has snc support.

Assume $K_X+\Delta$ is $\mathbb{Q}$-Cartier. Then we can write

$$K_{\tilde{X}} \equiv r^*(K_X+\Delta) + \sum a(E_i, X, \Delta)\cdot E_i$$

where the E_i are prime divisors over X. $a(E_i, X, \Delta) \in \mathbb{Q}$ is called the discrepancy of E_i for (X,Δ).

Note: If $E \subseteq X$ is an irreducible component of Δ, then $a(\tilde{E}, X, \Delta) = -\operatorname{coeff}_E \Delta$, $\tilde{E} = r_*^{-1}E$

Alternatively, one can write

$$K_{\tilde{X}} + r_*^{-1}\Delta \equiv r^*(K_X+\Delta) + \sum_{\substack{E \text{ exceptional} \\ \text{over } X}} a(E, X, \Delta) E$$

Notation: The log discrepancy of some divisor E over (X,Δ) is defined as $1 + a(E, X, \Delta)$

Notation: $\operatorname{Discrep}(X,\Delta) = \inf\{a(E,X,\Delta) \mid E \text{ exceptional over } X\}$

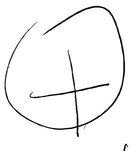
$\operatorname{Total\,discrep}(X,\Delta) = \inf\{a(E,X,\Delta) \mid E \text{ divisor over } X\}$

Exercise ([KM98, 2.3])

Either $\operatorname{discrep}(X,\Delta) = -\infty$ or

$-1 \leq \operatorname{totaldiscrep}(X,\Delta) \leq \operatorname{discrep}(X,\Delta)$

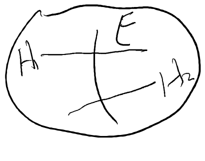
Example: $X = \mathbb{C}^2$, $\Delta = H_1 + H_2$, $H_1 = \{x=0\}$, $H_2 = \{y=0\}$



$\operatorname{totaldiscrep}(X,\Delta) = \operatorname{discrep}(X,\Delta) = -1$.

$\mathrm{discrep}(H_1, X, \Delta) = -1$, $\quad \mathrm{discrep}(H_2, X, \Delta) = -1$

Blowup $\{x = y = 0\}$, $r: \tilde{X} \to X$, with exceptional div E



$K_{\tilde{X}} = r^* K_X + E$

Thus $K_{\tilde{X}} = r^*(K_X + \Delta) + E - r^*\Delta$

Then $K_{\tilde{X}} = r^*(K_X + \Delta) + E - (r_{-1}^* H_1 + r_{-1}^* H_2 + 2E)$

$\Rightarrow K_{\tilde{X}} + r_{-1}^* H_1 + r_{-1}^* H_2 = r^*(K_X + \Delta) - E$

Note $K_{\tilde{X}} + r_{-1}^* H_1 + r_{-1}^* H_2 + E = r^*(K_X + \Delta)$

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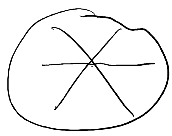
Example: $X = \mathbb{C}^2$, $H_1 = \{x = 0\}$, $H_2 = \{y = 0\}$, $H_3 = \{x + y = 0\}$

$\Delta = H_1 + H_2 + H_3$



Def (KM98, 2.3) (X, Δ) a pair, $\Delta \geq 0$, $K_X + \Delta$ $\mathbb{Q}$-Cartier

(X, Δ) is

- terminal if $\mathrm{discrep}(X, \Delta) > 0$
- canonical if $\mathrm{discrep}(X, \Delta) \geq 0$
- klt if $\mathrm{discrep}(X, \Delta) > -1$ and $\mathrm{coeff}\,\Delta < 1$
- plt if $\mathrm{discrep}(X, \Delta) > -1$
- log canonical if $\mathrm{discrep}(X, \Delta) \geq -1$

Note: klt means Kawamata log terminal

plt mean pure log terminal

Def: (X, Δ) is dlt $\Leftrightarrow$ $\exists$ a resolution $\tilde{X} \to X$ such that $\mathrm{discrep}(E, X, \Delta) > -1$ for all $E \subseteq \tilde{X}$.

Equivalently, it is the "disjoint" union of reduced simple normal crossing and klt.

(KM98, 2.3)

Note: plt is dlt, klt is plt.

Exercise (KM98, 2.3)

① Assume (X, Δ) klt, let $\Delta' \geq 0$ be a divisor. Then for $0 \leq \varepsilon \ll 1$, $(X, \Delta + \varepsilon\Delta')$ is again klt

(klt is an open property)

② Assume (X, Δ) dlt, $\Delta' \geq 0$, Δ' and $\lfloor \Delta \rfloor$ has no common component. Then $(X, \Delta + \varepsilon\Delta')$ is dlt for $\varepsilon \ll 1$. Here $\lfloor \Delta \rfloor$ means

$$\lfloor \sum a_i D_i \rfloor = \sum \lfloor a_i \rfloor D_i$$

IV Cone Theorem (Proved in the 1980s)

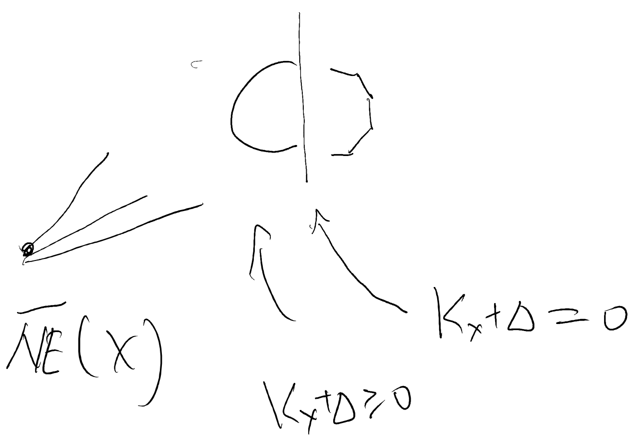
Proof: see [KM98, Sect 3] or [KMM87, Chapter 1-4]

Cone Theorem: (X, Δ) a klt pair, X proj, $\Delta \geq 0$

Then ① there are at most countably rational curves $C_j \subseteq X$ such that $0 < -(K_X + \Delta) \cdot C_j \leq 2 \dim X$

and

$$\overline{NE}(X) = \overline{NE}(X)_{(K_X+\Delta) \geq 0} + \sum \mathbb{R}^{\geq 0} [C_j]$$



② For any $\varepsilon>0$, any ample H

$$\overline{NE}(X) = \overline{NE}_{(K_X+\Delta+\varepsilon H)} + \sum_{\substack{\text{finitely}\\ \text{many}}} \mathbb{R}^{\geq 0}[C_j]$$

This means the rays $\mathbb{R}^{\geq 0}[C_j]$ only accumulates towards $\{K_X+\Delta=0\}$

In particular, away from $\{K_X+\Delta\}$, $\overline{NE}_{(K_X+\Delta<0)}$ is polyhedral

③ given a face $F \subseteq \overline{NE}_{(K_X+\Delta<0)}$, there is a projective morphism $\mathrm{cont}_F : X \longrightarrow Z$

such that a curve $C \subseteq X$ is contracted to a point in Z iff $[C] \in F$

Moreover, Z is normal, cont_F has connected fibers

④ $\mathrm{cont}_F : X \to Z$ as before. The Picard numbers satisfy $\rho(X) = \rho(Z) + \dim_{\mathbb{R}} F$

and a $\mathbb{Q}$-Cartier divisor $D \subseteq X$ satisfies $D \cdot C = 0$ for all $[C] \in F$

$\Longleftrightarrow$ $\exists$ $\mathbb{Q}$-Cartier divisor $G \subseteq Z$ such that $D \sim_{\mathbb{Q}} \mathrm{cont}_F^* G$.

For proof of rational curves, see

[Kollár 96, rational curves in alg var]

or [Debarre homepage:

Http://www.math.ens.fr/~debarre/grenoble.pdf.

on Mori's bend-and-break.

V MMP conjecture

Given a proj pair (X, Δ), with klt sing, $\Delta \geq 0$

If $K_X + \Delta$ is nef; we say that (X, Δ) is minimal. (Abundance conjecture)

Assume $K_X + \Delta$ is not nef, then there is some $R \subseteq \overline{NE}(X)_{K_X+\Delta<0}$ extremal ray

(that is a face of dim 1)

$R = \mathbb{R}^{\geq 0}[C]$ for some curve C with $(K_X+\Delta)\cdot C<0$

We can contract R by cone Thm

There are 3 kinds of contraction for $f = \mathrm{cont}_R$

- $\mathrm{ex}(f)$ has $\dim \leq \dim X - 2$, we say f is small
- $\mathrm{ex}(f)$ has $\dim = \dim X - 1$, we say f is divisorial
- $\mathrm{ex}(f)$ has $\dim = \dim X$, the f is called a Mori fibration

Here $\mathrm{ex}(f) = \{x \in X$ such that f is not an iso around $x\}$.

V 1 divisorial contraction

Prop: (X,Δ) proj klt pair, $\Delta \geq 0$. X $\mathbb{Q}$-factorial
(every Weil divisor is $\mathbb{Q}$-Cartier)

Assume that $f = \mathrm{cont}_R : X \longrightarrow Z$ is a divisorial contraction. Then

1. $\mathrm{Ex}(f)$ is a prime divisor
2. Z is again $\mathbb{Q}$-factorial
3. $(Z, f_*\Delta)$ is again klt.

Proof of 3:

Lemma (Negativity lemma)

Lemma (Negativity Lemma): Same assumption as before.

Let E be the exceptional divisor, then $-E$ is f-ample over Z

That is, for any curve C contracted by f

$$C \cdot (-E) > 0$$

With this lemma, we write $K_X + \Delta = (f^* K_Z + f_*\Delta) + aE$.

Then $f^* K_Z$ is trivial on every curve contracted by f

On the other hand, $K_X + \Delta$ is negative on every curve contracted by f.

Thus $a > 0$ by the negativity Lemma.

Thus $(K_X + \Delta) - aE = f^*(K_Z + f_*\Delta)$

Since discrepancy is an decreasing function on coeffs of the boundary (exercise)

we obtain $\operatorname{discrep}(Z, f_*\Delta) \geq \operatorname{discrep}(X, \Delta)$

Flips, MFS

(KM98, ~Sect 3.5, running the MMP)