Rigidity of Moduli Spaces

Benson Farb
University of Chicago

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Dedicated to
S.S. Chern (1911-2004)

Professor of Mathematics
University of Chicago
(1949-1960)

Algebro-geometric constructions $\rightsquigarrow$ Morphism $\Psi : \mathcal{M} \to \mathcal{N}$ of moduli spaces

or

Section Ψ of $\begin{array}{c}\mathcal{M}\\ \downarrow\\ \mathcal{N}\end{array}$

Guiding Principle: Ψ is a miracle.

Holomorphic Rigidity: Any nontrivial, holomorphic $F : \mathcal{M} \to \mathcal{N}$

$\Longrightarrow F = \Psi$.

Topological Rigidity: $F : \mathcal{M}_m \to \mathcal{N}_n$ nontriv. continuous

$$\Longrightarrow \begin{array}{l} m = m_0 \\ n = n_0 \\ F \sim \Psi \end{array}$$

- Regular
- Rational
  -
  ⋮

Guiding Princ. $\rightsquigarrow$ Many Conjectures / predictions
(mostly open!)

Why?

Today: 3 examples

methods: Geom. group thry,
H^*,
$K \le 0$,
rigidity a la Margulis, Yau, ...

I. <u>Resolving the Quartic</u>

A clue: $\psi: S_n \twoheadrightarrow S_m$ Sym groups, $n > m > 2$

$\iff (n,m) = (4,3)$

<u>Proof</u>: $\implies$: A_n is simple $\forall n \geq 5$
$\implies \psi(A_n) = 1$.

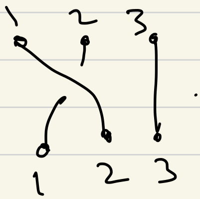
$\impliedby$: $S_4 \twoheadrightarrow \mathrm{Perm}\left\{\begin{matrix}(12)(34), (13)(24), \\ (23)(14)\end{matrix}\right\}$
$\cong S_3$

Why $(4,3)$?

A lift:

- Let B_n = braid group on n strands
  $:= \pi_1(\mathrm{UConf}_n(\mathbb{C}))$

- $B_n \twoheadrightarrow S_n$
  E. Artin.



$(12)(3)$

<u>Prop</u>: $\psi: B_n \twoheadrightarrow B_m \quad n > m > 2$

$\iff (n,m) = (4,3)$

$$\begin{tikzcd} B_4 \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, "\circlearrowleft"] & B_3 \arrow[d, two heads] \\ S_4 \arrow[r, two heads] & S_3 \end{tikzcd}$$

$\sigma_1, \sigma_3 \mapsto \sigma_1$

$\sigma_2 \mapsto \sigma_1$

A lift of the lift:

- $\mathrm{Poly}_n := \{ P(z) = z^n + a_1 z^{n-1} + \dots + a_n : P \text{ is square-free} \}$
  $\Updownarrow$
  $D_n(a_1, \dots, a_n) \neq 0$

- $\mathrm{Poly}_n \leftrightarrows \mathrm{UConf}_n \mathbb{C}$
  $P \mapsto \{\text{roots of } P\}$
  $P(z) = \prod (z - z_i) \leftarrow\!\shortmid \{z_1, \dots, z_n\}$

Thus $\quad \pi_1(\mathrm{Poly}_n) = B_n \quad \left(\begin{array}{c}\text{in fact} \\ K(B_n,1)\end{array}\right)$

Ferrari, 1545 "Resolving the Quartic"

$\exists$ morphism of varieties

$$\begin{aligned} \mathrm{Fer} : \quad & \mathrm{Poly}_4 \longrightarrow \mathrm{Poly}_3 \\ & \{a_1,\dots,a_4\} \longmapsto \{z_1,z_2,z_3\} \end{aligned}$$

$$\left.\begin{aligned} z_1 &:= (a_1-a_2-a_3+a_4)^2/4 \\ z_2 &:= (a_1-a_2+a_3-a_4)^2/4 \\ z_3 &:= (a_1+a_2-a_3-a_4)^2/4 \end{aligned}\right\} \quad \begin{array}{c}\text{distinct!} \\ z_i \neq z_j\end{array}$$

$$\mathrm{Fer}_* : B_4 \longrightarrow B_3 \qquad \rightsquigarrow \quad S_4 \rightarrow S_3$$

Exercise: Find a geometric way to describe Fer

Theorem (Lin, 2004): $F : \mathrm{Poly}_n \rightarrow \mathrm{Poly}_m$
$n > m > 2$, nonconstant, holomorphic. Then either

1. F is "trivial": factors

$$\begin{tikzcd} \mathrm{Poly}_n \arrow[r, "\Delta_n"'] \arrow[rr, bend right, "F"'] & \mathbb{C}^* \arrow[r] & \mathrm{Poly}_m \end{tikzcd}$$

or

2. $(n,m)=(4,3)$ and $F = F_{crr}$.

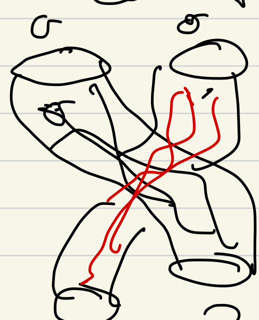
<u>Problem</u>: Classify (cont., holo, ...) maps
$\mathrm{Poly}_n \to \mathrm{Poly}_m$ $m > n$.

- $\exists$ $\mathrm{Poly}_n \to \mathrm{Poly}_{n+1}$ continuous
  $\{z_i\} \mapsto \{z_i\} \cup \{\sum |z_i| + 1\}$
- $\nexists$ any holomorphic
  $F: \mathrm{Poly}_n \to \mathrm{Poly}_{n+1}$
  s.t. $F(\{z_i\}) \cap \{z_{n+1}\} = \phi$
- $\exists$ many interesting homoms

Smooth
$\mathrm{Poly}_n \to \mathrm{Poly}_{2n}$

$B_n \to B_{2n}$ Cabling

$\sigma \mapsto \mathrm{Cable}(\sigma)$



holomorphic?

L. Chen, K. Kordek, D. Margalit

II. choosing points on cubic curves

Question (F, 2016): $\exists$? way to continuously (holo, algebraically, ...) choose a point, or an n-tuple of points, on a smooth cubic curve $C \subset \mathbb{P}^3$

i.e. $\exists$? section

$$(*) \quad C \longrightarrow E = \{(C,p) : C \in \mathcal{C},\ p \in C\}$$

$$\sigma \curvearrowright \big\downarrow \pi$$

$$\pi(C,p) := C \qquad \mathcal{C} : \{\text{smooth cubic curves } C \subset \mathbb{P}^2\}$$

n-tuples of distinct points

$$\mathrm{Uconf}_n C \to E_n := \{(C, \{p_1, \dots, p_n\}) : \ p_i \in C,\ p_i \neq p_j\}$$

$$\big\downarrow$$

$$\mathcal{C}$$

"n-multisection of $(*)$"

Recall: Any smooth cubic curve $C \subset \mathbb{P}^2$ has 9 distinct flex points ($I_p(C, T_pC) \geq 3$)

$\Rightarrow$ $\exists$ algebraic 9-multisection of ($*$):

$$\sigma(C) = \{\text{Flex points of } C\}$$

- $(C, p) \rightsquigarrow$ elliptic curve $\rightsquigarrow$ $\{K\text{-torsion points}\}$ $K \geq 1$

$\Rightarrow$ $\forall K \geq 1$ $\exists$ an algebraic $9K^2$ multisection!

- Can add multisections

$\rightsquigarrow$ "Torsion Construction"

I. Banerjee - Weiyan Chen BC

$\exists$ algebraic n-multisection of ($*$) for

$$n = 9 \sum_{m \in I} \left(m^2 \prod_{\substack{p \mid m \\ \text{prime}}} (1 - p^{-2}) \right)$$

$I \subseteq \mathbb{N}^+$ finite

$n = 9, 27, 36, 77, 81, \ldots$

Known / Conjectured using guiding principle(s):

1. Algebraic Rigidity (C. McMullen, 2021):

Any algebraic n-multisection is a torsion construction.

2. W. Chen, 2018: Continuous n-multi-section exists $\Longrightarrow 9 \mid n$ (no for $n=18$, e.g.)

BC

3. B–C: Any continuous 9-multi-sect. is homotopic $C \mapsto \mathrm{Flex}(C)$.

4. (B–C) $\forall m \geq 4$ $\exists$ smooth, n-multisec. that is <u>not</u> $\sim$ any holo. multisec.

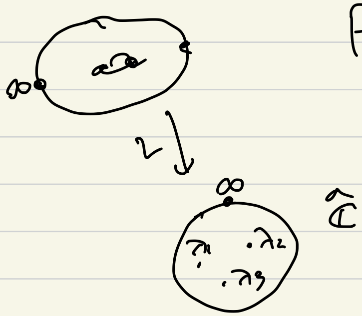
For $n = 18m^2 \prod_{p \mid m} (1 - p^{-2})$

$$\boxed{n = 216}$$

- deg δ hypersurfaces in $\mathbb{P}^n$

Remark: $\exists$ morphism $\forall k \geq 1$

$$\mathrm{Poly}_3 \longrightarrow \mathrm{Poly}_{k^2}$$
$$P(z) \longmapsto \{k\text{-torsion points}\} \text{ of } y^2 = P(z)$$



III. Rigidity of the Period Mapping

(Farb, archive)

$X :=$ smooth, genus $g \geq 2$ curve

$$\psi : H_1(X; \mathbb{Z}) \longrightarrow \Omega^1_{hol}(X)^*$$
$$\alpha \longmapsto \left(\theta \longmapsto \int_\alpha \theta \right)$$

$$\rightsquigarrow \quad \mathrm{Jac}\, X := \Omega^1_{hol}(X)^* / \mathrm{Im}(\psi) \cong \mathbb{C}^g / \Lambda$$

PPAV

$\Rightarrow \quad J : \mathcal{M}_g \longrightarrow \mathcal{A}_g$ Period mapping

$X \longmapsto \operatorname{Jac} X$

- injective morphism (Torelli Thm)

<u>Theorem (F, 2021)</u> Let $F : \mathcal{M}_g \longrightarrow \mathcal{A}_h$, $g \geq 4$, $g \geq h \geq 1$ be nonconstant, holomorphic.

Then $h = g$ and $F = J$.

<u>Problem</u>: classify all holo. $F : \mathcal{M}_g \to \mathcal{A}_h$, $h > g$

(all covers?)

<u>Prym map</u>: $\mathcal{P}_g = \{(X, \theta) : X \in \mathcal{M}_g,\ 0 \neq \theta \in H^1(X; \mathbb{F}_2)\}$

$$\begin{array}{ccc} \mathcal{P}_g & & \tilde{X} \\ {\scriptstyle 2^{2g-1}} \downarrow & & \downarrow \\ \mathcal{M}_g & & X \end{array}$$

noncontstant morphism $\operatorname{Prym} : \mathcal{P}_g \longrightarrow \mathcal{A}_{g-1}$

$X \longmapsto \operatorname{Jac}(\tilde{X}_\theta) / \operatorname{Jac} X$

Conjectures: $\Psi : \mathcal{P}_g \to \mathcal{A}_h$ $h \le g$ $g \ge 3$
noncont. holo.

$\Longrightarrow$ 1. $h = g-1$ and $\Psi = $ Prym, or
2. $h = g$ and

$$\begin{tikzcd} \mathcal{P}_g \arrow[r] \arrow[rr, bend right, "F"'] & \mathcal{M}_g \arrow[r, "J"] & \mathcal{A}_g \end{tikzcd}$$

Proof that $F : \mathcal{M}_g \to \mathcal{A}_h$ $g \ge 4$, $h \le g$
nonconst., holo

$\Longrightarrow$ $\cancel{\exists}$ $g = h$ and $F = J$

Period mapping

1. $F_* : \pi_1^{orb}(\mathcal{M}_g) \longrightarrow \pi_1^{orb}(\mathcal{A}_h)$

$$\begin{array}{ccc} \| & & \| \\ \mathrm{Mod}\,\Sigma_g & \longrightarrow & Sp(2h, \mathbb{Z}) \end{array}$$

Korkmaz

$\Longrightarrow$ $F_* = 0$ or $\cancel{\exists}$ $g = h$ and
$F_* = J_*$

$\Rightarrow F \sim J$

2. F, J holo, $F \sim J$
$\Longrightarrow F = J$