2021-10-18 Kähler geometry

Constant scalar curvature Kähler (cscK) metrics

First Bianchi identity

$$R_{ijkl} + R_{jkil} + R_{kijl} = 0$$

Second Bianchi identity

$$\underbrace{\nabla_i R_{jk}}\underline{{}_{lm}} + \underbrace{\nabla_j R_{ki}}\underline{{}_{lm}} + \underbrace{\nabla_k R_{ij}}\underline{{}_{lm}} = 0.$$

Riemannian

Kähler case

$$\nabla_i R_{j\bar{k}l\bar{m}} + \nabla_j R_{\bar{k}il\bar{m}} + \nabla_{\bar{k}} \underbrace{R_{ijl\bar{m}}}_{0} = 0$$

$$\nabla_i R_{j\bar{k}l\bar{m}} = \nabla_j R_{i\bar{k}l\bar{m}}$$

$$\nabla_i R_{j\bar{k}}{}^{p}{}_{p} = \nabla_j R_{i\bar{k}}{}^{p}{}_{p}$$

$$\nabla_i R_{j\bar{k}} = \nabla_j R_{i\bar{k}}$$

Ricci

scalar curvature

$$\nabla_i R_j{}^{i} = \nabla_j R_i{}^{i} = \nabla_j S$$

__Cor__ $S = \text{const} \iff \bar{\partial}^* \rho = 0.$

where

$$\rho = i R_{i\bar{j}}\, dz^i \wedge d\bar{z}^j \quad \text{Ricci form.}$$

Recall $\bar\partial \rho = 0$ $\left(\because\ d\rho = (\partial + \bar\partial)\rho = (\partial + \bar\partial)(-i\partial\bar\partial \log \det(g_{i\bar j})) = 0 \right)$

<u>Cor</u> $\quad Scal = const \iff \rho$ is a harmonic form.

Recall Hodge theory

$$\Delta_d = d^* d + d d^*, \quad \Delta_{\bar\partial} = \bar\partial^* \bar\partial + \bar\partial \bar\partial^*$$

$$\Delta_\partial = \partial^* \partial + \partial \partial^*$$

Kähler case

$$\Delta_d = 2\Delta_{\bar\partial} = 2\Delta_\partial$$

$$\ker \Delta_d = \ker \Delta_{\bar\partial} = \ker \partial.$$

$\alpha \in \ker \Delta_{\bar\partial}$ is called a harmonic form

$\iff \bar\partial \alpha = 0, \quad \underline{\bar\partial^* \alpha = 0.}$

$\circledast$ harmonic form is unique in each cohomology class

Suppose M is Fano, thus $c_1(M) > 0$

$$\begin{matrix} c_1(M) \\ \psi \\ \omega \end{matrix}$$

ω is a harmonic form in $c_1(M)$

$\rho \in c_1(M)$

$s = \text{const} \iff \rho$ is harmonic

$\iff \rho = \omega \iff \omega$ K-E.

K-E problem = cscK problem in $c_1(M)$

constant scalar curvature Kähler metric. (cscK metric)

Remark Riem $g^{ij} R_{\cdot i \cdot j}$ $\sum_i R(X, e_i, Y, e_i)$

Kähler $g^{i\bar{j}} R_{p\bar{j}q\bar{i}} = R_{p\bar{q}}$ ← Our Ricci

$R(X, \bar{f_i}, Y, f_i)$ $T'M$

$f_i = \frac{1}{\sqrt{2}}(e_i - iJe_i)$

$i = 1, \cdots, n.$

Note Our Ricci $= \frac{1}{2}$ (Riem Ricci)

Our Scal $= \frac{1}{4}$ Riem Scal

- $\Delta f = \lambda f \implies \lambda \leq 0$

$\Delta f = \nabla^i \nabla_i f$

$$\because \int \Delta f \cdot f = -\int \nabla^i f \cdot \nabla_i f$$

$$\lambda \|f\|_{L^2} \qquad\qquad -\|\nabla f\|_{L^2}$$

$$\lambda = 0 \iff f \text{ const.}$$

- Hodge decomp.

$$\begin{aligned} \alpha &= H\alpha + \Delta G\alpha \\ &= H\alpha + dd^* G\alpha + d^* d G\alpha. \\ &= \text{harmonic part} + d\text{-exact part} \\ &\quad + d^*(\qquad) \end{aligned}$$

Lichnerowicz - Matsushima theorem (1956)

cscK. KE

Theorem Let M be a compact Kähler manifold with constant scalar curvature. Then the Lie algebra $\underset{\sim}{\mathfrak{h}}(M)$ of all holomorphic vector fields is reductive, i.e. $\exists$ real Lie alg $\mathfrak{k}$ of a compact Lie group K such that

german h

$$\mathfrak{h}(M) = \mathfrak{k} \otimes \mathbb{C}$$

abeli semi-simple part

$$\iff \mathfrak{h} = \underline{\underline{\mathfrak{z}}} + [\mathfrak{h}, \mathfrak{h}]$$

Proof For simplicity we assume $\underline{H^1(M) = 0}$

$X = X^i \dfrac{\partial}{\partial z^i} \in \mathfrak{h}(M)$ holo vector field

$$\iff \underline{\nabla_{\bar{j}} X^i = 0} \qquad \underline{\nabla_{\bar{j}}} X^i = \underline{\frac{\partial X^i}{\partial \bar{z}_j}} + \underline{\Gamma^{i}_{\bar{j}} A}_{0}$$

$$\iff \underline{\nabla_{\bar{j}} X_{\bar{i}} = 0} \longleftarrow \underline{\nabla_{\bar{j}}} \left(\underline{g_{\bar{i}k}} \underline{X^k} \right)$$

Put $\alpha = g_{i\bar{k}} X^i d\overline{z^k} = X_{\bar{k}}\, d\overline{z^k}$

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(6)

Then $\bar\partial\alpha = \dfrac{\partial X_{\bar k}}{\partial \bar z^i} d\bar z^j \wedge d\bar z^k$

$$= \Big(\frac{\partial X_{\bar k}}{\partial \bar z^j} + \underbrace{\Gamma^{\bar l}_{\bar j \bar k}}_{\text{symmetric}} X_{\bar l} \Big) d\bar z^j \wedge d\bar z^k$$

$$= \nabla_{\bar j} X_{\bar k}\, d\bar z^j \wedge d\bar z^k = 0.$$

Since we assume $H^1(M) = 0$, we have

$$\alpha = \bar\partial u, \quad u \in C^\infty(M)\otimes\mathbb{C}$$

$(u \in \mathbb{C})$

$$\mathfrak{f}(M) \cong \Big\{ u \in C^\infty(M)\otimes\mathbb{C} \;\Big|\; \nabla_{\bar j}\nabla_{\bar k} u = 0, \ \int_M u\,\omega^n = 0 \Big\}$$

Lemma $\nabla_{\bar j}\nabla_{\bar k} u = 0 \iff \nabla^{\bar j}\nabla^{\bar k}\nabla_{\bar j}\nabla_{\bar k} u = 0.$

$\because$ $(\Longleftarrow)$

$$0 = \int \bar u \cdot \nabla^{\bar j}\nabla^{\bar k}\nabla_{\bar j}\nabla_{\bar k} u\ \omega^n$$

$$= -\int \nabla^{\bar j}\bar u \cdot \nabla^{\bar k}\nabla_{\bar j}\nabla_{\bar k} u\ \omega^n$$

integration by parts

$$= +\int \nabla^{\bar k}\nabla^{\bar j}\bar u \cdot \nabla_{\bar j}\nabla_{\bar k} u \cdot \omega^n$$

$$= \int g^{l\bar{j}} g^{m\bar{k}} \, \overline{\nabla_{\bar{l}} \nabla_{\bar{m}} u} \cdot \nabla_{\bar{j}} \nabla_{\bar{k}} u \; \omega^n$$

$$= \| \nabla'' \nabla'' u \|^2_{L^2} \qquad \nabla = \underset{(1,0)}{\nabla'} + \underset{(0,1)}{\nabla''}$$

$$\therefore \nabla'' \nabla'' u = 0$$

$$\therefore \nabla_{\bar{i}} \nabla_{\bar{j}} u = 0$$

($\Longrightarrow$) Trivial. ;)

Put $\quad Du = \nabla^{\bar{j}} \nabla^{\bar{k}} \nabla_{\bar{j}} \nabla_{\bar{k}} u$

"Lichnerowicz operator"

$$\therefore \mathcal{f}(M) = \{ u \in C^\infty(M) \otimes \mathbb{C} \mid Du = 0, \; \int u \, \omega^n = 0 \}$$

$$Du = \nabla^{\bar{i}} \left(\nabla^{\bar{k}} \nabla_{\bar{j}} (\nabla_{\bar{k}} u) \right)$$

Ricci formula

$$= \nabla^{\bar{i}} \left(\nabla_{\bar{j}} \nabla^{\bar{k}} \nabla_{\bar{k}} u - R^{\bar{k}}{}_{\bar{j}}{}^{\bar{l}}{}_{\bar{k}} \nabla_{\bar{l}} u \right)$$

$$= \Delta^2 u + \nabla^{\bar{i}} \left(R^{\bar{l}}{}_{\bar{k}\bar{j}}{}^{\bar{k}} \nabla_{\bar{l}} u \right)$$

$$= \Delta^2 u + \nabla^{\bar{i}} \left(R^{\bar{l} k}{}_{\bar{j} k} \nabla_{\bar{l}} u \right)$$

$$= \Delta^2 u + \nabla^{\bar{i}} \left(R^{\bar{l}}{}_{\bar{i}} \nabla_{\bar{l}} u \right)$$

$$= \Delta^2 u + R^{\bar\ell}{}_{\bar j} \nabla^{\bar j} \nabla_{\bar\ell} u + \nabla^{\bar j} R^{\bar\ell}{}_{\bar j} \nabla_{\bar\ell} u$$

$$= \underbrace{\Delta^2 u + R^{i\bar j} \nabla_i \nabla_{\bar j} u}_{Du} + \cancel{\nabla_i S \, \partial^i u}$$

$$\left(\Delta u = g^{i\bar j}\left(\frac{\partial^2 u}{\partial z^i \partial \bar z^j} + \overset{0}{\Gamma^A_{i\bar j}} \frac{\partial u}{\partial z^A} \right) \right)$$

$$\overline{\Delta u} = \Delta \bar u \qquad \left(\Delta_\partial = \frac{1}{2} \Delta_d \right)$$

$\because$ Δ is a real operator.

$$\overline{R^{i\bar j} \nabla_i \nabla_{\bar j} u} = R^{\bar i j} \nabla_{\bar i} \nabla_j \bar u$$

$$= R^{j\bar i} \nabla_j \nabla_{\bar i} \bar u$$

$$= R^{i\bar j} \nabla_i \nabla_{\bar j} \bar u$$

By assumption $S = \text{const}$ $\therefore \nabla_i S = 0$

So $\overline{Du} = D\bar u$

D is a real operator.

$$\begin{cases} u = f + ig & \text{real part + imaginary part} \\ Du = 0 \end{cases}$$

$$\Rightarrow \quad 0 = \overline{Du} = D\bar{u} = Df - iDg$$

$$0 = Du = Df + iDg$$

$$\therefore \quad Df = Dg = 0$$

$$D(ig) = 0 \iff \nabla'' \underline{\nabla'' ig} = 0$$

<u>Well known fact.</u>

For a purely imaginary function ig, if the gradient of ig generates a <u>holomorphic vector</u> field <u>it generates a Killing vector field.</u>

(Kobayashi's book Transformation groups in differential geometry)

$$\therefore \mathfrak{h}(M) = \{ X + iY \mid X, Y \text{ are Killing} \}$$

More precisely,

X : real vector field $\overset{\text{identify}}{\longleftrightarrow} \frac{1}{2}(X - iJX)$

If $i\bar{g}$ corresponds to holom vector field, then it corresponds to a killing vector.

This means

$$(i\bar{g})^i \frac{\partial}{\partial z^i} = \frac{1}{2}(X - iJX) \quad \text{holo}$$

$$\rightsquigarrow X \text{ is killing}$$

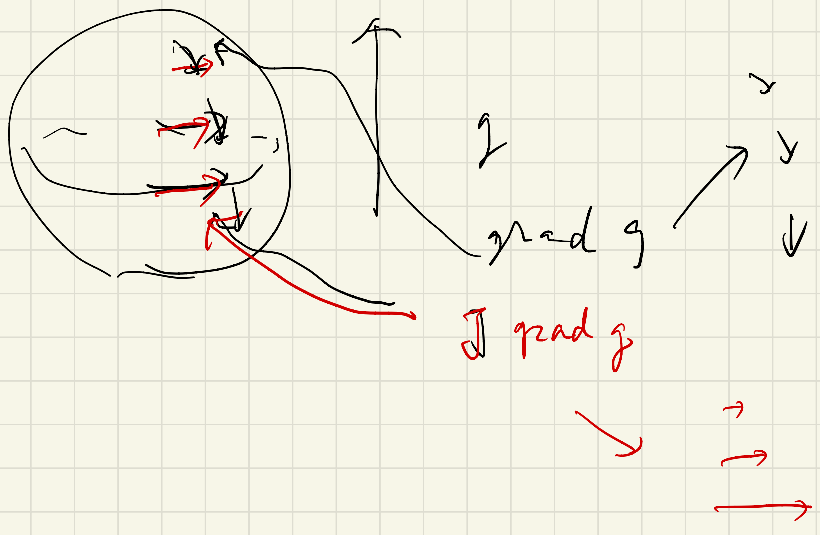
$$g^i \frac{\partial}{\partial z^i} = \frac{1}{2}(\overset{-JX}{Y} - i\overset{X}{JY}) \quad \text{holo}$$

$$\longrightarrow JY \text{ is killing}$$

$$\shortparallel$$

$$J\left(\frac{\partial g}{\partial x^i}\frac{\partial}{\partial x^i}\right) = J \operatorname{grad} g$$

Example : $S^2 = \mathbb{P}^1(\mathbb{C})$



$$f + ig = \underbrace{ig}_{\mathfrak{k}} + i(\underbrace{-if}_{\mathfrak{k}})$$

Killing.

$$\therefore\ f(M) = \mathfrak{k} \otimes \mathbb{C}$$

$\mathfrak{k}$ = Killing vector fields.

= Lie algebra of the Lie group of all isometries. K.

$K = \mathrm{Isom}_0(M)$ = compact if M is compact.

well known in Riem geom.

:)

$\exists$ cscK $\Longrightarrow$ $\mathfrak{h}(M)$ reductive.

Example

$E \subset \widehat{\mathbb{CP}}^2$ = the blow-up of $\mathbb{CP}^2$ at a point.

$$\begin{array}{ccc} E & \subset & \widehat{\mathbb{CP}}^2 \\ \downarrow & & \downarrow \pi \\ pt & & \mathbb{CP}^2 \end{array}$$

$$\begin{array}{ccc} \mathrm{Aut}(\widehat{\mathbb{CP}}^2) & \longrightarrow & \mathrm{Aut}(\mathbb{CP}^2) \\ \psi & & \\ \sigma & \longmapsto & \left(\begin{array}{c|cc} * & * & * \\ \hline 0 & * & * \\ 0 & * & * \end{array}\right) \end{array}$$

$\sigma(E) = E$

$\pi(E) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

not reductive

$\mathfrak{z}$ + ss.

Any Kähler class on $\widehat{\mathbb{CP}}^2$ cannot have a cscK metric.

$c_1(\widehat{\mathbb{CP}}^2) > 0$ but $\nexists$ KE.

$\mathrm{Aut}(\widehat{\mathbb{CP}}^2) = \left\{ \begin{matrix} * & 0 & * \\ 0 & * & * \\ 0 & 0 & * \end{matrix} \right\}$ not reductive

$\mathrm{Aut}(\widehat{\widehat{\widehat{\mathbb{CP}}}}^2) = \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \right\}$ reductive.

Siu, Nadel, Tian $\exists$ KE.

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Def $\mathfrak{g} := \{ u \in C^\infty(M) \otimes \mathbb{C} \mid Du = 0$

$\int_M u\, \omega^n = 0 \}$

$= \{ u \in C^\infty(M) \otimes \mathbb{C} \mid \nabla^i u \frac{\partial}{\partial z^i} \in \mathfrak{h}(M),$

$\int_M u\, \omega^n = 0 \}$

Lie alg in terms of Poisson bracket

$$\{u, v\} = u^i v_i - v^i u_i \in \mathfrak{g}$$

$\in \mathfrak{g}$

$$= g^{i\bar{j}} \left(\frac{\partial u}{\partial \bar{z}^j} \frac{\partial v}{\partial z^i} - \frac{\partial v}{\partial \bar{z}^j} \frac{\partial u}{\partial z^i} \right)$$

Rem If $H^1(M) \neq 0$ then

$\mathfrak{h}(M) = \mathfrak{a} + \mathfrak{g}$ as a Lie alg

$\uparrow$ abelian

If $H^1(M) = 0$ then $\mathfrak{h}(M) = \mathfrak{g}$.