Mini-course 4 Irreducible components of affine Deligne-Lusztig varieties

Recall: Commutative diagram

$$\begin{tikzcd} V_\mu \in \arrow[d, mapsto] & \mathrm{Coh}([\bullet/\hat{G}]) \arrow[r, "\mathrm{Sat}", "\sim"'] \arrow[d, "\pi^*"] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Perv}(\mathrm{Hk}^{\mathrm{loc}}_{\bar{k}}) \arrow[d, "\Phi^{\mathrm{loc},*}"] \\ \tilde{V}_\mu \in & \mathrm{Coh}([\hat{G}/\mathrm{Ad}_\sigma \hat{G}]) \arrow[r, "\exists S"] & \mathrm{Perv}^{\mathrm{Corr}}(\mathrm{Sht}^{\mathrm{loc}}_{\bar{k}}) \end{tikzcd}$$

$\Phi^{\mathrm{loc}}: \mathrm{Sht}^{\mathrm{loc}} \to \mathrm{Hk}^{\mathrm{loc}}$

- Understand homs: $\mathrm{Hom}_{[\hat{G}/\mathrm{Ad}_\sigma \hat{G}]}(\tilde{V}_\lambda, \tilde{V}_\mu) = \mathrm{Hom}_{\hat{G}}(\mathcal{O}_{\hat{G}} \otimes V_\lambda, \mathcal{O}_{\hat{G}} \otimes V_\mu)^{\mathrm{Ad}_\sigma \hat{G}}$

$$= \Gamma(\hat{G}, \mathcal{O}_{\hat{G}} \otimes V_\lambda^* \otimes V_\mu)^{\mathrm{Ad}_\sigma \hat{G}}$$

Definition For a $\hat{G}$-rep'n V, define $J(V) := \Gamma(\hat{G}, \mathcal{O}_{\hat{G}} \otimes V)^{\mathrm{Ad}_\sigma \hat{G}}$

$$= \{ f: \hat{G} \to V \text{ s.t. } f(hg\sigma(h)^{-1}) = h \cdot f(g) \}$$

For simplicity, assume that $\hat{G}$ is simply-connected + σ trivial.

Consider the analogue of Chevalley restriction

$$\begin{array}{ccc} J_{\hat{G}}(V) = \Gamma(\hat{G}, \mathcal{O}_{\hat{G}} \otimes V)^{\mathrm{Ad}\hat{G}} & \hookrightarrow & \Gamma(\hat{T}, V)^{\mathrm{Ad}\hat{N}} = \Gamma(\hat{T}, \underset{\substack{\uparrow \\ \text{weight 0 space}}}{V(0)})^W \\ f & \longmapsto & f|_{\hat{T}} \end{array}$$

injectivity because $\hat{G}^{ss} = \mathrm{Ad}_{\hat{G}}(\hat{T}) \subseteq \hat{G}$ is dense.

$$(*) \quad \begin{tikzcd} J_{\hat{G}}(V) \arrow[r, hook] & J_{\hat{T}}(V(0))^W = (\mathcal{O}_{\hat{T}} \otimes V(0))^W \\ J_{\hat{G}}(\mathbb{1}) \arrow[u, phantom, "\circlearrowleft"] \arrow[r, "\simeq"] & J_{\hat{T}}(\mathbb{1})^W \arrow[u, phantom, "\circlearrowleft"] \end{tikzcd}$$

$\leftarrow$ work of Balagović describes the image.

Theorem. Assume that $\hat{G}$ is semisimple and simply-connected. Write $X := \hat{G}/\mathrm{Ad}_\sigma \hat{G}$

(1) Then $J_{\hat{G}}(V)$ is a free module over $\mathcal{O}_{\hat{T}}^W$ of rank $\dim V(0)$

and $(*)$ is generically isom. over $J_{\hat{T}}(\mathbb{1})^W$.

(2) Consider

$$\begin{tikzcd} \mathrm{Hom}_X(\tilde{\mathbb{1}}, \tilde{V}_\mu) \times \mathrm{Hom}_X(\tilde{V}_\mu, \tilde{\mathbb{1}}) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_X(\tilde{\mathbb{1}}, \tilde{\mathbb{1}}) = \mathcal{O}_{\hat{G}}^{\mathrm{Ad}\hat{G}} = \mathcal{O}_{\hat{T}}^W \arrow[d, equal] \\ \boxed{J_{\hat{G}}(\tilde{V}_\mu) \times J_{\hat{G}}(\tilde{V}_\mu^*) \longrightarrow J_{\hat{G}}(\mathbb{1})} \quad (**) & \end{tikzcd}$$

$\downarrow \mathrm{Res} \qquad \downarrow \mathrm{Res} \qquad \wr\| \ \mathrm{Res}$

$$J_{\hat T}(\tilde V_\mu(0))^W \times J_{\hat T}(\tilde V^*_\mu(0))^W \longrightarrow J_{\hat T}(\mathbb{1})^W$$

Fixing a $J_{\hat G}(\tilde{\mathbb{1}})$-basis of $J_{\hat G}(\tilde V_\mu)$ and $J_{\hat G}(\tilde V^*_\mu)$

The determinant of (**) is $\mathrm{disc}_{long}^{m_\ell} \cdot \mathrm{disc}_{short}^{m_s} \cdot k^\times$

where $\mathrm{disc}_{long} = \pm \prod_{\substack{\alpha\in\Pi \\ \alpha \text{ long roots}}} (e^\alpha - 1) \in \mathbb{C}[\hat T]^W$; same for short roots.

Proof of (1). $J_{\hat G}(V) = (\mathcal{O}_{\hat G}\otimes V)^{\mathrm{Ad}\hat G} = \Big(\bigoplus_{\lambda\in X^\bullet(\hat T)^+} V_\lambda \otimes V_\lambda^* \otimes V\Big)^G$

$$= \bigoplus_{\lambda\in X^\bullet(\hat T)^+} \mathrm{Hom}_{\hat G}(\underset{\substack{\| \\ \mathrm{Ind}_{\hat B^-}^{\hat G} k_\lambda}}{V_\lambda}, V_\lambda\otimes V) = \bigoplus_{\lambda\in X^\bullet(\hat T)^+} \boxed{\mathrm{Hom}_{\hat B^-}(k_\lambda, V_\lambda \otimes V)} \qquad \varphi \mapsto c(\varphi)$$

$$\downarrow$$

$$V(0)$$

s.t. $\varphi(v_\lambda) = v_\lambda \otimes c(\varphi) +$ other terms

If $\lambda' \geq \lambda$, there is a natural map $\mathrm{Hom}_{\hat B^-}(k_\lambda, V_\lambda\otimes V) \longrightarrow \mathrm{Hom}_{\hat B^-}(k_{\lambda'}, V_{\lambda'}\otimes V)$, both mapping to $V(0)$.

This defines a filtration fil_ν on $V(0)$ defined by the images

The action of $\mathcal{O}_{\hat G}^{\mathrm{Ad}\hat G} = \bigoplus_{\delta\in X^\bullet(\hat T)^+} \boxed{\mathrm{Hom}_{\hat G}(V_\delta, V_\delta)}$ sends $\mathrm{Hom}_{\hat G}(V_\lambda, V_\lambda\otimes V)$ to $\mathrm{Hom}_{\hat G}(V_{\lambda+\delta}, V_{\lambda+\delta}\otimes V) \oplus$ "smaller terms"

<u>Fact</u>: $\mathrm{fil}_\bullet V(0)$ is a "nice" filtration so that

$$\dim V(0) = \sum_{\lambda\in X^\bullet(\hat T)^+} \dim \mathrm{gr}_\lambda V(0)$$

related to Satake cycles

<u>Fact</u>: $\exists$ "nice" basis of each of $\mathrm{gr}_\lambda V(0)$ s.t. $\exists$ lift $a_{\lambda,i} \in \mathrm{Hom}_{\hat G}(V_\lambda, V_\lambda\otimes V)$ & their images in $(\mathcal{O}_{\hat G}\otimes V)^{\mathrm{Ad}\hat G}$ form a basis over $\mathcal{O}_{\hat G}^{\mathrm{Ad}\hat G}$.

· <u>Case of isomorphism</u> $J_G(\mathbb{1}) \simeq \mathcal{O}_{\hat G}^{\mathrm{Ad}\hat G} \simeq C_c^\infty(G(\mathcal{O})\backslash G(F)/G(\mathcal{O}), \bar{\mathbb{Q}}_\ell)$

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\simeq} & {}^\sigma\mathcal{E} \\ \downarrow & & \downarrow \\ \mathcal{E}' & \xrightarrow{\simeq} & {}^\sigma\mathcal{E}' \end{array} \qquad [G(\mathbb{Z}_p)\backslash G(\mathbb{Q}_p)/G(\mathbb{Z}_p)] \qquad \mathrm{Hom}_{\mathrm{Perv}^{\mathrm{Corr}}(\mathrm{Sht}^{loc})}(\mathbb{1}_{\mathrm{Sht}_1^{loc}}, \mathbb{1}_{\mathrm{Sht}_1^{loc}})$$

$$\begin{tikzcd}[column sep=small] & \mathrm{Sht}^{loc}_{\mathbb{1}|\mathbb{1}} \arrow[dl] \arrow[dr] & \\ \mathrm{Sht}^{loc}_{\mathbb{1}} \arrow[d, phantom, "\parallel"] & & \mathrm{Sht}^{loc}_{\mathbb{1}} \arrow[d, phantom, "\parallel"] \\ \{\mathcal{E} \xrightarrow{\simeq} {}^{\sigma}\mathcal{E}\} = [\cdot/G(\mathbb{Z}_p)] & & [\cdot/G(\mathbb{Z}_p)] \end{tikzcd}$$

$$= \mathrm{Hom}_{[G(\mathbb{Z}_p)\backslash G(\mathbb{Q}_p)/G(\mathbb{Z}_p)]}(\bar{\mathbb{Q}}_\ell, \bar{\mathbb{Q}}_\ell)$$

$$= C_c^\infty(G(\mathbb{Z}_p)\backslash G(\mathbb{Q}_p)/G(\mathbb{Z}_p), \bar{\mathbb{Q}}_\ell)$$

$$\mathrm{Hom}_{[\hat{G}/\mathrm{Ad}\hat{G}]}(\tilde{\mathbb{1}}, \tilde{\mathbb{1}}) = J(\hat{G}) = \bigoplus_{\lambda \in X^\bullet(\hat{T})^+} \mathrm{Hom}_{\hat{G}}(V_\lambda, V_\lambda)$$

$\mathrm{id}_{V_\lambda} \in \mathrm{Hom}_{\hat{G}}(V_\lambda, V_\lambda)$ is supported on $\mathrm{Sht}^{loc,\lambda}_{\mathbb{1}|\mathbb{1}}$:

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\sim"] \arrow[d, "\lambda"] & {}^{\sigma}\mathcal{E} \arrow[d, "\sigma(\lambda)=\lambda"] \\ \mathcal{E}' \arrow[r, "\sim"'] & {}^{\sigma}\mathcal{E}' \end{tikzcd}$$

· <u>Geometric interpretation of $J_{\hat{G}}(\mathbb{1})$-basis of $J_{\hat{G}}(\tilde{V}_\mu)$</u>

Consider

$$\begin{tikzcd}[column sep=small] & \mathrm{Sht}^{loc,\nu}_{1|\mu} \arrow[dl] \arrow[dr] & \\ \mathrm{Sht}^{loc}_{\mu} & & \mathrm{Sht}^{loc}_{1} \end{tikzcd} \rightsquigarrow \text{classifies} \quad \begin{tikzcd} \mathcal{E}' \arrow[r, "\mu"] \arrow[d, "\nu"'] & {}^{\sigma}\mathcal{E}' \arrow[d, "\nu"] \\ \mathcal{E} \arrow[r, "\simeq"] \arrow[d, dashed, "\lambda"'] & {}^{\sigma}\mathcal{E} \arrow[d, dashed, "\lambda"] \\ \mathcal{E} \arrow[r, "\simeq"] & {}^{\sigma}\mathcal{E} \end{tikzcd}$$

action is given by appending this part.

$$\mathrm{Corr}\left((\mathrm{Sht}^{loc}_\mu, IC_\mu), (\mathrm{Sht}^{loc}_1, \bar{\mathbb{Q}}_\ell)\right) \circlearrowleft \mathrm{Corr}\left((\mathrm{Sht}^{loc}_1, \bar{\mathbb{Q}}_\ell), (\mathrm{Sht}^{loc}_1, \bar{\mathbb{Q}}_\ell)\right)$$

<u>Interpretation in terms of Shimura varieties:</u>

$$\begin{tikzcd}[column sep=small] & & I^{\leq \nu+\lambda} \arrow[dl, dashed] \arrow[dr, dashed] & & \\ & I^{\leq \nu} \arrow[dl] \arrow[dr] & & T_{p,\lambda} \arrow[dl] \arrow[dr] & \\ \mathrm{Sh}_G(K) & & \underbrace{\mathrm{Sh}_{G'}(K)}_{\text{discrete Shimura variety}} & & \mathrm{Sh}_{G'}(K) \end{tikzcd}$$

Hecke operator at p.

$\bigcup_\nu I^{\leq \nu}$ is an increasing union and different irred. components may be generated by applying the Hecke operators.

We need to find the support of basis $a_{\lambda,i}$'s.

- <u>Fibers of the correspondence</u>:

$$\begin{tikzcd} & X_{\mu^*}(1) \arrow[r] \arrow[d] & pt \arrow[d] \\ Sht^{loc}_{\mu} & Sht^{loc}_{\mu|1} \arrow[l] \arrow[r] & Sht^{loc}_{1} = [\bullet / G(\mathbb{Z}_p)] \end{tikzcd}$$

Explicitly, $X_{\mu^*}(1)$ classifies

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\leq \mu"] \arrow[d, "\alpha"'] & {}^{\sigma}\mathcal{E} \arrow[d, "\sigma(\alpha)"] \\ \mathcal{E}_{triv} \arrow[r, "id"] & {}^{\sigma}\mathcal{E}_{triv} = \mathcal{E}_{triv} \end{tikzcd}$$

$\alpha \longleftrightarrow g \in Gr$

s.t. $\sigma(g)^{-1} g \in [L^+G \backslash Gr_{\mu}]$

$\hookrightarrow g^{-1}\sigma(g) \in [L^+G \backslash Gr_{\mu^*}]$

More generally for $b \in LG$, define the Affine Deligne-Lusztig Variety associated to b & μ by

$$X_{\mu^*}(b) = \{ h \in Gr \; ; \; h^{-1} b \sigma(h) \in \overline{Gr_{\mu^*}} \}$$

$$\circlearrowleft$$

$$J_b(F) := \{ g \in G(\breve{F}) \, , \, g^{-1} b \sigma(g) = b \}$$

$$\breve{F} = \breve{\mathbb{Q}}_p^{ur} \text{, or } \overline{\mathbb{F}}_p((\varpi))$$

<u>Fact</u>: J_b is always an inner form of a Levi of G.

When $b = 1$, $J_b(F) = G(F)$.

When $b = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$, $J_b = D^{\times}_{\mathbb{Q}_p}$

When $b = \varpi^{\tau^*}$, $J_b(F) = M_\tau(F)$, M_τ = centralizer of τ.

simplifying condition

<u>Theorem</u>. When G is semisimple and adjoint, split $/\mathbb{Q}_p, \mathbb{F}_p((\varpi))$, assume $\tau \in X_*(T)^+$ s.t. $V_\mu(\tau) \neq 0$,

then there is a bijection $\mathrm{Irr}(X_{\mu^*}(\varpi^{\tau^*})) = \displaystyle\coprod_{\text{basis of } V_\mu(\tau)} M_\tau(F) / M_\tau(\mathcal{O})$

equidim of dim $\langle \rho, \mu - \tau \rangle$

Explicitly, $X_{\mu^*}(\varpi^{\tau^*}) = \displaystyle\bigcup_{\text{basis } \alpha \text{ of } V_\mu(\tau)} M_\tau(F) \overset{M_\tau(\mathcal{O})}{\times} X_{\mu^*}(\varpi^{\tau^*})^{\alpha}$

<u>Rmk</u>: The action of $M_\tau(\mathcal{O})$ on $X_{\mu^*}(\varpi^{\tau^*})^{\alpha}$ may not factors through $M_\tau(k)$

It would be interesting to study this from the point of view of Deligne-Lusztig theory.

<u>Rmk</u>: Chen-Zhu conjecture (proved by Sian Nie)

some max'l parahoric

description of $\mathrm{Irr}(X_{\mu^*}(b)) = \coprod_{\text{basis } \alpha \text{ of } V_\mu(\nu_b)} J_b(F)/\mathrm{Stab}_\alpha$ ← subgp described explicitly

↑ some weight close to Newton polygon of b $\square$

Finer structure on affine Grassmannian

$U(\breve{F}) \curvearrowright Gr$ ← unipotent of a Borel.

Fact: $Gr = \coprod_{\lambda \in X_*(T)} \underbrace{U(\breve{F})\varpi^\lambda G(\breve{O})/G(\breve{O})}_{= S_\lambda}$ "semiinfinite orbits"

(locally closed)

- $\overline{S_\lambda} = \coprod_{\lambda' \le \lambda} S_{\lambda'}$
- For $\lambda \in X_*(T)$ and $\mu \in X_*(T)^+$,

  $S_\lambda \cap Gr_\mu \neq \phi$ iff $\lambda \in \mathrm{Conv}(W\cdot\mu)$

  & in this case, $S_\lambda \cap Gr_\mu$ is equi-dimensional of dim $\langle \rho, \mu+\lambda\rangle$

  $$H_c^*(S_\lambda \cap Gr_\mu, IC_\mu) \simeq \begin{cases} V_\mu(\lambda) \leftarrow \text{weight } \lambda \text{ subspace of } V_\mu & \text{if } * = \langle\rho,\mu+\lambda\rangle \\ 0 & \text{if } * \neq \langle \rho, \mu+\lambda\rangle \end{cases}$$

  In particular, $\mathrm{Irr}(S_\lambda \cap Gr_\mu)$ form a basis of $V_\mu(\lambda)$, denoted by $MV_\mu(\lambda)$

Example: $\lambda = \mu \in X_*(T)^+$, $S_\mu \cap Gr_\mu$ is open in Gr_μ

$\lambda = w_0(\mu)$ $\quad S_{w_0(\mu)} \cap Gr_\mu = \{\varpi^{w_0(\mu)}\}$ is a point.

$G = GL_2$, $Gr_{\omega_1} = \mathbb{P}^1$, $S_{\omega_1} \cap Gr_{\omega_1} = \mathbb{A}^1$, $S_{(0,1)} \cap Gr_{(1,0)} = pt$

- $$\begin{array}{ccc} Gr_{\mu_1} \tilde{\times} Gr_{\mu_2} & \supseteq & \coprod_{\lambda = \lambda_1+\lambda_2} (S_{\lambda_1} \cap Gr_{\mu_1}) \tilde{\times} (S_{\lambda_2} \cap Gr_{\mu_2}) \\ \downarrow & \square & \downarrow \\ Gr_{\mu_1+\mu_2} & \supseteq & S_\lambda \cap Gr_{\mu_1+\mu_2} = \coprod_{\nu \le \mu_1+\mu_2} S_\lambda \cap Gr_\nu \end{array}$$

  ↳ twisted product over L^+U

- Will prove that for any $\nu \in X_*(T)$, $S_\nu \cap X_{\mu^*}(\varpi^{\tau^*})$ is equivariant of dim $\langle \rho, \mu - \tau\rangle$

  Recall: $S_\nu \cap X_{\mu^*}(\varpi^\tau)$ classifies

  $$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\le \mu} & {}^\sigma\mathcal{E} \\ \big|\alpha & \eta & \big|\sigma(\alpha) \end{array}$$

  α defines a point in S_ν

$$\mathcal{E}_{triv} \xrightarrow{\varpi^{\tau}} \sigma^*\mathcal{E}_{triv}$$

We can rewrite

$$\begin{tikzcd}
S_\nu \cap X_{\mu^*}(\varpi^{\tau^*}) \arrow[r] \arrow[d] & \left(S_{\sigma(\nu)-\tau} \tilde{\times} \overset{S_\tau}{\overset{\|}{S_{\nu-(\sigma(\nu)-\tau)}}}\right) \cap \left(Gr \tilde{\times} Gr_\mu\right) = S_{\nu-\tau} \tilde{\times} \underbrace{\left(S_\tau \cap Gr_\mu\right)}_{\text{Irred. components form basis of } V_\mu(\tau)} \arrow[d, "m \times pr_1"] \\
S_\nu \arrow[r, "(id,\, \varpi^{-\tau}\sigma)"] & S_\nu \times S_{\sigma(\nu)-\tau}
\end{tikzcd}$$

For each $\mathfrak{a} \in Irr(S_\tau \cap Gr_\mu)$, denote $Y_\nu^{\mathfrak{a}} :=$ preimage of $S_{\nu-\tau} \tilde{\times} (S_\tau \cap Gr_\mu)^{\mathfrak{a}} \subseteq S_\nu \cap X_{\mu^*}(\varpi^{\tau^*})$

<u>Remark</u>: multiplication by ϖ^λ (from the left) defines an isom. $Y_\nu^{\mathfrak{a}} \underset{\sim}{\xrightarrow{\varpi^\lambda\cdot}} Y_{\nu+\lambda}^{\mathfrak{a}}$

(subtlety: when σ is nontrivial, need $\lambda \in X_\cdot(T)^{\sigma=1}$.)

Need to understand

$$\begin{tikzcd}
\tilde{Y}_\nu^{\mathfrak{a}} \arrow[rrr, bend left] \arrow[d] \arrow[dr] & & & S_{\nu-\tau} \times (S_\tau \cap Gr_\mu)^{\mathfrak{a}} \arrow[dll, "L^+U"'] \arrow[d] \\
Y_\nu^{\mathfrak{a}} \arrow[r] \arrow[d] & S_{\nu-\tau} \tilde{\times} (S_\tau \cap Gr_\mu)^{\mathfrak{a}} \arrow[d] & & S_{\nu-\tau} \times S_\tau \arrow[dll, "L^+U"] \\
S_\nu \arrow[r] & S_{\nu-\tau} \tilde{\times} S_\tau & LU \arrow[ull] \arrow[ur, bend right]
\end{tikzcd}$$

$u \mapsto u\varpi^\nu$

$$S_{\nu-\tau} \tilde{\times} S_\tau = \varpi^{\nu-\tau} LU \overset{L^+U}{\times} S_\tau = LU\varpi^{\nu-\tau} \overset{L^+U}{\times} S_\tau$$

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "(\sigma(u)\varpi^\nu)^{-1}\varpi^\tau u\varpi^\nu"] \arrow[d, "u\varpi^\nu"'] & \sigma^*\mathcal{E} \arrow[d, "\sigma(u)\varpi^\nu"] \\
\mathcal{E}_{triv} \arrow[r, "\varpi^\tau"] & \sigma^*\mathcal{E}_{triv}
\end{tikzcd}$$

$LU \longrightarrow S_\tau$ is given by $\varpi^{-\nu}\sigma(u)^{-1}\varpi^{\tau}u\varpi^{-\tau}\cdot\varpi^{\nu+\tau}$

$$\begin{tikzcd}
& & S_\tau \\
LU \arrow[r] & LU \arrow[ur]
\end{tikzcd}$$

$u \mapsto \sigma(u)^{-1}\varpi^\tau u \varpi^{-\tau}$; $x \mapsto \varpi^{-\nu} x \varpi^{\nu+\tau}$ "affine morphism", fiber dim $= \langle 2\rho, -\tau\rangle$

$\hookrightarrow$ Artin-Schreier map, a torsor under $U_\tau(F) = U(F) \cap M_\tau(F)$.

<u>Already proved</u>: $Y_\nu^{\mathfrak{a}}$ is equi-dimension of dim $= \langle \rho, \mu - \tau\rangle$

and $Irr(X_{\mu^*}(\varpi^{\tau^*}))/T(F)\cdot U_\tau(F) \overset{bij.}{\longleftrightarrow} Irr(S_\tau \cap Gr_\mu)$

<u>Key</u>: understand the monodromy: $\exists\, \nu_{\mathfrak{a}}$ s.t. $\mathfrak{a} \in fil_\nu(V_\mu(\tau))$

will show that
$$\begin{tikzcd}
Y_\nu^{\mathfrak{a}} \cap Gr_\nu \arrow[d] & \widetilde{Y_\nu^{\mathfrak{a}}} \cap Gr_\nu \arrow[l] \arrow[r] \arrow[d] & (S_\lambda \cap Gr_\mu)^{\mathfrak{a}} \arrow[d] \\
{} & {} & {}
\end{tikzcd}$$

$$S_\nu \cap Gr_\nu \longleftarrow L^r U \longrightarrow L^r U \longrightarrow S_\lambda$$

$$u \longmapsto \sigma(u)^{-1} \varpi^{\tau} u \varpi^{-\tau}$$

with Galois gp $U_\tau(\mathcal{O}/\varpi^r)$

Need to show that the surjectivity of the monodromy map

$$\begin{tikzcd}
\pi_1\left((\overline{S_\lambda \cap Gr_\mu})^a\right) \arrow[r, "?"] \arrow[dr, "??"'] & U_\tau(\mathcal{O}/\varpi^r) \arrow[d] \\
& U_\tau(\mathcal{O}/\varpi^r)^{ab}
\end{tikzcd}$$

use crystal theory to do so $\longrightarrow$??