# Square-tiled surfaces and interval exchanges: geometry, dynamics, combinatorics and applications

## Lecture 10. Formula for the sum of the Lyapunov exponents of the Kontsevich–Zorich cocycle

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# Siegel–Veech constants

## Configurations of cylinders filled with closed geodesics

Closed regular geodesics on flat surfaces appear in families of parallel closed geodesics sharing the same length. Every such family fills one or several *maximal cylinders* having conical points on each of the boundary components. The combinatorial geometry of the resulting decomposition of the surface into a necklace is encoded by a *configuration* $\mathcal{C}$ of cylinders, with homologous waste curves sharing the same length and direction.

Denote by $N_{area}(S, L)$ the sum of areas of all cylinders spanned by all regular closed geodesics of length at most $L$ on a translation surface $S$ of area $1$.

For any $L > 0$ denote by $N_{\mathcal{C}}(S, L)$ the number of occurrences of a specific configuration $\mathcal{C}$ of cylinders filled with closed regular geodesics of length at most $L$ on a translation surface $S$ of area $1$.

## Example of a configuration of several cylinders

## Quadratic asymptotics for counting functions

**Theorem (A. Eskin and H. Masur, 2001).** *For almost all flat surfaces* $S$ *of unit area in any connected component* $\mathcal{H}$ *of any stratum of Abelian differentials, the counting functions* $N_{\mathcal{C}}(S,L)$*, and* $N_{area}(S,L)$ *have exact quadratic asymptotics*

$$\lim_{L\to\infty}\frac{N_{\mathcal{C}}(S,L)}{\pi L^2} = c(\mathcal{C},\mathcal{H}) \qquad \lim_{L\to\infty}\frac{N_{area}(S,L)}{\pi L^2} = c_{area}(\mathcal{H})\,,$$

*where the Siegel–Veech constants* $c(\mathcal{C},\mathcal{H})$*, and* $c_{area}(\mathcal{H})$ *depend only on the ambient component* $\mathcal{H}$ *(and on the configuration* $\mathcal{C}$ *in case of* $c(\mathcal{C},\mathcal{L})$*).*

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**Theorem (W. Veech, 1998).** *Let $\mathcal{H}$ be a component of a stratum of Abelian differentials; let $d\nu$ be the associated ergodic probability measure on $\mathcal{H}$. The following ratio is equal to the constant $c(\mathcal{C},\mathcal{H})$ (and, in particular, does not depend on the value of a positive parameter $L$)*:

$$\frac{1}{\pi L^2}\int_{\mathcal{H}} N_{\mathcal{C}}(S,L)\,d\nu = c(\mathcal{C},\mathcal{H})\,.$$

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## Siegel–Veech formula

To every closed regular geodesic $\gamma$ on a flat surface $S$ we associate a vector $\vec{v}(\gamma)$ in $\mathbb{R}^2$ having the length and the direction of $\gamma$. In other words, $\vec{v} = \int_\gamma \omega$, where we consider a complex number as a vector in $\mathbb{R}^2 \simeq \mathbb{C}$. Applying this construction to all closed regular geodesic on $S$ we construct a discrete set $V(S) \subset \mathbb{R}^2$. Consider the following operator $f \mapsto \hat{f}$ from functions with compact support on $\mathbb{R}^2$ to functions on the stratum $\mathcal{H}_1(\beta) = \mathcal{H}_1(d_1, \dots, d_n)$:

$$\hat{f}(S) := \sum_{\vec{v} \in V(S)} f(\vec{v})$$

Function $\hat{f}(S)$ generalizes the counting functions $N_{\mathcal{C}}(S, L)$. Namely, when $f = \chi_L$ is the characteristic function $\chi_L$ of the disc of radius $L$ with the center at the origin of $\mathbb{R}^2$, the function $\hat{\chi}_L(S)$ counts the number of regular closed geodesics of length at most $L$ on a flat surface $S$.

## Siegel–Veech formula

**Theorem (W. Veech, 1998).** *Let $\mathcal{H}$ be a component of a stratum of Abelian differentials; let $d\nu$ be the associated ergodic probability measure on $\mathcal{H}$. For any sufficiently regular function $f : \mathbb{R}^2 \to \mathbb{R}$ with compact support the following equality is valid:*

$$\int_{\mathcal{H}} \hat{f}(S)\, d\nu = c \int_{\mathbb{R}^2} f(x,y)\, dx\, dy \ ,$$

*where the constant $c$ does not depend on the function $f$.*

The same theorem is valid for configurations of homologous saddle connections. The constant $c$ depends only on the topological type of the configuration $\mathcal{C}$ used in the definition of the Siegel–Veech transform $f \to \hat{f}$.

All the above theorems can be generalized to any affine $\mathrm{SL}(2,\mathbb{R})$-invariant submanifold endowed with the associated $\mathrm{SL}(2,\mathbb{R})$-invariant ergodic probability measure.

## Evaluation of Siegel–Veech constants

**Theorem (A. Eskin, H. Masur, A. Zorich, 2003)**

$$c_{area} = \frac{1}{\dim_{\mathbb{C}} \mathcal{H}(\alpha) - 1} \cdot \sum_q q \cdot \sum_{\substack{Configurations\ \mathcal{C} \\ containing\ q\ cylinders}} c(\mathcal{C})$$

where

$$c(\mathcal{C}) = \lim_{\varepsilon \to 0} \frac{1}{\pi \varepsilon^2} \frac{\operatorname{Vol}(\text{“}\varepsilon\text{-neighborhood of the cusp } \mathcal{C}\text{ ”})}{\operatorname{Vol} \mathcal{H}_1(\beta)} =$$

$$= \big(\text{explicit combinatorial factor}\big) \cdot \frac{\prod_{j=1}^{k} \operatorname{Vol} \mathcal{H}_1(\beta'_k)}{\operatorname{Vol} \mathcal{H}_1^{comp}(\beta)}$$

# Formula for the sum of the Lyapunov exponents

## Abelian differentials

**Theorem (A. Eskin, M. Kontsevich, A. Zorich, 2014).** *The Lyapunov exponents $\lambda_i$ of the bundle $H^1_{\mathbb{R}}$ along the Teichmüller flow restricted to an $\mathrm{SL}(2,\mathbb{R})$-invariant suborbifold $\mathcal{L} \subseteq \mathcal{H}_1(d_1,\dots,d_n)$ satisfy:*

$$\lambda_1 + \lambda_2 + \cdots + \lambda_g \;=\; \frac{1}{12}\cdot\sum_{i=1}^{n}\frac{d_i(d_i+2)}{d_i+1} + \frac{\pi^2}{3}\cdot c_{area}(\mathcal{L})\,.$$

*where $c_{\text{area}}(\mathcal{L})$ is the Siegel–Veech constant. The top exponent $\lambda_1$ is equal to one, $\lambda_1 = 1$.*

● One can write analogous formulae for the sum of exponents of any Hodge $*$-invariant covariantly constant subbundle of the bundle $H^1_{\mathbb{R}}$.
In those cases, when the bundle $H^1_{\mathbb{R}}$ has maximal possible splitting, one can compute all individual exponents.

## Quadratic differentials

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$$\lambda_1 + \lambda_2 + \cdots + \lambda_g \;=\; \frac{1}{24}\cdot\sum_{i=1}^{n}\frac{d_i(d_i+4)}{d_i+2} + \frac{\pi^2}{3}\cdot c_{area}(\mathcal{L})\,.$$

*where $c_{area}(\mathcal{L})$ is the Siegel–Veech constant. This time the top exponent $\lambda_1$ is strictly smaller than one, $\lambda_1 < 1$.*

● One can write analogous formulae for the sum of exponents of any Hodge $*$-invariant covariantly constant subbundle of the Hodge bundle.
In those cases, when the Hodge bundle has maximal possible splitting, one can compute all individual exponents.

## Siegel–Veech constant for an arithmetic Teichmüller disc

Consider a connected square-tiled surface $S_0$ in a stratum $\mathcal{H}_1(m_1, \dots, m_n)$. For every square-tiled surface $S_i$ in its $\mathrm{SL}(2,\mathbb{Z})$-orbit consider the decomposition of $S_i$ into maximal cylinders $cyl_{ij}$ filled with closed regular horizontal geodesics. Let $w_{ij}$ be the length of the corresponding closed horizontal geodesic and let $h_{ij}$ be the height of the cylinder $cyl_{ij}$.

**Theorem (A. Eskin, M. Kontsevich, A. Zorich, 2014).** *The Siegel–Veech constant of an arithmetic Teichmüller disc $\mathcal{T}$ is represented by the formula:*

$$\frac{\pi^2}{3} \cdot c_{area}(\mathcal{T}) = \frac{1}{\mathrm{card}(\mathrm{SL}(2,\mathbb{Z}) \cdot S_0)} \sum_{S_i \in \mathrm{SL}(2,\mathbb{Z}) \cdot S_0} \quad \sum_{\substack{\textit{horizontal} \\ \textit{cylinders } cyl_{ij} \\ \textit{such that} \\ S_i = \sqcup cyl_{ij}}} \frac{h_{ij}}{w_{ij}}.$$

**Conjecture (A. Eskin, M. Kontsevich, A. Zorich, 2014).** *For any $\mathrm{SL}(2,\mathbb{R})$-invariant suborbifold $\mathcal{L}$ in any stratum of Abelian differentials the corresponding Siegel–Veech constant $\pi^2 \cdot c_{area}(\mathcal{L})$ is a rational number.*

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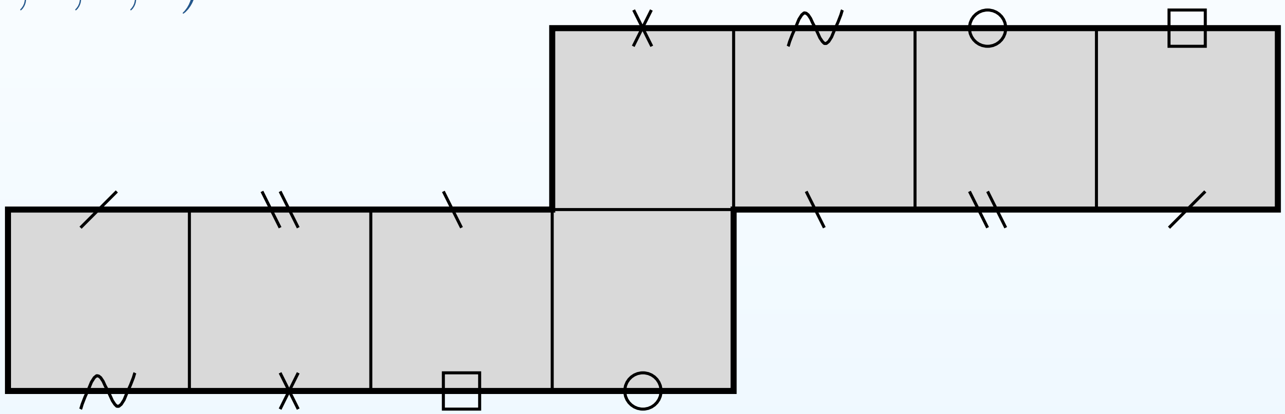
## Eierlegende Wollmilchsau

The following square-tiled surface is $\mathrm{SL}(2, \mathbb{Z})$-invariant. It belongs to the stratum $\mathcal{H}(1, 1, 1, 1)$.



Hence, the sum of the Lyapunov exponents for the corresponding Teichmüller disc equals

$$1 + \lambda_2 + \lambda_3 = \frac{1}{12} \cdot \sum_{i=1}^{4} \cdot \frac{1(1+2)}{1+1} + \frac{1}{1}\left(\frac{1}{4} + \frac{1}{4}\right) = \frac{1}{2} + \frac{1}{2} = 1$$

This implies that $\lambda_2 = \lambda_3 = 0$. (This result was first proved by G. Forni, who used symmetry arguments).

## Non degeneracy of the Lyapunov spectrum in high genera

**Corollary** *For any regular* $\mathrm{SL}(2,\mathbb{R})$*-invariant suborbifold in in any stratum of Abelian differentials in genus* $g \geq 7$ *the Lyapunov exponents* $\lambda_2 \geq \cdots \geq \lambda_k$ *are strictly positive, where* $k = \left[\frac{(g-1)g}{6g-3}\right] + 1$*.*

*For any regular* $\mathrm{SL}(2,\mathbb{R})$*-invariant suborbifold in the principal stratum* $\mathcal{H}_1(1\ldots1)$ *of Abelian differentials in genus* $g \geq 5$ *the Lyapunov exponents* $\lambda_2 \geq \cdots \geq \lambda_k$ *are strictly positive, where* $k = \left[\frac{g-1}{4}\right] + 1$*.*

*Proof.* Note that $1 = \lambda_1 > \lambda_2 \geq \lambda_3 \geq \cdots \geq 0$. Since $c_{area} > 0$, as soon as the expression

$$\frac{1}{12}\sum_{i=1}^{n}\frac{m_i(m_i+2)}{m_i+1}$$

in the formula for the sum of Lyapunov exponents is greater than or equal to some positive integer $k$, we get at least $k+1$ positive Lyapunov exponents $\lambda_1, \ldots, \lambda_k$. It remains to compute the minimum of this expression over all partitions $m_1 + \cdots + m_n$ of $2g-2$.

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# Application. Volumes of strata in genus zero

## Sum of Lyapunov exponents in genus zero

The sum of the Lyapunov exponents for any stratum $\mathcal{Q}_1(d_1, \dots, d_k)$ of meromorphic quadratic differentials with at most simple poles on a Riemann surface of genus zero vanishes for $H^1(\mathbb{C}\mathrm{P}^1) = 0$. Thus:

$$0 \;=\; \frac{1}{24} \cdot \sum_{i=1}^{n} \frac{d_i(d_i+4)}{d_i+2} + \frac{\pi^2}{3} \cdot c_{area}\big(\mathcal{Q}(d_1, \dots, d_k)\big)$$

The Siegel–Veech constant $c_{area}$ is expressed in terms of volumes of the initial stratum $\mathcal{Q}(d_1, \dots, d_k)$ and volumes of some simpler strata. Hence, our formula gives a recursive relation for volumes of strata in genus zero!

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## Volumes of strata in genus zero

An explicit computation gives the following answer.

Let

$$v(n) := \frac{n!!}{(n+1)!!} \cdot \pi^n \cdot \begin{cases} \pi & \text{when } n \geq -1 \text{ is odd} \\ 2 & \text{when } n \geq 0 \text{ is even} \end{cases}$$

By convention we set $(-1)!! := 0!! := 1$ , which implies that $v(-1) := 1$ and $v(0) := 2$.

**Theorem. (J. Athreya, A. Eskin, A. Zorich, 2016)** *The volume of any stratum $\mathcal{Q}_1(d_1, \ldots, d_k)$ of meromorphic quadratic differentials with at most simple poles on $\mathbb{C}\mathrm{P}^1$ (i.e. $d_i \in \{-1\,;\,0\} \cup \mathbb{N}$ for $i = 1, \ldots, k$, and $\sum_{i=1}^k d_i = -4$) is equal to*

$$\operatorname{Vol} \mathcal{Q}_1(d_1, \ldots, d_k) = 2\pi \cdot \prod_{i=1}^{k} v(d_i)$$

This formula was conjectured by M. Kontsevich about 2006.

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# Relation with Lyapunov exponents of the Rauzy–Veech induction

## Time scale for the first return map

Let $f_t : X^n \to X^n$ be a flow ergodic with respect to a finite measure $\mu$. Let $Y^{n-1} \subset X^n$ be a section to the flow $f_t$ and $G : Y^{n-1} \to Y^{n-1}$ be the induced first return map. Suppose that the induced $G$-invariant measure $\eta$ on $Y^{n-1}$ is finite.

Consider a trajectory launched from a point $y$ of the section $Y$ and define $t(y)$ to be the time of the first return of the trajectory back to the section $Y^{n-1}$. By ergodic theorem for almost any $y \in Y^{d-1}$ we have the following asymptotic formula for the time $t_N(y)$ of the $N$-th return:

$$\lim_{N\to+\infty} \frac{t_N}{N} = \lim_{N\to+\infty} \frac{t(y) + t(f(y)) + \cdots + t(f^{(N-1)}(y))}{N} = t_{mean}$$

where $t_{mean} = const.$

Thus, trajectory $f_t(y)$ reaches the point $G^{(N)}(y) \in Y^{d-1}$ of $N$-th return to $Y^{d-1}$ at a time $t_N \approx N \cdot t_{mean}$ for large $N$.

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## Cocycle induced by the first return map

In other words, the “discrete time” $N$ of the discrete dynamical system $G^{(N)}$ is asymptotically proportional to the “continuous time” $t$ of the continuous dynamical system $f_t$ with coefficient $t_{mean}$.

We have seen that a multiplicative cocycle $a(x,t)$ related to the flow $f_t$ on $X^n$ induces a multiplicative cocycle $\mathcal{A}(y,N)$ for the map $G : Y^{n-1} \to Y^{n-1}$: for a point $y \in Y^{n-1}$ we have

$$\mathcal{A}(y,N) := a(y, t_N(y))$$

Clearly the Lyapunov exponents $\theta_j$ of the cocycle $\mathcal{A}(y,N)$ are proportional to the Lyapunov exponents $\nu_j$ of the continuous cocycle $a(x,t)$:

$$\theta_k = t_{mean} \cdot \nu_k \qquad k = 1, 2, \dots,$$

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We have seen that a multiplicative cocycle $a(x,t)$ related to the flow $f_t$ on $X^n$ induces a multiplicative cocycle $\mathcal{A}(y,N)$ for the map $G : Y^{n-1} \to Y^{n-1}$: for a point $y \in Y^{n-1}$ we have

$$\mathcal{A}(y,N) := a(y, t_N(y))$$

Clearly the Lyapunov exponents $\theta_j$ of the cocycle $\mathcal{A}(y,N)$ are proportional to the Lyapunov exponents $\nu_j$ of the continuous cocycle $a(x,t)$:

$$\theta_k = t_{mean} \cdot \nu_k \qquad k = 1, 2, \dots,$$

## Relation to accelerated Rauzy–Veech induction

Now let us apply these results to the particular case when:

- $X$ is a fundamental in the space of zippered rectangles (which is a finite covering over some $\mathcal{H}_1(d_1, \dots, d_n)$);
- $g_t$ is the Teichmüller geodesic flow on it;
- the cocycle $a(x,t)$ is the holonomy of the vector bundle $H^1(S, \{saddles\}; \mathbb{C})$ along trajectories of the flow;
- the section $Y \subset X$ of codimension one is the section used to construct the accelerated Rauzy—Veech induction;
- the induced cocycle $\mathcal{A}(y, N)$ is defined by a matrix-valued function $A(y)$ representing the Rauzy—Veech induction on zippered rectangles:

$$\begin{pmatrix} \lambda \\ h \end{pmatrix} \mapsto \begin{pmatrix} B^{-1} & 0 \\ 0 & B^T \end{pmatrix} \cdot \begin{pmatrix} \lambda \\ h \end{pmatrix} \qquad A = \begin{pmatrix} B^{-1} & 0 \\ 0 & B^T \end{pmatrix} \qquad y = \begin{pmatrix} \lambda \\ h \end{pmatrix}$$

## Relation to accelerated Rauzy–Veech induction

Recall that the matrix $B$ is completely defined by the interval exchange transformation $(\lambda, \pi)$ corresponding to a zippered rectangle. The top $g$ Lyapunov exponents $\theta_1, \dots, \theta_g$ of cocycles $B^{-1}(\lambda, \pi)$ and $B^T(\lambda, \pi)$ coincide.

According to our general considerations these Lyapunov exponents are related with the top $g$ Lyapunov exponents $1+\nu_1, \dots, 1+\nu_g$ of the Teichmüller geodesic flow by the following formula:

$$\theta_k = t_{mean} \cdot \nu_k \qquad k = 1, 2, \dots, g$$

Here $t_{mean}$ is an average return time of a Teichmüller geodesic to the section $Y$. One has the following formula for $t_{mean}$:

$$t_{mean} = \int_Y \left| \log(1-\lambda_m) - \log(1-\lambda_{\pi^{-1}(m)}) \right| d\eta$$

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## Exercise: Levy constant

**Exercise** Consider the Rauzy–Veech induction in the torus case applying it to interval exchange transformations of two subintervals. We have seen that in this case the space of interval exchange transformations is just an interval $(0,1)$. Find an explicit formulae for the Rauzy–Veech renormalization map $\mathcal{T} : (0,1) \to (0,1)$ and for the "fast" renormalization map $\mathcal{G} : (0,1) \to (0,1)$. Explain why the invariant measure is infinite for the map $\mathcal{T}$. Find a relation between $\mathcal{G}$ and the Gauss map $g : x \mapsto \left\{\frac{1}{x}\right\}$. Let

$$\frac{p_s}{q_s} = \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{\cdots + \cfrac{1}{n_s}}}}$$

## Exercise: Levy constant

be the $s$-th best rational approximation of a real number $x \in (0,1)$. In the torus case the spectrum of Lyapunov exponents reduces to a single pair $\theta_1 > -\theta_1$. Show that for almost all $x \in (0,1)$ the Lyapunov exponent $\theta_1$ of the cocycle $B^{-1}$ (called in number theory the *Lévy constant*) is responsible for the growth rate of the denominator of the continued fraction expansion of $x$:

$$\lim_{s\to\infty} \frac{\log q_s}{s} = \theta_1 = \frac{\pi^2}{12\log 2}$$

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It is relatively easy to compute Lyapunov exponents $\theta_1, \ldots, \theta_g$ experimentally using the cocycle $B^{-1}(\lambda, \pi)$. The results give a perfect match with a theoretical prediction.

## Some experimental values of Lyapunov exponents

| Types of zeros $\mathbf{d}$ | Hyperelliptic or spin structure | Lyapunov exponents | | |
|---|---|---|---|---|
| | | $\lambda_2$ | $\lambda_3$ | $\sum_{j=1}^{g} \lambda_j$ |
| $(4)$ | *hyperelliptic* | $0.6156$ | $0.1844$ | $9/5$ |
| $(4)$ | *odd* | $0.4179$ | $0.1821$ | $8/5$ |
| $(1,3)$ | $-$ | $0.5202$ | $0.2298$ | $7/4$ |
| $(2,2)$ | *hyperelliptic* | $0.6883$ | $0.3117$ | $4/2$ |
| $(2,2)$ | *odd* | $0.4218$ | $0.2449$ | $5/3$ |

## Large genus asymptotics

However, for further strata the fractions become more complicated. Say, for $\mathcal{H}(1,1,1,1,1,1,1,1)$ one already gets

$$\nu_1 + \cdots + \nu_5 = \frac{235\,761}{93\,428}$$

To complete, I state the following Theorem (obtained using a result of A. Eskin, M. Kontsevich, M. Möller and A. Zorich, 2018) and a Conjecture.

**Theorem (Fei Yu 2018).** *For the hyperelliptic components $\mathcal{H}^{hyp}(2g-2)$ and $\mathcal{H}^{hyp}(g-1,g-1)$ and for any fixed $k$ one has*

$$\lim_{g\to\infty} \nu_k = 1\,.$$

**Conjecture (M. Kontsevich and A. Zorich, 1996).** *For all other components of all other strata in the moduli spaces of holomorphic differentials one has*

$$\lim_{g\to\infty} \nu_2 = \frac{1}{2}$$

*uniformly for all strata in genus $g$.*