Recent developments in Seiberg-Witten theory

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- Lecture 1 : Basics of SW theory
- Lecture 2 : Diff vs. Homeo in 4D
- Lecture 3 : SW Floer homotopy theory

Today.

- §1. Introduction
- §2. SW equations
- (§3. Finite-dim approximation.)

§1

History.

non-linear PDE

- Donaldson theory (1983 ~)

  ... used Yang-Mills theory (anti-self-dual equation)

  to 4-dim. topology

- SW theory (1994 ~)

  ... used SW equations to 4-dim topology and geometry

called (mathematical) gauge theory.

Framework (of SW)

Step 1.

X^4: oriented closed smooth 4-manifold

Take a Riemannian metric on X

(and fix a spin^c structure on X).

$$\rightsquigarrow \begin{cases} F_A^+ = \sigma(\phi,\phi), \\ \not{D}_A \phi = 0 \end{cases}$$: SW equations

Step 2

$\mathcal{M}(X) := \{\text{solutions to the SW eq}\} / \text{gauge symmetry}$

moduli space

this is "generically" a finite dim compact smooth manifold.

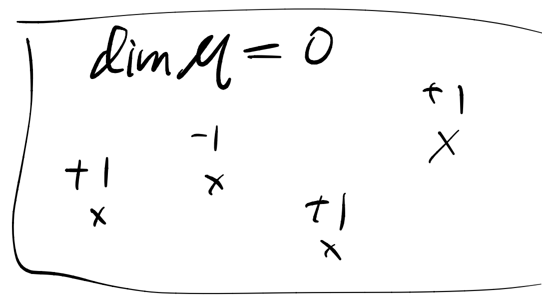
Feature of SW

Step 3. Two ways to use $\mathcal{M}(X)$:

(i) (Invariant) $SW(X) :=$ "$\# \mathcal{M}(X)$" $\in \mathbb{Z}$

SW invariant

can be used to detect "exotic" 4-mfds.



(ii) (Constraint) From $\mathcal{M}(X)$, get some "constraint" on a classical invariant of X.

intersection form

- Donaldson's diagonalization
- Furuta's 10/8-ineq.

Def.

X : oriented closed 4-manifold

$$Q_X : H^2(X;\mathbb{Z})/\mathrm{Tor} \times H^2(X;\mathbb{Z})/\mathrm{Tor} \to \mathbb{Z}$$

$$(\alpha, \beta) \longmapsto \langle \alpha \cup \beta, [X] \rangle$$

Q_X is symmetric bilinear form over $\mathbb{Z}$

is also "non-degenerate". form

Def.

$b^+(X) := \#\{\text{positive eigenvalues of } Q_X\}$

$b^-(X) := \#\{\text{negative} \text{ ———— }\}$

$\sigma(X) := b^+(X) - b^-(X)$.

signature

e.g.

	b^+	b^-
$Q_{\mathbb{C}P^2} = (1)$ $H_2(\mathbb{C}P^2) = \mathbb{Z}[\mathbb{C}P^1]$	1	0
$Q_{S^2\times S^2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $H_2(S^2\times S^2) = \mathbb{Z}\cdot[S^2\times pt] \oplus \mathbb{Z}[pt\times S^2]$	1	1
$Q_{K3} = 2(\underbrace{-E_8}) \oplus 3\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ↑ negative-definite, rank 8	3	19

even (i.e. $Q(\alpha,\alpha) \equiv 0 \mod 2$)
$\forall \alpha$

<u>Thm</u> (Donaldson, 1983)

X : oriented closed <u>smooth</u> 4-mfd

If X is negative-definite (i.e. $b^+(X) = 0$),

then $$Q_X \underset{/\mathbb{Z}}{\cong} \begin{pmatrix} -1 & & \\ & \ddots & \\ & & -1 \end{pmatrix}.$$

Remark: $\exists$ many definite forms $/\mathbb{Z}$

e.g. $\#\{\text{definite forms } /\mathbb{Z} \text{ of rank } 32\} \geq 10^7$.

Freedman (1982) : $^\forall$forms / $\mathbb{Z}$ can be realized by topological 4-mfds.

$-E_8$ Casson handle

Other topics

- Positive scalar curvature
- symplectic 4-mfds
- connected sum

§2. SW equations

- spin^c structure
- Dirac operator
- A quadratic term

- $\underline{\text{spin}^c \text{ structure}}$

<u>Fact.</u>

$$\pi_1 SO(n) = \begin{cases} \mathbb{Z}, & n=2 \\ \mathbb{Z}/2, & n \geq 3 \end{cases} \rightsquigarrow \exists! \text{ double covering of } SO(n).$$

$$\text{Spin}(n)$$

e.g. $\mathrm{Spin}(3) = Sp(1) = \{ q \in \mathbb{H} \mid |q| = 1 \} \cong S^3$

$$\begin{array}{ccc}
\mathrm{Spin}(3) & & q \\
\big\downarrow \mathbb{Z}/2 & & \big\downarrow \\
SO(3) = SO(\mathrm{Im}\,\mathbb{H}) & \ni & \left(\begin{array}{ccc} \mathrm{Im}\,\mathbb{H} & \longrightarrow & \mathrm{Im}\,\mathbb{H} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ v & \longmapsto & q v q^{-1} \end{array} \right)
\end{array}$$

$$\mathbb{H} = \mathbb{R} \oplus \underbrace{i\mathbb{R} \oplus j\mathbb{R} \oplus k\mathbb{R}}_{\mathrm{Im}\,\mathbb{H}}$$

$$\begin{array}{ccc}
\mathrm{Spin}(4) = Sp(1) \times Sp(1) & \ni & (q_+, q_-) \\
\big\downarrow \mathbb{Z}/2 & & \big\downarrow \\
SO(4) = SO(\mathbb{H}) & \ni & \left(\begin{array}{ccc} \mathbb{H} & \longrightarrow & \mathbb{H} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ v & \longmapsto & q_- v q_+^{-1} \end{array} \right)
\end{array}$$

$\lrcorner$

$$\mathrm{Spin}^c(n) := \frac{\mathrm{Spin}(n) \times U(1)}{\{\pm 1\}} \qquad \{\pm 1\} \hookrightarrow \mathrm{Spin}(n) \twoheadrightarrow SO(n)$$

Def.

X : oriented n-manifold
equipped a Riemannian metric.

A <u>spin structure</u> $\mathfrak{s}$ on X is a pair of
a principal $\mathrm{Spin}(n)$-bdle $\begin{array}{c} P \\ \downarrow \\ X \end{array}$ and

an isomorphism $P \times_{\mathrm{Spin}(n)} SO(n) \cong Fr(X)$ as $SO(n)$-bundle.

$\mathrm{Spin}(n) \to SO(n)$

oriented
frame bundle

spin^c str : defined similarly
(Spin$^c(n)$ in place of Spin(n)).

<u>Focus on dim 4:</u>

$$\begin{tikzcd} & U(2) \\ Spin^c(4) \arrow[ur, "\Delta_+"] \arrow[r, "\Delta_-"] \arrow[dr, "\det"'] & U(2) \\ & U(1) \end{tikzcd}$$

$$\left\{ \begin{array}{l} \Delta_\pm : Spin^c(4) = \dfrac{\overbrace{Sp(1) \times Sp(1)}^{SU(2) \cong} \times U(1)}{\pm 1} \longrightarrow U(2) \\ \qquad\qquad\qquad\qquad\quad \rotatebox{90}{$\in$} \qquad\qquad\qquad\qquad\quad \rotatebox{90}{$\in$} \\ \qquad\qquad\quad [A_+, A_-, \lambda] \longmapsto A_\pm \cdot \lambda \\ \\ \det : Spin^c(4) \longrightarrow U(1) \\ \qquad\quad \rotatebox{90}{$\in$} \qquad\qquad\qquad \rotatebox{90}{$\in$} \\ \quad [A_+, A_-, \lambda] \longmapsto \lambda^2 \end{array} \right.$$

X^4 : 4-mfd, $\mathfrak{s}$: a spin^c structure on X

$$\updownarrow$$

$$\left(\begin{tikzcd} \mathrm{Spin}^c(4) \arrow[r] & P \arrow[d] \arrow[r] & \mathrm{Fr}(X) \arrow[dl] \\ & X & \end{tikzcd} \right)$$

$$\rightsquigarrow \quad \begin{tikzcd} S^{\pm} := P \times_{\Delta_{\pm}} \mathbb{C}^2 \arrow[d] \\ X \end{tikzcd} \quad , \quad \begin{tikzcd} L = P \times_{\det} \mathbb{C} \arrow[d] \\ X \end{tikzcd}$$

spinor bundles , determinant line bundle.

- Self-duality.

$* : \Lambda^2 T^*X \circlearrowleft, \quad ** = 1$

$\rightsquigarrow \Lambda^2 T^*X = \Lambda^+ \oplus \Lambda^-$

$\Lambda^+ \circlearrowleft\ * = 1$ self-dual

$\Lambda^- \circlearrowleft\ * = -1$ anti-self-dual

Fact.

$\underline{\mathbb{R}} \oplus \Lambda^+ \cong P \times_{Spin^c(4)} \mathbb{R}^4$, where $Spin^c(4) \curvearrowright \mathbb{R}^4 = \mathbb{H}$

$\mathbb{R} \oplus \mathrm{Im}\,\mathbb{H} = \mathbb{H}$

$$\frac{Sp(1)\times Sp(1)\times U(1)}{\pm 1}$$

$$[\xi_+, \xi_-, \lambda] \mapsto \boxed{\begin{pmatrix} \mathbb{H} \to \mathbb{H} \\ \xi \mapsto \xi_+ \xi \xi_+^{-1} \end{pmatrix}}$$

Summary:

T^*X, $\underline{\mathbb{R}} \oplus \Lambda^+$, S^+, S^- are associated vector bundles of real rank 4 from P.

$$\begin{array}{ccc} T^*X, & \underline{\mathbb{R}} \oplus \Lambda^+ & S^\pm \\ \updownarrow & \updownarrow & \updownarrow \\ \mathbb{H}_{TX}, & \mathbb{H}_{\Delta}, & \mathbb{H}_{\Delta^\pm} \end{array}$$

: copies of $\mathbb{H}$, as $Spin^c(4)$-representation.

• <u>Dirac operator</u>

Clifford multiplication

$$T^*X \otimes_{\mathbb{R}} S^+ \longrightarrow S^-$$

$\rightsquigarrow$ induced from

$$\begin{tikzcd}
\mathbb{H}_{TX} \times \mathbb{H}_{\Delta^+} \arrow[r] & \mathbb{H}_{\Delta^-} \\[-20pt]
\rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} \\[-20pt]
(q, q') \arrow[r, mapsto] & qq'
\end{tikzcd}$$

(one may check this map is $\mathrm{Spin}^c(4)$-equivariant.)

$$A \in \mathcal{A}(L) := \{ U(1)\text{-connections on } L \}.$$

$$\rightsquigarrow \not{D}_A : \Gamma(S^+) \to \Gamma(S^-) \quad \text{is induced as follows:}$$

$$Spin^c(4) = \frac{Spin(4) \times U(1)}{\pm 1}$$

$$\rightsquigarrow \mathfrak{spin}^c(4) \cong \underbrace{\mathfrak{spin}(4)}_{\substack{\cong \\ \mathfrak{so}(4)}} \times \underbrace{\mathfrak{u}(1)}$$

Lie alg

Levi-Civita

A determines a connection form valued at here

$\rightsquigarrow$ a connection on P.

$\rightsquigarrow$ a connection on S^+, denoted by
induce

$$\nabla_A : \Gamma(S^+) \longrightarrow \Gamma(T^*X \otimes S^+).$$

$$\not{D}_A : \Gamma(S^+) \xrightarrow{\nabla_A} \Gamma(T^*X \otimes S^+) \xrightarrow{\text{Clifford multiplication}} \Gamma(S^-).$$

<u>Dirac operator</u>.

- A quadratic term :

A map $\sigma : \begin{array}{ccc} S^+ & \longrightarrow & \Lambda^+ \ (\subset \underline{\mathbb{R}} \oplus \Lambda^+) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \phi & \longmapsto & \sigma(\phi, \phi) \end{array}$

is induced from

$$\begin{tikzcd}
S^+ \arrow[d, leftrightarrow] \arrow[rrr, bend left] & & \underline{\mathbb{R}} \oplus \Lambda^+ \arrow[d, leftrightarrow] & \supset \Lambda^+ \arrow[d, leftrightarrow] \\
\mathbb{H}_{\Delta+} \arrow[rr] & & \mathbb{H}_\Lambda & \supset \mathrm{Im}\,\mathbb{H} \\
q \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[rr, mapsto] & & q i \bar{q} \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[ur] &
\end{tikzcd}$$

(may check $Spin^c(4)$-equiv.)

$$\Omega^2 = \Omega^+ \oplus \Omega^-$$
$$\downarrow p^+ \text{ : projection}$$

$A \in \mathcal{A}(L)$

$$F_A^+ := p^+(\underbrace{F_A}_{= dA}) \in \Omega^+(X, \underbrace{u(1)}_{= i\mathbb{R}}) = i\Omega^+(X)$$

SW equations

For $\underbrace{(A, \varphi)}_{\text{unknow function}} \in \mathcal{A}(L) \times \Gamma(S^+)$,

$$\begin{cases} F_A^+ = \sigma(\varphi, \varphi), \\ \not{D}_A \varphi = 0 \end{cases}$$

$$\underset{\substack{\uparrow \\ \text{form}}}{\not{D}_{A+a}}\varphi = \not{D}_A \varphi + \frac{a}{2} \underset{\substack{\uparrow \\ \text{Clifford multi.}}}{\cdot} \varphi$$

Gauge group

say, C^∞-maps

$$u \in \mathcal{G} = \mathcal{G}(X) = \mathrm{Map}(X, U(1)) \ : \text{ gauge group}$$

$\circlearrowright$ $\mathcal{A}(L) \times \Gamma(S^+)$ by $u \cdot (A, \phi) := (A + u^{-1}du, u \cdot \phi)$

$\mathcal{G} \curvearrowright \Omega^+$: trivially, $\mathcal{G} \curvearrowright \Gamma(S^-)$ by $u \cdot \phi$.

$\phi \in \Gamma(S^-)$

$$SW : \mathcal{A}(L) \times \Gamma(S^+) \longrightarrow i\Omega^+ \times \Gamma(S^-) \ : \ \mathcal{G}\text{-equivariant.}$$

$$(A, \phi) \longmapsto (F_A^+ - \sigma(\phi, \phi), \not{D}_A \phi)$$

$\mathcal{M}(X,\mathfrak{s}) := SW^{-1}(0)/\mathcal{G}$. moduli space.

↑ depends on a metric we choose.

"Thm."

After perturbing SW,

$\mathcal{M}(X,\mathfrak{s})$ is finite-dim, compact, orientable smooth mfd.

may get a canonical orientation by fixing orientation of $H^1(X;\mathbb{R}) \oplus H^+(X)$.

↑ calculated by index thm:

$$\dim \mathcal{M}(X,\mathfrak{s}) = \frac{1}{4}\left(c_1(L)^2 - 2\chi(X) - 3\sigma(X)\right).$$

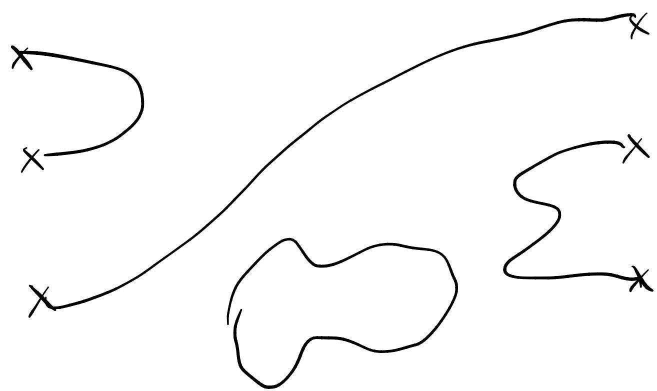
$$\left\{\begin{array}{l} \Omega^0 \xrightarrow{\substack{\mathfrak{g}\ \sim \\ \updownarrow \\ d}} \Omega^1 \xrightarrow{\substack{F_A^+ \\ \updownarrow \\ d^+}} \Omega^+, \quad \overset{\text{Fredholm index}}{\rightsquigarrow} \quad b_0 - b_1 + b^+ \\[2em] \qquad\qquad \Gamma(S^+) \xrightarrow{\not{D}_A} \Gamma(S^-) \underset{\mathbb{R}}{\rightsquigarrow} \frac{1}{4}\left(c_1(L)^2 - \sigma(X)\right), \end{array}\right.$$

To get a well-def invariant by "$\#\mathcal{M}(X,\mathfrak{s})$",

we need $\underline{b^+(X) \geq 2}$.

metrics

$$\rightsquigarrow \quad \mathcal{M}(X,\mathfrak{s},g_0) \overset{g_t}{\underset{\text{cobordant}}{\sim}} \mathcal{M}(X,\mathfrak{s},g_1)$$



$b^+ = 0$: $M(A,S)$ has a singular point.

$\Updownarrow$

reducible solution

$\shortparallel$

$\{(A,0)\}$

$\circlearrowleft$

$\mathcal{G}$

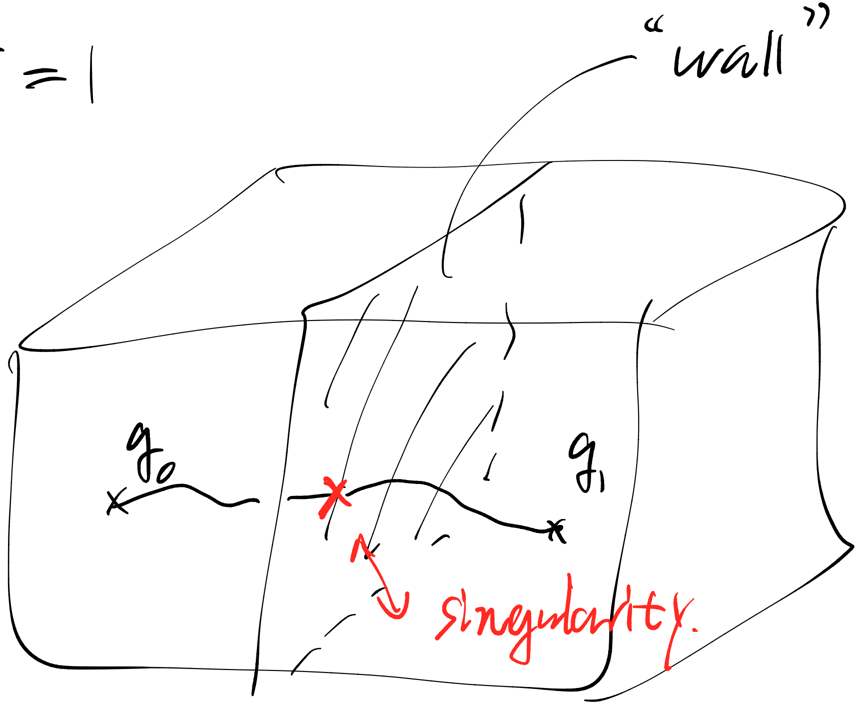
$$\boxed{\Omega^+ = \operatorname{Im} d^+ \oplus \mathcal{H}^+}$$

$\Updownarrow$

harmonic self-dual 2-forms

$$\boxed{\dim \mathcal{H}^+ = b^+}$$

$b^+ = 1$



$\mathrm{Met}(X) = \{\text{metrics}\}$

$$sw : \mathcal{V} \longrightarrow \mathcal{W}$$

$$\rightsquigarrow f : V^+ \longrightarrow W^+ \quad \text{finite-dim}$$

$$\circlearrowleft S^1 \qquad \circlearrowleft S^1$$

$$\mathcal{G}_0 \hookrightarrow \mathcal{G} \longrightarrow S^1$$

$$\underbrace{f^{-1}(0)}/S^1 \quad \underset{\text{cobordant}}{\sim} \quad M(X, \mathfrak{s})$$

↑

framed bordism class (well-defined indep. of approx.)

$\mathcal{L} : V \longrightarrow W$ Fredholm

$\Downarrow$

$0 = L : \operatorname{Ker}\mathcal{L} \longrightarrow \operatorname{Im}\mathcal{L}^{\perp}.$