Densities of coalescing particle systems

Jinjiong Yu
East China Normal University

THU-PKU-BNU Probability Webinar

Nov 4, 2020

Coalescing particle systems

- Let $(\widetilde{X}_t^x)_{t\geq 0}$ be a random walk or random motion in a space E, starting from $\widetilde{X}_0 = x$.
- Let $\{\widetilde{X}^{x_n}\}_{n=1}^{\infty}$ be independent. Define

$$X^{x_1} = \widetilde{X}^{x_1},$$

$$\tau_n = \inf\{t : \widetilde{X}_t^{x_n} = X_t^{x_k} \text{ for some } k < n\} \quad (n \geq 2),$$

$$K_n = \inf\{k : \widetilde{X}^{x_n} = X^{x_k}\} \quad (n \geq 2),$$

$$X^{x_n}(t) = \begin{cases} \widetilde{X}^{x_n}(t) & t < \tau_n, \\ X^{x_{K_n}}(t) & t \geq \tau_n, \end{cases} \qquad (n \geq 2).$$

  $(X^{x_n} : n \geq 1)$ is a coalescing particle system.
- Write $\xi_t^A := \{X_t^x : x \in A\}$ where $A \subset E$.

Coalescing particle systems

Question:

As $t \to \infty$, what is the density of the particle system $(\xi_t^A)_{t \geq 0}$ at given points $y_1, \dots, y_n$?

- If E is discrete such as $E = \mathbb{Z}^d$, then
$$\rho_n(y_1, \dots, y_n; t) = \mathbb{P}(y_1, \dots, y_n \in \xi_t^A)$$
- If E is continuous such as $E = \mathbb{R}^d$, then
$$\rho_n(y_1, \dots, y_n; t)\mathrm{d}y_1 \dots \mathrm{d}y_n = \mathbb{P}(\mathrm{d}y_1, \dots, \mathrm{d}y_n \in \xi_t^A)$$

As $t \to \infty$,

$$\rho_n(y_1, \dots, y_n; t) \sim c(y_1, \dots, y_n) t^{-\alpha(n)}?$$

Coalescing Brownian motions in $\mathbb{R}$

- $\xi_t^{\mathbb{Z}} = \{B_t^x : x \in \mathbb{Z}\}$, where B^x is a Brownian motion with $B_0^x = x$.
- $\rho_1(t) \sim Ct^{-1/2}$.

Coalescing Brownian motions in $\mathbb{R}$

- $\xi_t^{\mathbb{Z}} = \{B_t^x : x \in \mathbb{Z}\}$, where B^x is a Brownian motion with $B_0^x = x$.
- $\rho_1(t) \sim Ct^{-1/2}$.

Coalescing simple random walks $(\xi_t^A)_{t\geq 0}$ is dual to a voter model $(\zeta_t^B)_{t\geq 0}$ in the sense that

$$\mathbb{P}(B \cap \xi_t^A \neq \emptyset) = \mathbb{P}(\zeta_t^B \cap A \neq \emptyset),$$

where the generator of ζ^B is

$$A \mapsto A \cup \{x\} \quad (x \notin A) \quad \text{at rate } \tfrac{1}{2}\Big|\{y \in A : |y-x| = 1\}\Big|,$$
$$A \mapsto A \setminus \{x\} \quad (x \in A) \quad \text{at rate } \tfrac{1}{2}\Big|\{y \in A^c : |y-x| = 1\}\Big|.$$

Coalescing Brownian motions in $\mathbb{R}$

- $\xi_t^{\mathbb{Z}} = \{B_t^x : x \in \mathbb{Z}\}$, where B^x is a Brownian motion with $B_0^x = x$.
- $\rho_1(t) \sim Ct^{-1/2}$.

Coalescing simple random walks $(\xi_t^A)_{t\geq 0}$ is dual to a voter model $(\zeta_t^B)_{t\geq 0}$ in the sense that

$$\mathbb{P}(B \cap \xi_t^A \neq \emptyset) = \mathbb{P}(\zeta_t^B \cap A \neq \emptyset),$$

where the generator of ζ^B is

$$A \mapsto A \cup \{x\} \quad (x \notin A) \quad \text{at rate } \tfrac{1}{2}\Big|\{y \in A : |y - x| = 1\}\Big|,$$
$$A \mapsto A \setminus \{x\} \quad (x \in A) \quad \text{at rate } \tfrac{1}{2}\Big|\{y \in A^c : |y - x| = 1\}\Big|.$$

$$\left.\begin{array}{l} \mathbb{P}(0 \in \xi_t^{\mathbb{Z}}) = \mathbb{P}(\zeta_t^0 \neq \emptyset) \\ (|\zeta_t^0|)_{t\geq 0} \text{ is a simple random walk} \end{array}\right\} \Rightarrow \rho_1(t) \sim Ct^{-1/2}.$$

Coalescing Brownian motions in $\mathbb{R}$

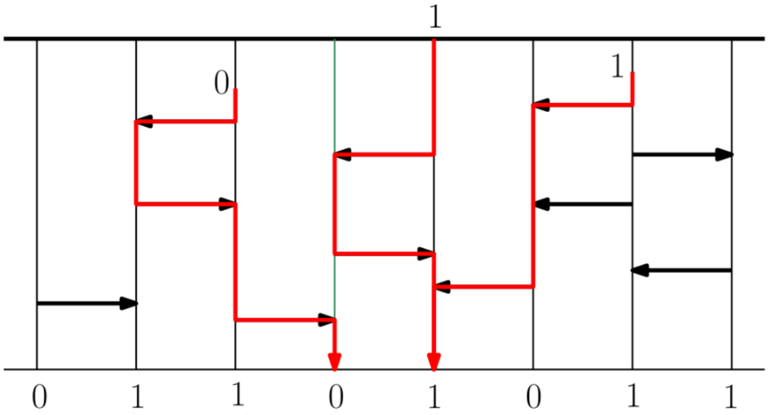


Figure: Duality between coalescing random walks and the voter model

Coalescing Brownian motions in $\mathbb{R}$

- [MRTZ '06] Given $y_1, \ldots, y_n$, there exist $C_1 < C_2$ such that

$$C_1 \leq \frac{\rho_n(y_1, \ldots, y_n; t)}{t^{-n/2-n(n-1)/4} \prod_{i<j} |y_i - y_j|} \leq C_2.$$

Coalescing Brownian motions in $\mathbb{R}$

- [MRTZ '06] Given $y_1, \ldots, y_n$, there exist $C_1 < C_2$ such that

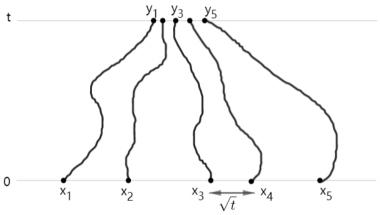
$$C_1 \le \frac{\rho_n(y_1, \ldots, y_n; t)}{t^{-n/2 - n(n-1)/4} \prod_{i<j} |y_i - y_j|} \le C_2.$$

Key tool: The Karlin-McGregor formula.

Transition kernel for non-intersecting Brownian motions

$$K_t(\mathbf{x}, \mathbf{y}) = \begin{vmatrix} K_t(x_1, y_1) & \ldots & K_t(x_1, y_n) \\ \vdots & & \vdots \\ K_t(x_n, y_1) & \ldots & K_t(x_n, y_n) \end{vmatrix}$$

where $K_t(x, y) = (2\pi t)^{-1/2} e^{-|y-x|^2/2t}$.



Coalescing Brownian motions in $\mathbb{R}$

- [TZ '11] If $\sup_i |y_i| \ll t^{1/2}$, then as $t \to \infty$,

  $$\rho_{2n}(y_1, \ldots, y_{2n}; t) \sim (4\pi)^{-n/2}\mathrm{Pf}(J^{(2n)}(\phi))t^{-n-n(2n-1)/2}\prod_{i<j}|y_i - y_j|,$$

  where $\phi(x) = x\exp\{-x^2/4\}$, $J^{(2n)}(\phi)$ is the $2n \times 2n$ anti-symmetric matrix with entries ($1 \le i < j \le 2n$)

  $$J_{ij}^{(2n)}(\phi) = (-1)^{j-1}\frac{1}{(i-1)!(j-1)!}\frac{\mathrm{d}^{i+j-2}\phi}{\mathrm{d}x^{i+j-2}}(0),$$

  and $\mathrm{Pf}(A)$ is the Paffian of the matrix A given by

  $$\mathrm{Pf}(A) = \sum_{\sigma\in\Sigma_{2n}} \mathrm{sgn}(\sigma)a_{i_1 j_1}\ldots a_{i_n j_n},$$

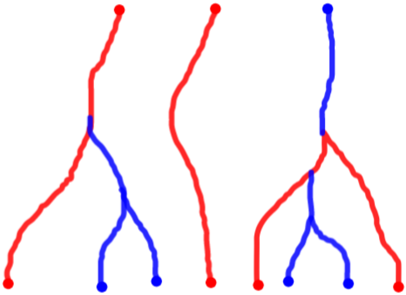
  where Σ_{2n} is the set of permutations of $\{1, \ldots, 2n\}$ with $\sigma(2k-1) = i_k,\ \sigma(2k) = j_k,\ i_k < j_k$ and $i_1 < \ldots < i_n$.

Coalescing Brownian motions in $\mathbb{R}$

1) A thinning relation between cBMs and annihilating BMs.
   cBMs: $B + B \to B$ aBMs: $B + B \to \emptyset$

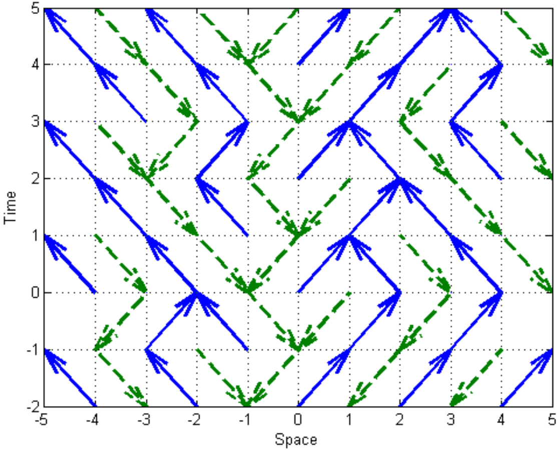
   Let $\Theta(A)$ be the random subset of A by thinning at rate 1/2.

   $$\Theta(\xi_t^A) \stackrel{\mathrm{d}}{=} \eta_t^{\Theta(A)} : \text{ aBMs at time } t \text{ with initial condition } \Theta(A)$$



Coalescing Brownian motions in $\mathbb{R}$

2) Duality between forward and backward Brownian webs.



Coalescing Brownian motions in $\mathbb{R}$

$1) + 2) \Rightarrow$ Duality formulas:

For $x_1 < \ldots < x_{2m}$ and $y_1 < \ldots < y_{2n}$, let

$$A = \{x_i : 1 \leq i \leq 2m\} \quad \text{and} \quad B = \{y_i : 1 \leq i \leq 2n\},$$

Let $I(B) = (y_1, y_2) \cup \ldots \cup (y_{2n-1}, y_{2n})$.

$$\mathbb{P}(\xi_t^A \cap I(B) = \emptyset) = \mathbb{P}(I(\hat{\eta}_t^B) \cap A = \emptyset),$$

By thinning the points in A,

$$\mathbb{E}\left[(-1)^{|\eta_t^A \cap I(B)|}\right] = \mathbb{E}\left[(-1)^{|I(\hat{\eta}_t^B) \cap A|}\right],$$

Coalescing Brownian motions in $\mathbb{R}$

3) Pfaffian formulas for aBMs. Let

$$m_t^{(2n)}(y_1, \ldots, y_{2n}) = \mathbb{E}\left[\prod_{x \in \hat{\eta}_t^B} g(x)\right],$$

where g is a bounded measurable function. Then,

$$m_t^{(2n)}(y_1, \ldots, y_{2n}) = \mathrm{Pf}\left(m_t^{(2)}(y_i, y_j) : 1 \leq i < j \leq 2n\right).$$

Coalescing Brownian motions in $\mathbb{R}$

3) Pfaffian formulas for aBMs. Let

$$m_t^{(2n)}(y_1, \ldots, y_{2n}) = \mathbb{E}\left[\prod_{x \in \hat{\eta}_t^B} g(x)\right],$$

where g is a bounded measurable function. Then,

$$m_t^{(2n)}(y_1, \ldots, y_{2n}) = \mathrm{Pf}\big(m_t^{(2)}(y_i, y_j) : 1 \le i < j \le 2n\big).$$

$1) + 2) + 3) \Rightarrow$

Differentiating w.r.t. $y_1, y_3, \ldots, y_{2n-1}$ and letting $y_{2i} \downarrow y_{2i-1}$, we get

$$\mathbb{P}(\mathrm{d}y_1, \mathrm{d}y_3, \ldots, \mathrm{d}y_{2n-1} \in \xi_t^{\mathbb{Z}}) = \mathbb{P}(\mathrm{d}y_1, \mathrm{d}y_3, \ldots, \mathrm{d}y_{2n-1} \in \xi_t^{\mathbb{R}})$$

Annihilating Brownian motions in $\mathbb{R}$

Using thinning relation, one can obtain results on aBMs if the corresponding results on cBMs are known.

- Maximal Entrance Law

$$\xi_t^{\frac{1}{n}\mathbb{Z}+\{0,\frac{1}{2n}\}} \to \xi_t^{\mathbb{R}}, \quad \xi_t^{\frac{1}{n}\mathbb{Z}+\{0,\frac{1}{n^2}\}} \to \tilde{\xi}_t^{\mathbb{R}} \quad \Rightarrow \quad \xi_t^{\mathbb{R}} \overset{\mathrm{d}}{=} \tilde{\xi}_t^{\mathbb{R}}$$

$$\eta_t^{\frac{1}{n}\mathbb{Z}+\{0,\frac{1}{2n}\}} \to \eta_t^{\mathbb{R}}, \quad \eta_t^{\frac{1}{n}\mathbb{Z}+\{0,\frac{1}{n^2}\}} \to \tilde{\eta}_t^{\mathbb{R}} \quad \Rightarrow \quad \eta_t^{\mathbb{R}} \overset{\mathrm{d}}{\neq} \tilde{\eta}_t^{\mathbb{R}}$$

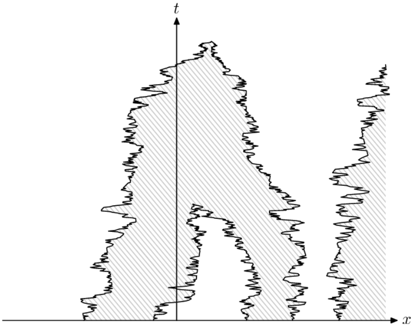
Annihilating Brownian motions in $\mathbb{R}$

Using thinning relation, one can obtain results on aBMs if the corresponding results on cBMs are known.

- Maximal Entrance Law

$$\xi_t^{\frac{1}{n}\mathbb{Z}+\{0,\frac{1}{2n}\}} \to \xi_t^{\mathbb{R}}, \quad \xi_t^{\frac{1}{n}\mathbb{Z}+\{0,\frac{1}{n^2}\}} \to \tilde{\xi}_t^{\mathbb{R}} \quad \Rightarrow \quad \xi_t^{\mathbb{R}} \stackrel{\mathrm{d}}{=} \tilde{\xi}_t^{\mathbb{R}}$$

$$\eta_t^{\frac{1}{n}\mathbb{Z}+\{0,\frac{1}{2n}\}} \to \eta_t^{\mathbb{R}}, \quad \eta_t^{\frac{1}{n}\mathbb{Z}+\{0,\frac{1}{n^2}\}} \to \tilde{\eta}_t^{\mathbb{R}} \quad \Rightarrow \quad \eta_t^{\mathbb{R}} \stackrel{\mathrm{d}}{\neq} \tilde{\eta}_t^{\mathbb{R}}$$



Annihilating Brownian motions in $\mathbb{R}$

- Identify aBMs with a measure-valued process $(\mu_t)_{t\geq 0}$.

  Let $\mu_t(\mathrm{d}x) = U_t(x)m(\mathrm{d}x)$ with $U_t : \mathbb{R} \to [0,1]$ the density w.r.t. the Lebesgue measure.
  Let $(1-\mu)_t(\mathrm{d}x) = (1 - U_t(x))m(\mathrm{d}x)$.
  Define an equivalence relation $\sim$ identifying μ and $1-\mu$. Then, there is a bijection between $(\mu_t)_{t\geq 0}$ with $U_t \in \{0,1\}$ and aBMs (η_t^A) with A a discrete set.

Annihilating Brownian motions in $\mathbb{R}$

- Identify aBMs with a measure-valued process $(\mu_t)_{t\geq 0}$.

  Let $\mu_t(\mathrm{d}x) = U_t(x)m(\mathrm{d}x)$ with $U_t : \mathbb{R} \to [0,1]$ the density w.r.t. the Lebesgue measure.
  Let $(1-\mu)_t(\mathrm{d}x) = (1-U_t(x))m(\mathrm{d}x)$.
  Define an equivalence relation $\sim$ identifying μ and $1-\mu$. Then, there is a bijection between $(\mu_t)_{t\geq 0}$ with $U_t \in \{0,1\}$ and aBMs (η_t^A) with A a discrete set.

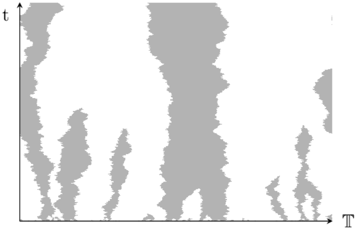
  μ_t is an random element in $\mathcal{M}_1(\mathbb{R})$ endowed with the vague topology.
  Define the quotient space $\mathcal{V} = \mathcal{M}_1(\mathbb{R})/\sim$.
  ABMs starting from a discrete set is a process taking values in

  $$\mathcal{V}_{\mathrm{d}} := \{\mu \in \mathcal{V} : U(x) = 0 \text{ or } 1\}.$$

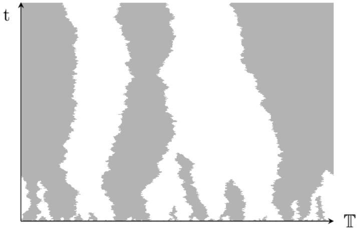
Annihilating Brownian motions in $\mathbb{R}$

- [HOV '18] For any entrance law $(\nu_t)_{t>0}$ for the semigroup $(P_t)_{t\geq 0}$ of aBMs there exists a sequence $(\nu^n)_{n\in\mathbb{N}}$ of probability measures on $\mathcal{V}_{\mathrm{d}}$ such that
$$\nu_t = \lim_{n\to\infty} \nu^n P_t, \quad t > 0.$$

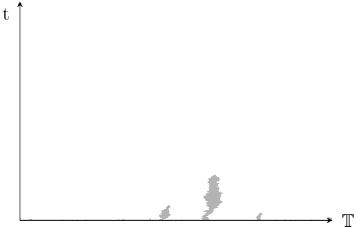
Annihilating Brownian motions in $\mathbb{R}$



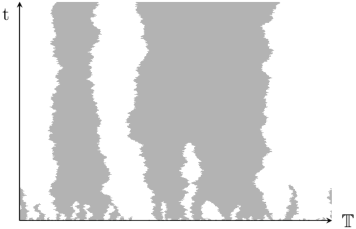
(a) $\mathbf{x}_n^{(1)} = \frac{1}{n}\mathbb{Z}$



(b) $\mathbf{x}_n^{(2)} \sim PPP(n)$



(c) $\mathbf{x}_n^{(3)} = \frac{1}{n}\mathbb{Z} + \{0, \frac{1}{n^2}\}$



(d) $\mathbf{x}_n^{(4)} = \frac{1}{n}\mathbb{Z} + \{0, \frac{1}{4n}\}$

(a) (b) $[U] = [\frac{1}{2}]$ (c) [U]=[0] (d) $[U] = [\frac{1}{4}]$

Coalescing simple random walks on $\mathbb{Z}^d$

$$\rho_n(y_1, \ldots, y_n; t) = \mathbb{P}(y_1, \ldots, y_n \in \xi_t^{\mathbb{Z}^d})$$

- [BG '80] $\rho_1(t) \sim \begin{cases} \frac{1}{\pi} \frac{\log t}{t}, & d = 2; \\ \frac{1}{\gamma_d} \frac{1}{t}, & d \geq 3, \end{cases}$

  where $\gamma_d = \mathbb{P}(X_t^1 \neq 0 \text{ for all } t > 0)$.

  An analysis on survival probability $\mathbb{P}(|\zeta_t^0| > 0)$ of the dual voter model which starts from the origin.

- [BK '00, BK '02] redo $\rho_1(t)$ for $d \geq 3$ by building the *effective rate equation* $\frac{\mathrm{d}}{\mathrm{d}t}\rho_1(t) = -\lambda \rho_1^2(t)$.

Coalescing simple random walks on $\mathbb{Z}^d$

- [LTZ '18] For $d \geq 2$,

$$\rho_n(y_1, \dots, y_n; t) \sim \rho_1^n(t)\, p_{\mathrm{NC}}(y_1, \dots, y_n; t),$$

where p_{NC} is the noncollision probability

$$\mathbb{P}(\text{no pair of random walks meets by time } t).$$

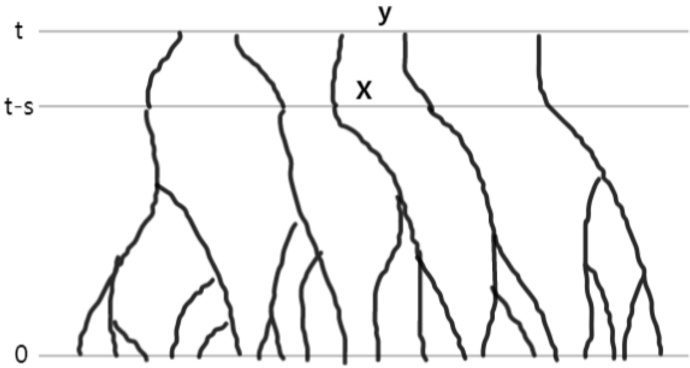
In particular, for $d = 2$,

$$\rho_n(y_1, \dots, y_n; t) \sim \frac{c(y_1, \dots, y_n)}{\pi^n} t^{-n} (\log t)^{n - n(n-1)/2}.$$

It is conjectured that

$$c(y_1, \dots, y_n) = \prod_{i<j} \log\left(|y_i - y_j|^2\right).$$

Coalescing simple random walks on $\mathbb{Z}^d$



Coalescing simple random walks on $\mathbb{Z}^d$

For $1 \ll s \ll t$,

$$
\begin{aligned}
\rho_n(\mathbf{y}; t) &= \sum_{\mathbf{x}\in\mathbb{Z}^d, x_i\neq x_j} \rho_n(\mathbf{x}; t-s)\, p_{\mathrm{NC}}(\mathbf{y},\mathbf{x}; s) \\
&\approx \sum_{\mathbf{x}\in\mathbb{Z}^d, x_i\neq x_j} \rho_n(\mathbf{x}; t-s)\, p_s(\mathbf{y},\mathbf{x})\, p_{\mathrm{NC}}(s) \\
&\approx \sum_{\mathbf{x}\in\mathbb{Z}^d} \rho_n(\mathbf{x}; t-s)\, p_s(\mathbf{x})\, p_{\mathrm{NC}}(s) \\
&= \mathbb{E}\Big[\Big(\sum_{x\in\mathbb{Z}} p_s(x) 1_{\{x\in\xi_{t-s}\}}\Big)^n\Big]\, p_{\mathrm{NC}}(s) \\
&\approx \rho_1^n(t)\, p_{\mathrm{NC}}(t).
\end{aligned}
$$

The noncollision probability $p_{\mathrm{NC}}(t) \sim c(y_1,\ldots,y_n)(\log t)^{-n(n-1)/2}$ is obtained in [CMP '10, LTZ '18].

Coalescing heavy-tailed random walks in $\mathbb{R}$

The underlying random walk X is a heavy-tailed random walks with increments in the domain of attraction of the α-stable law for some $\alpha \in (0, 2)$. Initial condition is $\mathbb{Z}$.

- [Y. '20$^+$] For $\alpha \in (0, 1]$,

$$\rho_n(y_1, \ldots, y_n; t) \sim \rho_1^n(t)\, p_{\mathrm{NC}}(y_1, \ldots, y_n; t).$$

- For $\alpha \in (1, 2)$, $\rho_n(y_1, \ldots, y_n; t)$ is unknown.
  1. The system is not integrable.
  2. The n-point correlation is crucial for $\alpha > 1$.
     Asymptotics for $p_{\mathrm{NC}}(y_1, \ldots, y_n; t)$ is not known. Indeed, it is not known even for $p_{\mathrm{NC}}(y_1, y_2, y_3; t)$.

Coalescing stable processes in $\mathbb{R}$

The underlying random motion X is an α-stable process with $\alpha > 1$.

- [EMS '13] $(\xi_t^{\mathbb{R}})_{t\geq 0}$ has the phenomenon of 'coming down from infinity', i.e., $\xi_t^{\mathbb{R}}$ is locally finite for any $t > 0$.

Observations:

- There exist $\beta, p > 0$ such that
$$|x_1 - x_2| < \varepsilon \quad \Rightarrow \quad \mathbb{P}(X_s^1 = X_s^2 \text{ for some } 0 \leq s \leq \beta\varepsilon^\alpha) \geq p.$$
- Assume the processes are living in the circle S^1 with perimeter C.
$$|A| > n \quad \Rightarrow \quad |x_1 - x_2| \leq Cn^{-1}.$$
The collision probability for some pair of ξ_t^A within $[0, \beta C^\alpha n^{-\alpha}]$ is at least p.
- $\mathbb{P}(\tau_n^{n+1} \geq k\beta C^\alpha n^{-\alpha}) \leq (1-p)^k$.

Coalescing stable processes in $\mathbb{R}$

- $\mathbb{P}(\tau_n^A \geq t) \leq t^{-1} \sum_{m=n}^{|A|} \mathbb{E}[\tau_m^{m+1}] \leq ct^{-1} \sum_{m \geq n} m^{-\alpha} \leq ct^{-1} n^{1-\alpha}$.
- Choose finite $A_n \uparrow A$, where A is a countable dense subset of S^1. Then, $\xi_t^{A_n} \uparrow \xi_t^{S^1}$ with $\mathbb{P}(\tau_K^{S^1} \geq t) \leq ct^{-1} K^{1-\alpha}$.
- To replace the initial condition S^1 by an bounded interval $I \subset \mathbb{R}$, one can choose $I \subset I_t \subset I_{2t} \subset \ldots$ such that $|I_{(n+1)t}|/|I_{nt}| = 1 + o(1)$, and $\xi_{nt}^I \subset I_{nt}$ with high probability.
- To start from the whole real line $\mathbb{R}$, find disjoint bounded intervals $\cup_i J_i = \mathbb{R}$. If J_i is far away from J, then it is unlikely that
$$\xi_t^{J_i} \cap J \neq \emptyset \quad (t \leq T).$$

Coalescing Brownian motions in the Sierpinski triangle

Coming down from infinity holds.

Thank You!