Lecture 10. The Kontsevich integral and formulae for the Vassiliev invariants

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今日唐诗

Das Lied von der Erde - Der Pavillon aus Porzellan

Gustav Mahler

Mitten in dem kleinen Teiche steht ein Pavillon aus grünem und aus weißem Porzellan.

Wie der Rücken eines Tigers wölbt die Brücke sich aus Jade zu dem Pavillon hinüber.

In dem Häuschen sitzen Freunde, schön gekleidet, trinken, plaudern; manche schreiben Verse nieder.

Ihre seidnen Ärmel gleiten rückwärts, ihre seidnen Mützen hocken lustig tief im Nacken.

Auf des kleinen Teiches stiller Wasserfläche zeigt sich alles wunderlich im Spiegelbilde:

Alles auf dem Kopfe stehend in dem Pavillon aus grünem und aus weißem Porzellan.

Wie ein Halbmond steht die Brücke, umgekehrt der Bogen. Freunde, schön gekleidet, trinken, plaudern.

宴陶家亭子

李白

曲巷幽人宅，高门大士家。
池开照胆镜，林吐破颜花。
绿水藏春日，青轩秘晚霞。
若闻弦管妙，金谷不能夸。

Link to the song.

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Feynman diagrams

There is another interesting algebra $\mathcal{A}^t$ that is in fact isomorphic to $\mathcal{A}^c$.

Definition 1.1

A *Feynman diagram*[a] is a finite connected graph of valency three at each vertex with an oriented cycle (circle)[b] on it. All vertices not lying on the circle are called *interior* vertices. Those lying on the circle are *exterior vertices*. Each interior vertex should be endowed with a cyclic order of outgoing edges.

[a] Also called *Chinese diagram* or *circular diagram*.

[b] This circle is also called the *Wilson loop*; we shall not use this term.

Remark 1.2

Feynman diagrams on the plane are taken to have the counterclockwise orientation of the circle and counterclockwise cyclic order of outgoing edges at each interior vertex.

Definition 1.3

The *degree* of a Feynman diagram is half the number of its vertices.

Obviously, all chord diagrams are Feynman diagrams; in this case the two definitions of the degree coincide.

Consider the formal linear space of all Feynman diagrams of degree n. Let us factorise this space by the STU–relation that is shown in Fig. 1.

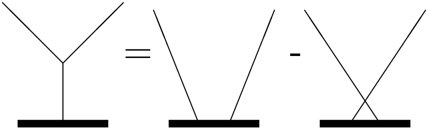

Figure 1: STU–relation

Denote this space by $\mathcal{A}^t_n$. Note that the STU–relation implies the $4T$–relation for the elements from $\mathcal{A}^c$.

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Theorem 1.4

There exists a natural isomorphism $f\colon \mathcal{A}^t_n \to \mathcal{A}^c_n$ which is identical on $\mathcal{A}^c_n$. Moreover, the STU–relation implies the following relations for $\mathcal{A}^t$:

1. *Antisymmetry; see Fig. 2.*
2. *IHX–relation; see Fig. 3.*

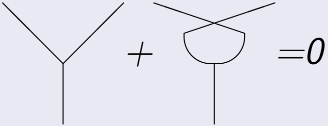

Figure 2: *The antisymmetry relation*

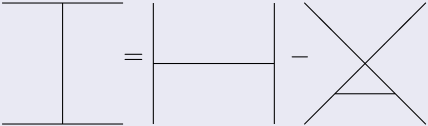

Figure 3: *The IHX–relation*

Proof of Theorem 1.4

First, let us prove that the algebras $\mathcal{A}^t$ and $\mathcal{A}^s$ are isomorphic. Let us construct now the isomorphism f. For all elements from $\mathcal{A}^c \subset \mathcal{A}^t$ we decree f to be the identity map. To define f on all Feynman diagrams, we shall use induction on the number x of interior vertices. For $x = 0$, there is nothing to prove.

Suppose f is well defined for all Feynman diagrams of degree d. Let K be a Feynman diagram of degree $d + 1$.

Obviously, there exists an interior vertex V of K that is adjacent to some exterior vertex by an edge v. Thus, we can apply the STU–relation to this vertex and obtain two diagrams of degree d. However, this operation is not well defined: it depends on the choice of such a vertex V and the edge v. Suppose there are two such pairs V, v and U, u, where $V \neq U, v \neq u$. In this case, we can prove that our operation is well–defined by applying the STU–relation twice; see Fig. 4.

Proof of Theorem 1.4

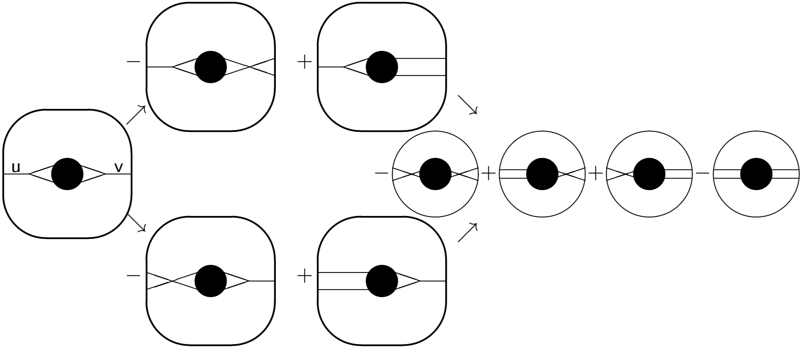

Figure 4: Applying the STU–relation twice

In the case when $U = V$ and $u \neq v$, we can try to find another pair. Namely, the pair W, w, where W is a vertex adjacent to an exterior vertex and w is an edge connecting this vertex with the circle. Then we prove that the result for U, u equals that for W, w and then it equals that for U, v.
Finally, we should consider the case when $U = V$, $u \neq v$, and U is the only interior vertex adjacent to the circle. In this case, we are going to show that our diagram is equivalent to zero modulo the STU–relation. In this case, we can indicate some domain containing all interior vertices except one. This domain has only one connection with exterior vertices , namely the connection via the vertex U and one of the chords u, v. In this case, the two possible splittings are equals because the product on Feynman diagrams is well defined. By the induction hypothesis, we see that the product of two Feynman diagrams of total degree d is well defined and commutative. By using this commutativity, we can move the vertex with the small domain from one point to the other one. Thus, we see that each of both splittings gives us zero. The concrete calculations are shown in Fig. 5.

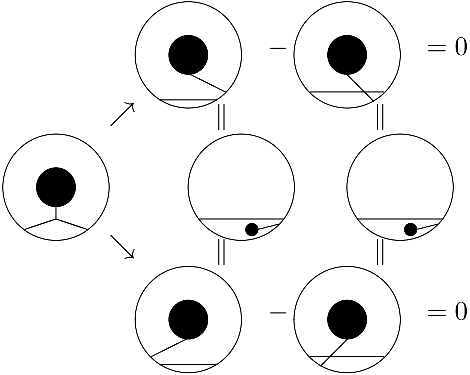

Figure 5: STU-reduction to zero diagram

Let us prove now that *STU* implies the antisymmetry relation. Applying the *STU*–relations many times, one can reduce the antisymmetry relation to the case when all chords outgoing from the given interior points finish at exterior points. In this case, the antisymmetry relation follows straightforwardly; see Fig. 6.

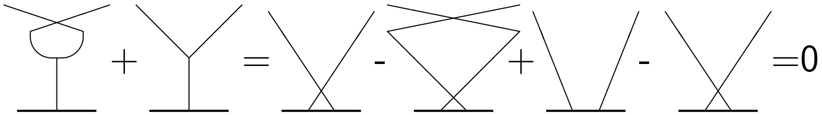

Figure 6: STU–relation implies antisymmetry

The proof of the fact that the *IHX*–relation holds can be reduced to the case when one of the four vertices (say, lower left) is an exterior one. This can be done by taking the lower left vertex for all diagrams that have to satisfy the *IHX*–relation and then splitting all interior vertices between this vertex and the circle in the same manner for all diagrams. Then we repeat this procedure for all obtained diagrams.

Finally, we get many triples of diagrams for each of which we have to check the *IHX* relation. For each of them, we have to consider only the partial case. The last step is shown in Fig. 7.

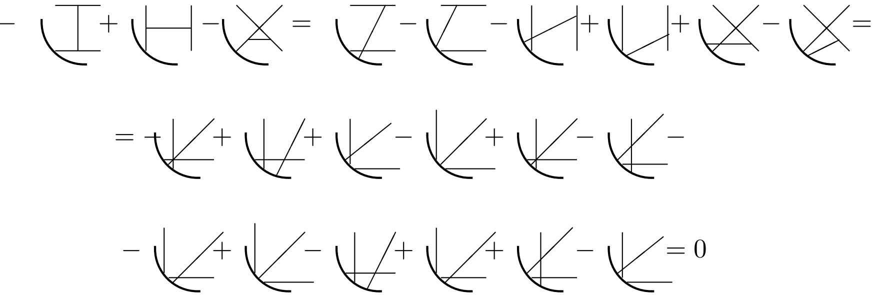

Figure 7: The STU–relation implies the IHX relation

Remark 1.5

Note that the 1–term relation does not spoil the bialgebra structure; the corresponding bialgebra is obtained by a simple factorisation.

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Coproduct of chord diagrams

The chord diagram algebra $\mathcal{A}^c$ has, however, very sophisticated structures. It is indeed a bialgebra. The coalgebra structure of $\mathcal{A}^c$ can be introduced as follows.

Let C be a chord diagram with n chords. Denote the set of all chords of the diagram C by $\mathcal{X}$. Let $\Delta(C)$ be

$$\sum_{s\in 2^{\mathcal{X}}} C_s \otimes C_{\mathcal{X}\setminus s},$$

where the sum is taken over all subsets s of $\mathcal{X}$, and C_y denotes the chord diagram consisting of all chords of C belonging to the set y. Now, let us extend the coproduct Δ linearly.

Now we should check that this operation is well defined. Namely, for each four diagrams $A =$ , $B =$ , $C =$ , $D =$  such that $A - B + C - D = 0$ is the $4T$–relation, one must check that $\Delta(A) - \Delta(B) + \Delta(C) - \Delta(D) = 0$.

Actually, let A, B, C, D be four such diagrams (A differs from B only by a crossing of two chords, and D differs from C in the same way). Let us consider the comultiplication Δ. We see that when the two "principal" chords are in different parts of $\mathcal{X}$, then we have no difference between A, B as well as between C, D. Thus, such subsets of $\mathcal{X}$ give no impact. And when we take both chords into the same part for all A, B, C, D, we obtain just the $4T$–relation in one part and the same diagram at the other part. Thus, we have proved that Δ is well–defined.

Now, let us give the formal definition of the bialgebra.[1].

Definition 2.1

An algebra A with algebraic operation μ and unit map e and with coalgebraic operation Δ and counit map ϵ is called a bialgebra if

1. e is an algebra homomorphism;
2. ϵ is an algebra homomorphism;
3. Δ is an algebra homomorphism.

Definition 2.2

An element x of a bialgebra B is called *primitive* if $\Delta(x) = x \otimes 1 + 1 \otimes x$.

[1] In [29] this is also called a *Hopf algebra*. One usually requires more constructions for the algebra to be a Hopf algebra, see e.g. [15, 27]. However, the bialgebras of chord and Feynman diagrams that we are going to consider are indeed Hopf algebras: the antipode map is defined by induction on the number of chords. We shall not use the antipode and its properties.

Obviously, for the case of $\mathcal{A}^c$ with natural e, ϵ and endowed with the product and coproduct Δ, e and ϵ are homomorphic. The map Δ is monomorphic: it has the empty kernel because for each $x \neq 0$, $\Delta(x)$ contains the summand $x \otimes 1$. Thus, $\mathcal{A}^c$ is a bialgebra.

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Now, let us define the coproduct in $\mathcal{A}^t$ of Feynman diagrams.

Remark 2.3

Within this section, we consider the algebras not factorised by the $1T$–relation.

As shown above, these algebras are isomorphic. Thus, $\mathcal{A}^t$ has a Hopf algebra structure as well. Let us describe this structure explicitly.
Let D be a Feynman diagram and let $V(D)$ be *connected components* of the diagram; i.e., connected components of the graph obtained from D by deleting the circle. Let $J \subset V(D)$ be a subset of $V(D)$. This subset defines a Feynman diagram C_J, whose connected components lie in $V(D)$; i.e., the Feynman diagram consisting of the circle of the diagram D and those connected components of the graph $V(D)$ belonging to J.
Let us define the *coproduct* $\tilde{\mu}(D)$ as

$$\tilde{\mu}(D) = \sum_{J \subset V(D)} C_J \otimes C_{V(D) \setminus J}.$$

Example 2.4

In Fig. 8 we illustrate the coproduct operation for a Feynman diagram.

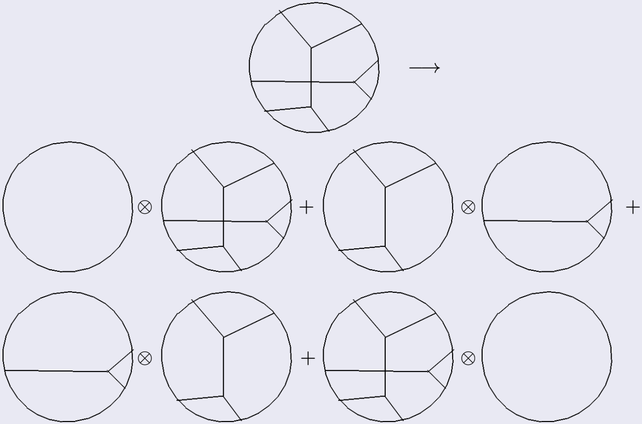

Figure 8: *Coproduct of a Feynman diagram*

Theorem 2.5

The coproduct defined below coincides with that for $\mathcal{A}^c$; i.e. $\mu \equiv \tilde{\mu}$.

Proof. We have to show that for each Feynman diagram D, its coproduct coincides with the linear combination of coproducts of chord diagrams that D can be decomposed into.
We shall use induction on the number k of interior vertices of the diagram. For $k = 0$, there is nothing to prove.
Suppose that the statement is true for all Feynman diagrams with n interior vertices. Consider a Feynman diagram D with $(n+1)$ interior vertices. We have to show that $\mu(D) = \tilde{\mu}(D)$. According to the *STU*–relation, the diagram D can be represented as a difference $D_+ - D_-$, as shown in Fig. 21; in this case the connected component of D corresponds to a pair of connected components for each of D_+, D_-.

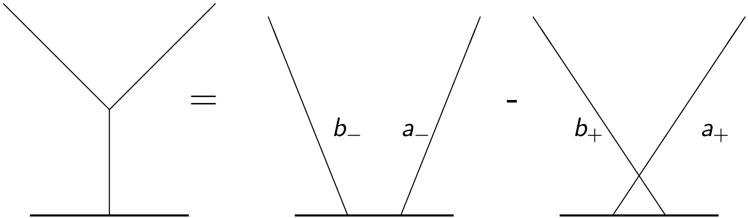

Figure 9: STU-reduction of a diagram

Let us choose the components (a_+, b_+) and (a_-, b_-) for the diagrams D_+ and D_-. These components are obtained by resolving a point of D. Each of $D_\pm$ has n interior vertices.

Thus, the claim of the theorem is true for them: $\mu(D_\pm) = \tilde{\mu}(D_\pm)$. Let us now write $\mu(D) = \mu(D_+) - \mu(D_-)$. In each of these coproducts we have only those terms where the components (a_+, b_+) (respectively, (a_-, b_-)) lie on the same side with respect to the $\otimes$ sign. Obviously, these terms collected together give $\tilde{\mu}(D)$ (in the previous sense of the coproduct). It is easy to see that the remaining terms give us zero. In fact, suppose we have a splitting of the Feynman diagram D_+ into two diagrams, where a_+ belongs to one of them and b_+ belongs to the other. Then if we divide D_- in just the same way as we did with D_+ with respect to all other connected components and take a_- to be the first multiplicator of the tensor product and b_- to be the second one, we obtain two coinciding tensor products.

Collecting all previous statements together, we obtain the statement of the theorem.

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Definition 3.1

The completion $\overline{\Delta}$ of $\Delta = \oplus_{m=0}^{\infty} \Delta_m$ is the set of all formal series $\sum_m c_m a_m$, where $c_m \in \mathbb{C}$ are numeric coefficients, and $a_m \in \Delta_m$ are elements of the space of degree m chord diagrams.

Let us think of the space $\mathbb{R}^3$ as the Cartesian product of $\mathbb{C}^1$ with the coordinate z and $\mathbb{R}^1$ with the coordinate t.
Given an oriented knot K in $\mathbb{R}^3 = \mathbb{C}_z \times \mathbb{R}_t$, by a small perturbation in $\mathbb{R}^3$ (without changing the knot isotopy type), we can make the coordinate t a *simple Morse function* on the knot K. This means that all critical points of t on the knot K are regular and all critical points have different critical values. See Fig. 10.

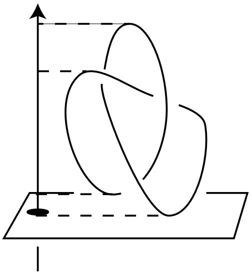

Figure 10: A Morse knot

Remark 3.2

Later on, such embeddings will be called *Morse knots*.

Definition 3.3

The *preliminary Kontsevich integral* of a knot K is the following element of $\overline{\Delta}$:

$$Z(K) = \sum_{m=0}^{\infty} \frac{1}{(2\pi i)^m} \int_{\substack{c_{min}<t_1<\cdots<t_m<c_{max} \\ t_j:\text{non-critical}}} \sum_{P=\{(z_j,z_j')\}} (-1)^{\downarrow} D_P \bigwedge_{j=1}^{m} \frac{dz_j - dz_j'}{z_j - z_j'} \tag{1}$$

We decree the coefficient of the "empty" chord diagram to be equal to one.

Let us discuss the formula (1) in more detail.
The real numbers c_{min} and c_{max} are maximal and minimal values of the function t on the knot K.
The integration domain is an n–simplex $c_{min} < t_1 < \cdots < t_m < c_{max}$.
This domain is divided into connected components. Herewith, z_i and dz_i should be understood as functions of the corresponding t_i.
For instance, for the unknot shown in Fig. 11 and $m = 2$, the integration domain consists of 6 components and looks as shown in Fig. 11.

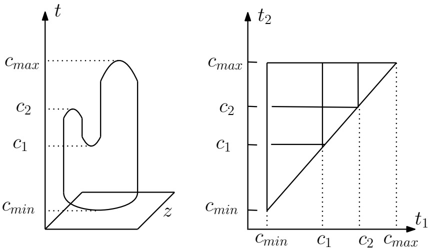

Figure 11: Integration domains for the Kontsevich integral

The number of summands is constant for each connected component, but it can vary when passing from one component to another. The part of the knot lying inside the margin between two adjacent critical levels is a set of curves; each of these curves is uniquely parametrised by t.
Let us fix m and choose m horizontal planes $\{t = t_i\}, i = 1, \ldots, m$, each of which does not contain critical points and lies between the minimal and the maximal levels. Later, we shall take the sum over all natural m. At each plane $\{t = t_i\} \subset \mathbb{R}^3$, let us choose an unordered pair (z_i, t_i), $(z_i^{'}, t_i)$ of different points lying on K. Denote by $P = \{(z_i, z_i')\}$ the system of m such pairings. Fix a pairing P. If we think of a knot as a circle and then connect the points of the circle corresponding to z_i, z_i' of the same pair (according to P) we obtain a chord diagram. Denote this diagram by D_P.

Now, under the integral we have the sum of such diagrams corresponding to different pairings P. The coefficients are obtained in the following way. Choosing any arbitrary connected component, the choice of P means that for each t_i, some pair of knot branches is taken. Thus, choosing m planes, we get m pairs of points.

As a matter of fact, after we have chosen all pairings, the diagram D_P is defined; thus we should integrate not chord diagrams, but only the form $(-1)^{\downarrow} \wedge_{j=1}^{m} \frac{dz_j - dz_j'}{z_j - z_j'}$. The obtained integral will give us the coefficient of our chord diagram D_P. Later, we shall collect similar terms.

In the example shown above, the connected component $\{c_{min} < t_1 < c_1, c_2 < t_2 < c_{max}\}$ corresponds to a unique pair of points at the levels $\{t = t_1\}$ and $\{t = t_2\}$. In this case, the desired sum consists of a unique summand. For the component $\{c_{min} < t_1 < c_1, c_1 < t_2 < c_2\}$, we have a unique choice at the level $\{t = t_1\}$, but the plane $\{t = t_2\}$ intersects the knot at four points; thus we have $C_4^2 = 6$ possible pairings (z_2, z_2'), and the total number of summands equals six.
For the component $\{c_1 < t_1, t_2 < c_2\}$ we have 36 summands, among them the most interesting case of  appears. In each part of Fig. 12, we choose exactly one pairing and show the corresponding chord diagram. It is easy to see that in all cases except $\{c_1 < t_1 < t_2 < c_2\}$ we obtain the chord diagram  with two non–intersecting chords. These diagrams are equal to zero modulo one–term relation. Thus, the integration can be reduced to the small simplex $\{c_1 < t_1 < t_2 < c_2\}$.

The symbol $\downarrow$ for a given set choice of P denotes the number of points (z, t_i) or (z', t_i) of P, where the coordinate t is decreasing while moving along the knot according to its orientation. In Fig. 12, the diagrams corresponding to different integration domains are shown.

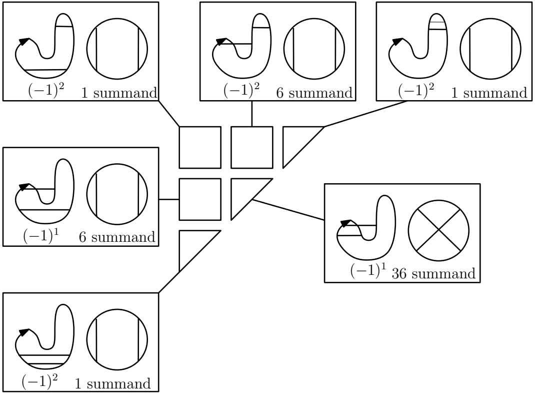

Figure 12: Integration domain and chord diagrams

Now we have the following questions to answer.

1. Do the coefficients of $\bar{\Delta}$ in the formula (1) converge?
2. Is the obtained element a knot invariant?
3. How is it related to the Vassiliev invariants?
4. How do we calculate this integral?

Theorem 3.4 ([4], see also [11])

All coefficients of the formula (1) *are finite.*

Definition 3.5

A *horizontal deformation* is an isotopy of a Morse embedding of a curve in $\mathbb{R}^3$ that does not change the setup of singular points.

The horizontal deformation can be expressed as a composition of moves shown in Fig. 13.

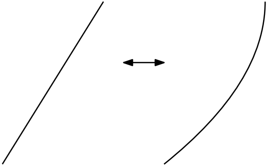

Figure 13: Horizontal deformation

Theorem 3.6 ([11])

The function $Z(K)$ is invariant under horizontal deformations of a knot and under the transformation shown in Fig. 14, but not invariant under the transformation (*), shown in Fig. 15.

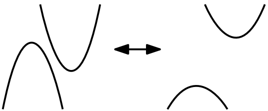

Figure 14: Moving critical values

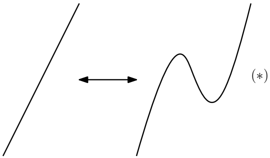

Figure 15: Forbidden transformation

Denote the knot representing the closure of the arc shown in Fig. 15 by T_{UD}.

We can consider the simplest realisation of the unknot (with one minimum and one maximum) and the realisation given by ∞; see Fig. 16. It is easy to see that $Z(K)$ for the simplest realisation is equal to one (i.e., the series consisting of the only diagram without chords with coefficient one). Moreover, $Z(\infty)$ is not equal to $1 = \bigcirc$.

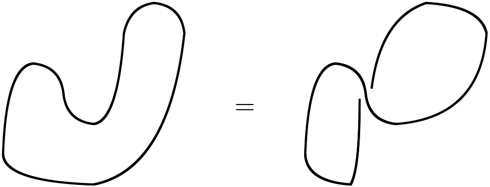

Figure 16: The "∞" knot

Thus we see that Z is not a knot invariant.

On the other hand, one can prove the following theorem.

Theorem 3.7

If the knot K' is obtained from K by using $(*)$, then $Z(K') = Z(K) \cdot Z(\infty)$.

Proof. First, let us note that ∞ is obtained from the knot T_{UD} by using allowed moves, thus $Z(\infty) = Z(T_{UD})$.
Let us consider now the connected sum of K with a “small” knot ∞ in such a way that the interval of the coordinate t, corresponding to the knot T_{UD}, has no critical points of the knot K. In this case, just two new critical points — one maximum and one minimum are added to this knot; see Fig. 17.

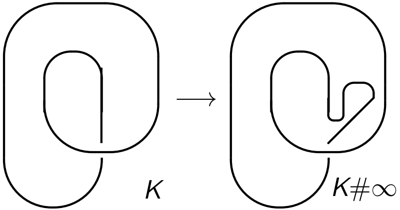

Figure 17: Transformation (*)

By virtue of the previous theorems, the Kontsevich integral of the obtained knot coincides with the Kontsevich integral for K' that is obtained from $K\#\infty$ by using horizontal deformations. Comparing the Kontsevich integral for the initial knot and for the knot K, we see that each term for the integral of the knot K corresponds to the same term multiplied by the Kontsevich integral for T_{UD} (in the integral of K'). Consequently, $Z(K') = Z(K) \cdot Z(\infty)$. $\square$

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Thus, the change of the preliminary integral $Z(\cdot)$ under (*) is not difficult: the value is just multiplied by $Z(\infty)$. Now let K be a Morse embedding of S^1 in $\mathbb{R}^3$, and c be the number of critical points of t on K. Let us consider now the preliminary Kontsevich integral as a formal series. Hence this series consists of elements of a graded algebra and its initial element is the unit element of this algebra. Then one can inverse such rows by

$$(1+a)^{-1} = 1 - a + a^2 - a^3 + \dots,$$

where a^i is the formal series for the i–th power of the series a. Furthermore, one can formally multiply such series.

Definition 4.1

The *universal Vassiliev–Kontsevich invariant* of a knot K is the following element of the completion of the chord diagram algebra:

$$I(K) = \frac{Z(K)}{Z(\infty)^{\frac{c}{2}-1}}. \tag{2}$$

Remark 4.2

Here the degree $(\frac{c}{2} - 1)$ is taken for the following majors. In the case of the simplest embedding representing the unknot we wish to have $I(\bigcirc) = 1$. For one maximum and one minimum we have $\frac{c}{2} - 1 = 0$.

Remark 4.3

Obviously, if the formula (1) converges, then the formula (2) makes sense: it is just the fraction of two series.

Thus we obtain the following theorem.

Theorem 4.4

The Kontsevich integral $I(\cdot)$ is a knot invariant.

Proof. Indeed, by virtue of Theorem 3.7 we see that $Z(K)$ depends not on the configuration of critical points, but only on their quantity.
It is easy to check that two Morse embeddings represent the same knot if and only if one can be transformed to the other by means of moves not changing the setting of critical points and moves shown in Figs. 14 and 15.
Taking into account the invariance of Z under all moves but the last one, we obtain the statement of the theorem. □

The invariant $I(\cdot)$ is called *the universal Vassiliev–Kontsevich invariant*.

Now it remains to formulate and to prove the most important theorem. Let W be a weight system[2] of degree m. Decree that $W(d) = 0$ for all diagrams d with the number of chords not equal to m.

Theorem 4.5 (Kontsevich, see also [11])

The invariant $W(I(\cdot))$ is a Vassiliev invariant with symbol W; i.e.,

$$V(W)(K) = W(I(K))$$

for each knot K.

This theorem implies the second (difficult) part of the Vassiliev–Kontsevich theorem about the existence of Vassiliev invariants corresponding to any given weight system.
We shall prove Theorems 3.4, 3.6 and 4.5 in Section 5.

[2] Each linear function on chord diagrams of order n, satisfying $1T$- and $4T$-relations, is said to be a weight system (of order n). See Lecture 14.

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Remark 5.1

By $Z_m(K)$ and $I_m(K)$ we mean the m–th graded summand of $Z(K)$ and $I(K)$, respectively.

First, let us prove Theorem 3.4 which states that the series for each coefficient at each term of (1) converges.

Proof. Consider a Morse knot K in $\mathbb{R}^3$. Let us fix $m \in \mathbb{N}$ and choose some m planes not intersecting K at critical points.

Choose some chord diagram D and consider the coefficient at this diagram. It is obtained by integrating the form

$$\bigwedge_{j=1}^{m} \frac{dz'_j(t_j) - dz_j(t_j)}{z'_j(t_j) - z_j(t_j)}$$

over the part of the m-simplex $\{c_{min} < t_1 < \cdots < t_m < c_{max}\}$ corresponding to the chord diagram D.

Proof of Theorem 3.4 continued

Let us consider the singular points of the form, namely, those where the condition $z_j = z_j'$ holds for some j. The integral of the form might diverge only in the neighbourhood of these points. Consider such pairs of points z_j, z_j' closed to the singular position.

Then we have the two possibilities:

1. The arc between z_j and z_j' contains other ends z_k of chords (as shown in Fig. 18). Then the integration domain (where we integrate $z_j - z_j'$) has smallness of higher order than $z_j - z_j'$ because the singular point is not degenerate. Consequently, this part of the formula (2) gives no divergence.

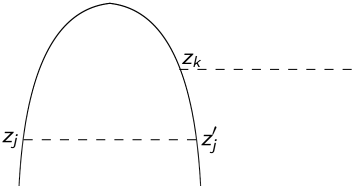

Figure 18: The arc between z_j and z_j' contains other ends z_k of chords.

Proof of Theorem 3.4 continued

2. The arc between z_j and z'_j has no other chord ends; see Fig. 19. Then the chord $z_j z'_j$ of the diagram D is isolated; thus, the diagram D equals zero modulo 1T–relation.

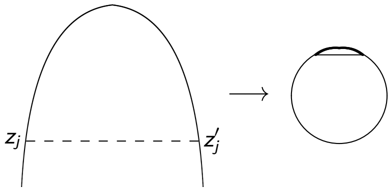

Figure 19:

This completes the proof of the theorem. □

Contents

In order to prove the remaining two theorems 3.6 and 4.5, we shall have to integrate holonomies and introduce the so called *Knizhnik–Zamolodchikov connection*. First, let us recall some constructions.

Definition 5.2

Let X be a smooth manifold and let $\mathcal{U}$ be an associative topological algebra with the unit element (considered over $\mathbb{R}$ or $\mathbb{C}$).
Then a *$\mathcal{U}$–connection* Ω on the manifold X is a 1–form Ω on X with coefficients from $\mathcal{U}$.
The *curvature* of the connection Ω is the 2–form

$$F_\Omega = d\Omega + \Omega \wedge \Omega.$$

The connection is *flat* if its curvature equals zero.

Definition 5.3

Let $B : [a, b] \to X$ be a smooth mapping of the interval $[a, b]$ to the space X. Let Ω be a $\mathcal{U}$–connection on X.
Let us define the *holonomy* $h_{B,\Omega}$ of the form Ω along the path B as the solution of the differential equation

$$\frac{\partial}{\partial t} h_{B,\Omega}(t) = \Omega(B'(t)) \cdot h_{B,\Omega}(t), t \in [a, b],$$

with the initial condition $h_{B,\Omega}(a) = 1$.

Remark 5.4

It is easy to show that if the connection Ω is flat, then the holonomy is defined only by the ends of a path and the homotopy type of this path. This is quite analogous to the Gauss–Ostrogradsky formula (or called Divergence Theorem) in the commutative case. Then the multiplicative integral is just the exponent of the Riemmanian integral for the logarithmic function. In the non–commutative case the extra term $\Omega \wedge \Omega$ arises.

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The solution to such an equation (holonomy) often exists. It is called the *product* or *multiplicative integral* of the form Ω. In many cases, the holonomy can be calculated according to the following *iterated formula*:

$$h_{B,\Omega}(t) = 1 + \sum_{m=1}^{\infty} \int_{a \le t_1 \le t_2 \le \dots \le t_m \le t} (B^*\Omega)(t_m) \dots (B^*\Omega)(t_1). \qquad (3)$$

In order to clarify the situation, let us consider the following simple construction.

Example 5.5

Let

$$Y' = AY$$

be a differential equation with the initial condition $Y(0) = 1$, say, in $n \times n$ matrices.[a] *Obviously, its solution $Y(t)$ is the product of "infinitely many" elements "infinitely close to the unit element". This is naturally called the* product integral *of A and denoted by*

$$Y(x) = \int_0^{x\curvearrowright} (E + A(t)dt).$$

Note that in order to calculate $Y(x)$, one can use the following formula

$$Y(x) = E + \int_0^x A(t_1)dt_1 + \int_0^x A(t_1) \int_0^{t_1} A(t_2)dt_2dt_1 + \cdots \qquad (4)$$

if the series (4) converges. Actually, while integrating the series, each next term becomes equal to the previous one multiplied by A.
Each term of the iterated integral (4) can be considered as an integral over some simplex.

Remark 5.6

In all “normal” cases this series actually converges.

The formula (3) is completely analogous to the formula (4).
The theory of product integration is well described in [19, 23, 26].

Remark 5.7

In the normal (convergent) case it is obvious that the formula actually gives a solution to the differential equation. The initial condition evidently holds. The derivative of the m–th integral gives the $(m-1)$–th integral with coefficient $\Omega(\dot{B})$.

Contents

Denote by $\mathcal{D}_n^{KZ}$ the set of all diagrams consisting of n ascending infinite arrows (in Fig. 20 they are shown by thick lines) and a finite number of edges such that:

1. each end point of each edge either lies on the arrow or is a trivalent vertex (with two other ends of edges);
2. any point on the arrow is incident to no more than one interval (only one end of this edge can coincide with this point).

Such diagrams are considered up to combinatorial equivalence.

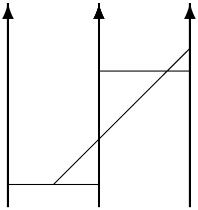

Figure 20: An element $\mathcal{A}^{KZ}$

Let $\mathbb{C}$ be the main field. Consider the set $\mathcal{A}_n^{KZ} = span(\mathcal{D}_n^{KZ})/\{STU - \text{relations}\}$. The STU–relation means the same as for the Feynman diagrams (by "multiplication" of all "partial" integrals), where we consider a part of an arrow instead of part of an oriented circle. Note that the STU–relation is local. When we finally close the "arrow" diagrams in order to obtain the Feynman diagram, we get the STU–relation as well.

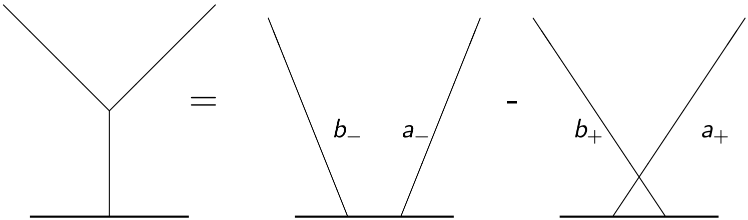

Figure 21: STU-reduction of a diagram

For a fixed n, the set $\mathcal{A}_n^{KZ}$ admits an algebraic structure: the product means the juxtaposition of one diagram over the other.

Example 5.8

For $n=3$ such a multiplication for $\mathcal{A}_3^{KZ}$ is shown in Fig. 22.

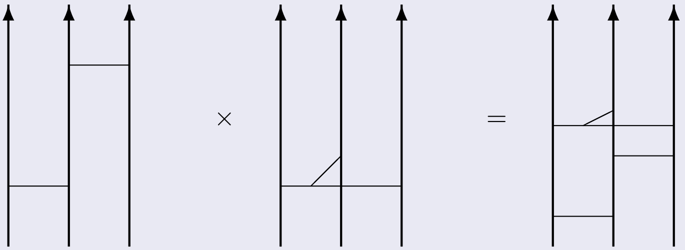

Figure 22: *Multiplication in $\mathcal{A}^{KZ}$*

For a fixed n, the algebra $\mathcal{A}_n^{KZ}$ is graded: the order of an element is equal to half of the total number of vertices.
For $1 \leq i, j \leq n$, let us define $\Omega_{ij} \in \mathcal{A}_n^{KZ}$ as the element with only one edge connecting the arrows i and j.

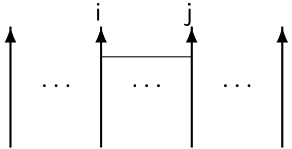

Figure 23: The element Ω_{ij}

Remark 5.9

It is easy to see that if $\{i, j\} \bigcap \{k, l\} = \emptyset$, then Ω_{ij} and Ω_{kl} commute, i.e. $\Omega_{ij} \times \Omega_{kl} = \Omega_{kl} \times \Omega_{ij}$.

Let X_n be the configuration space of n pairwise different points on $\mathbb{C}^1$.
Let ω_{ij} be the following 1–form on X_n:

$$\omega_{ij} = d(\ln(z_i - z_j)) = \frac{dz_i - dz_j}{z_i - z_j}.$$

Let us define the formal Knizhnik–Zamolodchikov connection Ω_n with coefficients in $\mathcal{A}_n^{KZ}$ as $\Omega_n = \sum_{1\le i<j\le n} \Omega_{ij}\omega_{ij}$ on X_n.

Theorem 5.10

This connection is flat. More precisely, $\Omega_n \wedge \Omega_n = 0$ *and* $d\Omega_n = 0$.

Proof. The last statement is evident. Indeed, $d\omega_{ij} = d^2(\ln(z_i - z_j))$ and this vanishes by definition of d.
Let us prove the first statement. Consider the element

$$\Omega_n \wedge \Omega_n = \sum_{i<j;k<l} \Omega_{ij}\Omega_{kl}\omega_{ij}\omega_{kl} \tag{5}$$

and the set $\{i, j, k, l\}$.

Proof of Theorem 5.10 continued

If this set consists of two or four elements, then the corresponding term of the sum equals zero (this case is commutative). Consequently, the desired sum equals the sum along all i, j, k, l, where the set $\{i, j, k, l\}$ consists of three elements. Consider, e.g., the set $\{i, j, k, l\} = \{1, 2, 3\}$ and all corresponding terms in the sum (5).

In this case we get:

$$\sum_{\{i,j,k,l\}=\{1,2,3\}} \Omega_{ij}\Omega_{kl}\omega_{ij}\omega_{kl} = (\Omega_{12}\Omega_{23}-\Omega_{23}\Omega_{12})\omega_{12}\wedge\omega_{23}+\langle\text{cyclic permutations}\rangle.$$

By using the *STU*–relation, we see that $-\Omega_{123} = \Omega_{12}\Omega_{23} - \Omega_{23}\Omega_{12}$ and the desired sum equals

$$-\Omega_{123}(\omega_{12} \wedge \omega_{23} + \omega_{23} \wedge \omega_{31} + \omega_{31} \wedge \omega_{12}),$$

where Ω_{123} is the element shown in Fig. 24.

Proof of Theorem 5.10 continued

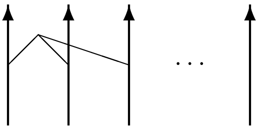

Figure 24: The element Ω_{123}

Exercise 5.11

Verify the equality $-\Omega_{123} = \Omega_{12}\Omega_{23} - \Omega_{23}\Omega_{12}$.

Exercise 5.12 (V.I. Arnold's identity.)

Show that $\omega_{12} \wedge \omega_{23} + \omega_{23} \wedge \omega_{31} + \omega_{31} \wedge \omega_{12} = 0$.

Proof of Theorem 5.10 continued

Remark 5.13

This identity appeared in [7] when Arnold studied the cohomologies of the pure braid group.

Thus if some set $\{i, j, k, l\}$ consists of precisely three different members, then it gives no contribution. We have considered all possible cases. Thus, $\Omega_n \wedge \Omega_n = 0$. $\square$

Remark 5.14

The connection Ω_n can be slightly modified for the case of the algebra $\mathcal{A}_{nn}^{KZ}$. This algebra is generated by arrow diagrams with $2n$ arrows (the first n arrows oriented upwards and the last n arrows oriented downwards). The *STU*–relation for such diagrams depends on the direction of the chord that the relation has to be applied to.

Let $\Omega_{nn} = \sum_{1\leq i<j\leq n} s_i s_j \Omega_{ij}\omega_{ij}$, where s_i equals 1 for $i \leq n$ and -1 for $i > n$.

Exercise 5.15

Show that the connection Ω_{nn} is flat.

Proof of Theorem 3.6

Now, let us prove the invariance theorem. Let us reformulate Theorem 3.6.

Theorem (Theorem 3.6)

The function $Z(K)$ is invariant under horizontal deformations of a knot and under the transformation shown in Fig. 25, but not invariant under the transformation (*), shown in Fig. 26.

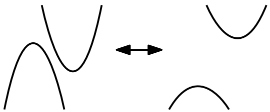

Figure 25: Moving critical values

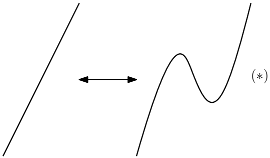

Figure 26: Forbidden transformation

Proof of Theorem 3.6 continued

Proof. First, let us prove that the preliminary Kontsevich integral $Z(K)$ is invariant under the transformation preserving the critical points.
The point is that the Kontsevich integral for the whole knot can be decomposed into a product of similar integrals for *parts* of this knot; each of these parts represents an element of some Knizhnik–Zamolodchikov algebra with ascending and descending arrows; being connected together, they constitute a normal chord diagram. Thus, the product of elements in some $\mathcal{A}^{KZ}$ is thought to be an element of $\bar{\Delta}$.
Let $c_{min} \leq a < b \leq c_{max}$. Let us define $Z(K, [a, b])$ just as was done in the formula (1), but taking the integration domain to be $\{a < t_1 < \cdots < t_n < b\}$, and replacing the chord diagrams with elements of the Knizhnik–Zamolodchikov algebra.

Proof of Theorem 3.6 continued

Although $Z(K,[a,b])$ does not belong to $\overline{\mathcal{A}^c} \subset \bar{\Delta}$, the corresponding series (evaluated at a knot) converges for the same reasons as Z. Since the interval (a,b) has no critical points, the intersection of the knot with the margin $\mathbb{C} \times (a,b)$ is a set of oriented curves without horizontal tangent lines. Suppose that the number of such curves equals $2n$. Obviously, n curves of them are ascending and the other n are descending. Let us fix the lower points $a_1,\dots,a_{2n}$ and the corresponding upper points $b_1,\dots,b_{2n}$, where the first n coordinates correspond to ascending curves and the other ones correspond to descending curves. The convergence of the integral can be proved in the same manner as before. One should, however, introduce an analogue of the one–term relation taking all diagrams with a "solitary" chord (with one end on an ascending chord and one end on a descending arc) to zero.

Proof of Theorem 3.6 continued

Now, the integral $Z(K[a,b])$ can be represented as the holonomy of the connection Ω_{nn} along the path from $(a_1,\dots,a_{2n})$ to $(b_1,\dots,b_{2n})$ by virtue of the iteration formula (3). Actually, the m–th term of the iteration formula for Ω_{nn} corresponds to the m–th term of the Kontsevich integral because in both cases we integrate the form

$$\sum_{P=\{(z_j,z_j')\}} (-1)^{\downarrow}\Omega_{jj'} \bigwedge_{j=1}^{m} \frac{dz_j - dz_j'}{z_j - z_j'}, \tag{6}$$

where $(-1)^{\downarrow}$ corresponds to the sign of the product $s_j s_j'$.
Recall that the STU–relation for Feymann diagrams is "the same" as the $4T$–relation for chord diagrams. This is just the place when we use the $4T$–relation (in its STU–form).

Proof of Theorem 3.6 continued

Since the curvature of the connection Ω_{nn} is zero, the integral (6) is invariant under homotopies of the integration path with fixed endpoints; i.e., under horizontal isotopies of the part of the knot lying inside $t \in (a, b)$.

It is not difficult to show that for arbitrary $a < b < c$ (possibly, critical), we have $Z(K, [a, c]) = Z(K, [a, b]) \cdot Z(K, [b, c])$. Thus we conclude that the integral $Z(K)$ which is a product $Z(K, [c_i, c_{i+1}])$, where c_i, c_{i+1} are all pairs of "adjacent" critical points, is invariant under horizontal deformation in the intervals not containing critical points. Now, let us consider the cases when critical points are moving during the knot isotopy.

Proof of Theorem 3.6 continued

1. The critical point is moving, but the disposition of all critical points stays the same; see Fig. 27.

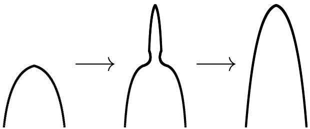

Figure 27: Move of a critical point

2. The order of heights of two critical points changes; see Fig. 28.

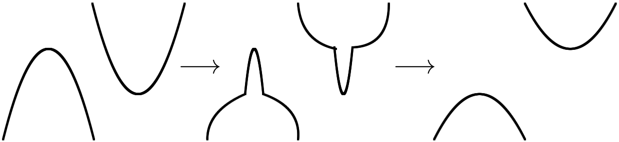

Figure 28: Change of order of heights of two critical points

Proof of Theorem 3.6 continued

As shown in Figs. 27 and 28, one can first perform the transformation that does not change $Z(K)$ to obtain a knot with a thin "needle". Let us show that the removal of this needle changes the m–th graduation term of the Kontsevich integral by some infinitely small ϵ depending on the diameter of the needle.

Actually, let K be a knot and let K' be the knot obtained from the knot K by means of adding a vertical needle somewhere.

Obviously, the difference $Z(K) - Z(K')$ contains only the terms corresponding to the diagrams with ends lying inside the needle. Suppose that the width of the needle equals ϵ. Let us show that $Z_m(K) - Z_m(K') = O(\epsilon)$.

Proof of Theorem 3.6 continued

Actually, consider all the chords incident to the needle. If the upper chord has both ends on the needle, then the chord diagram equals zero modulo $1T$–relation. If there are no chords with all ends lying on the margin, then the situation is quite simple as well: the term shown in Fig. 29 should have smallness of order ϵ: while integrating the left and the right part, the numbers $\uparrow$ have difference 1; thus we obtain a contraction because for each term there exists a "mirror" term; see Fig. 29.

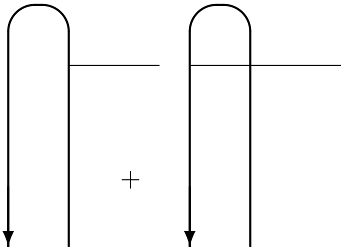

Figure 29: No chords with all ends lying on the margin

Proof of Theorem 3.6 continued

Thus, we only have to consider the case when the upper chord (z_i, z_i') has one end lying on the needle, and there are k chords lying under this with both ends on the needle. Suppose the lowest one is (z_{j_1}, z_{j_1}') and the upper one is (z_{j_k}, z_{j_k}'); see Fig. 30.

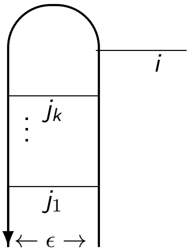

Figure 30: k chords lying under this with both ends on the needle

Proof of Theorem 3.6 continued

We may assume that (z_i, z_i') is the only chord such that one end of it lies on the needle. If we delete such chords, we multiply the final integral by some number bounded from zero and the infinity.
Let $\delta_\alpha = |z_{j_\alpha} - z_{j'_\alpha}|$. Then the difference $Z(K') - Z(K)$ is bounded by some constant multiplied by

$$\int_0^\epsilon \frac{d\delta_1}{\delta_1} \int_0^{\delta_1} \frac{d\delta_2}{\delta_2} \cdots \int_0^{\delta_{k-1}} \frac{d\delta_k}{\delta_k} \int_{z_{j_k}}^{z_{j'_k}} \frac{dz_i - dz_i'}{z_i - z_i'}.$$

The integral has smallness of the order $\tilde{\epsilon}$. Actually, the last integral has smallness of the order of δ_k. Consequently, the term δ_k is reduced in the penultimate integral, so this integral has smallness of δ_{k-1}, and so on. Finally, the total integral has smallness of $\delta_1 \sim \epsilon$.
Since ϵ is arbitrarily small, we conclude the desired invariance. □

Thus, we have proved that $I(\cdot)$ is a knot invariant.

Proof of Theorem 4.5

Now, let us prove Theorem 4.5 that for each weight system W, the function $W(I(\cdot))$ generates a Vassiliev invariant with symbol W.

Theorem (Theorem 4.5)

The invariant $W(I(\cdot))$ is a Vassiliev invariant with symbol W; i.e.,

$$V(W)(K) = W(I(K))$$

for each knot K.

Proof. Without loss of generality, we might assume that our knots are not only Morse embedded in $\mathbb{R}^3$ but their projections on some vertical plane (say, Oxz) represent planar knot diagrams (in the ordinary sense).

Proof of Theorem 4.5 continued.

Let W be a weight system of order m. In order to prove the theorem, we have to show that if D is a chord diagram of degree m and K_D is a Morse embedding of the singular curve (curve with intersection) in $\mathbb{C}_z \times \mathbb{R}_t$ (the singular knot corresponds to D), then we have

$$I(K_D) = \bar{D} + \langle \text{terms of order} \geq m \rangle,$$

where $\bar{D}$ is the equivalence class of the chord diagram D and $I(K_D)$ is defined to be the alternating sum of I evaluated at 2^m knots generating the singular knot K_D.

If two Morse knots K_1 and K_2 in $\mathbb{C}_z \times \mathbb{R}_t$ coincide everywhere except for a small part, where the branches of K_1 form an overcrossing (with respect to the projection on a vertical plane) and those of K_2 form an undercrossing, then the values $Z(K_2)$ and $Z(K_1)$ differ only in those chord diagrams, for which some point(s) on this branches is (are) paired with other point(s).

Proof of Theorem 4.5 continued.

By virtue of Vassiliev's relation, the singular knot K_D is an alternating sum of 2^m knots that differ in small neighbourhoods of m points. Note that the sign of this alternating sum is regulated by the multiplicator $(-1)^{\downarrow}$ in the formula (1).
Arguing as above, we conclude that $Z(K_D)$ has non-zero coefficients only at those chord diagrams obtained by pairing points for each neighbourhood. Thus, chord diagrams with non-zero coefficients must have at least m chords.
For chord diagrams of degree m this coefficient is not equal to zero only for the diagram K_D.
Let us calculate this coefficient.
At each of m vertices we obtain the difference of the integrals of the differential form $\frac{dz_i - dz_i'}{z_i - z_i'}$.

Proof of Theorem 4.5 continued.

This difference equals the integral of $\frac{dz}{z}$ along the circuit passing once around zero. According to Cauchy's theorem, this integral equals $2\pi i$. Because the number of such contours equals m, the coefficients should be multiplied.

Thus we obtain the multiplication factor $(2\pi i)^m$ that is cancelled by the denominator of (1). This means that

$$Z(K_D) = \bar{D} + \langle \text{terms of order} \geq m \rangle.$$

Taking into account $I(K) = \frac{Z(K)}{Z(\infty)^{\frac{c}{2}-1}}$, we have

$$I(K_D) = \bar{D} + \langle \text{terms of order} \geq m \rangle.$$

Consequently, $W(I(K_D)) = W(D)$, and the Vassiliev invariant $W(I(\cdot))$ of order m has the symbol W. This completes the proof. □

The calculation of the Kontsevich integral is, however, very difficult. For instance, it was quite a complicated problem to calculate the integral (preliminary) of ∞. The form (in the Feynman diagram) of the integral was conjectured by Bar–Natan, Garoufalidis, Rozansky, Thurston [9] and finally proved in [14].

The formula is represented in terms of Feynman diagrams. It looks like

$$I(\infty) = exp\sum_{n=0}^{\infty} b_{2n}w_{2n} = 1 + \left(\sum_{n=0}^{\infty} b_{2n}w_{2n}\right) + \frac{1}{2}\left(\sum_{n=0}^{\infty} b_{2n}w_{2n}\right)^2 + \dots.$$

Here b_{2n} are modified Bernoulli numbers; i.e., the coefficients of the Taylor series:

$$\sum_{n=0}^{\infty} b_{2n}x^{2n} = \frac{1}{2}\ln\frac{e^{x/2}-e^{-x/2}}{x/2},$$

and w_{2n} are wheels.

Each wheel w_{2n} is $\frac{1}{(2n)!}$ multiplied by the sum of $(2n!)$ Feynman diagrams. Each of these diagrams consists of one exterior circle, one interior circle (treated just as a circular set of interior edges), and $2n$ chords connecting fixed $2n$ points on the first one with fixed $2n$ points on the second one. These points can be connected according to arbitrary permutation from S_{2n}. Thus, we have $(2n)!$ summands and take their average.

Each wheel w_{2n} is $\frac{1}{(2n)!}$ multiplied by the sum of $(2n!)$ Feynman diagrams. Each of these diagrams consists of one exterior circle, one interior circle (treated just as a circular set of interior edges), and $2n$ chords connecting fixed $2n$ points on the first one with fixed $2n$ points on the second one. These points can be connected according to arbitrary permutation from S_{2n}. Thus, we have $(2n)!$ summands and take their average. For instance, if we consider w_4, we see that eight summands represent the diagram D_x shown in the left part Fig. 31 and another sixteen summands represent the diagram D_y, see the right part of Fig. 31.

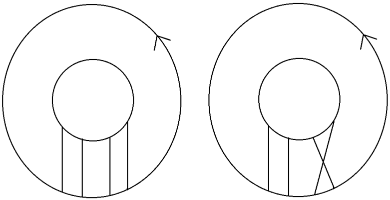

Figure 31: Wheels w_4

In the terms of chord diagrams w_4 can be represented as follows:

$$w_4 =$$  $$- \frac{10}{3}$$  $$+ \frac{4}{3}$$  .

Analogously (in fact, even more easily) one can find the expression for w_2 and w_2^2.

Exercise 5.16

Prove the formulae above.

The first terms of the final result look like:

$$I(\infty) = 1 + \frac{1}{48} w_2 + \frac{1}{4608} w_2^2 - \frac{1}{5760} w_4 + \dots$$

or, in terms of chord diagrams,

$I(\infty) = 1 - \frac{1}{24}$  $- \frac{1}{5760}$  $+ \frac{1}{1152}$  $+ \frac{1}{2880}$  $+ \dots$

Besides this, Le and Murakami [25] constructed a generalisation of the Kontsevich integral for the case of so–called *tangles* — one–dimensional manifolds lying between two horizontal planes and incident to these planes only at a finite number of points. A tangle is a common generalisation of both knots and braids, and the computation of the Kontsevich integral for the case of braids is much easier. In fact, tangles appeared indirectly in the text while calculating $Z[a, b]$ for some interval $[a, b]$. By using their own techniques, they calculated $Z(\infty)$.

Contents

Exercises

1. Let ω_{ij} be the following 1–form on X_n:

$$\omega_{ij} = d(\ln(z_i - z_j)) = \frac{dz_i - dz_j}{z_i - z_j}.$$

   Show that $\omega_{12} \wedge \omega_{23} + \omega_{23} \wedge \omega_{31} + \omega_{31} \wedge \omega_{12} = 0$.

2. Let $\Omega_{nn} = \sum_{1 \leq i < j \leq n} s_i s_j \Omega_{ij} \omega_{ij}$, where s_i equals 1 for $i \leq n$ and -1 for $i > n$. Show that the connection Ω_{nn} is flat.

Contents

Research problems:

1. **How the Kontsevich integral is related to 3-free braids?**
2. **How to present the G_n^3 and G_n^4-invariants (and other invariants where singular points are related to three points, not to two) in terms of integrals of holonomies?**

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