Holomorphic Floer Theory and exponential integrals

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December 8,2022, Colloquium talk at YMSC,Tsinghua University

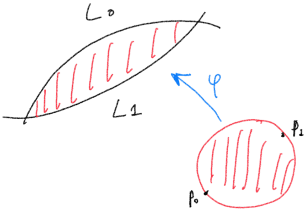
Motivation: Morse and Floer complexes in the holomorphic setting

Let X be a compact real smooth manifold and $f : X \to \mathbb{R}$ a smooth Morse-Smale function. Fix an auxiliary generic Riemannian metric. Then one associates to a pair (X, f) a cochain complex (Morse complex).

Morse complex as a graded vector space is $M(X, f) = \oplus_{x \in Crit(f)} \mathbb{C} \cdot [x]$, where each critical point x gives a generator $[x]$ of degree equal to $-ind(x)$, the Morse index of x. The differential $\partial_{Morse}([x]) = \sum_{y, ind(y) = ind(x)+1} n_{xy}[y]$, where n_{xy} is the number of gradient lines of the function $-f$ starting at x and ending at y. Cohomology of this complex is isomorphic to the cohomology of X, computed e.g. via the de Rham complex. In particular the cohomology does not depend on a choice of the Morse function and the Riemannian metric.

Then one can say that the Morse theory assigns to a pair (X, f) a cohomological invariant which is the complex $M(X, f)$. It also says that all these invariants are equivalent for different functions f (i.e. complexes are quasi-isomorphic).

Equivalently one can consider two Lagrangian submanifolds $L_0 = X$ and $L_1 = graph(df)$ of the symplectic manifold T^*X, and compute the same cohomology as the cohomology of the Floer complex $Hom(L_0, L_1) = \oplus_{x \in L_0 \cap L_1} \mathbb{C} \cdot [x]$ (grading is by Maslov index which coincides with the Morse index in this case). Floer differential ∂_{Floer} is defined in terms of the number of pseudo-holomorphic discs with boundary on $L_0 \cup L_1$. In order to define ∂_{Floer} one choses a generic almost complex structure which is compatible with the standard symplectic form on T^*X. The cohomology of the Floer complex does not depend on the choices.



As a motivation for my talk I start with the question in the spirit of Vladimir Arnold: what is a complex (= holomorphic) analog of the above story? In a sense Arnold himself gave an answer: complex analog of the Morse theory is Picard-Lefchetz theory. Indeed, instead of crossing the critical level of the function as in the real case one can walk around its critical fiber.
As we will see there is another structure in the complex framework, which admits a generalization far beyond Picard-Lefschetz theory. With Kontsevich we call it the wall-crossing structure. Taking it for a moment as a black box let me explain how it appears in our story.

Let us consider a complex manifold $X, dim_{\mathbb{C}}X = n$, and a holomorphic function $f : X \to \mathbb{C}$ which has finitely many isolated Morse critical points $\{x_1, ..., x_k\}$ with different critical values $\{z_1, ..., z_k\}$. Then automatically X is non-compact. Let us assume that X is an affine algebraic variety and f is a regular function. Since all points x_i have the same Morse index n, the Morse differential on the complex $M(X, Re(f))$ is trivial.

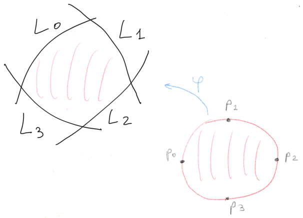
From the point of view of the Floer theory, the differential is trivial on the Floer complex of the pair of complex Lagrangian submanifolds $L_0 = X$ and $L_1 = graph(df)$ of the symplectic manifold T^*X (but all considered in C^∞ sense). Indeed generically there are no pseudo-holomorphic discs with boundary on $L_0 \cup L_1$. The wall-crossing structure recovers the lost interaction between the critical points of f (equivalently, points of $L_0 \cap L_1$).

Consider the 1-parameter family of functions $f/\hbar, \hbar \in \mathbb{C}^*$. Then the Morse differential for the Morse complex associated with $Re(f/\hbar)$ can be non-trivial for some values of $\hbar \in \mathbb{C}^*$. Namely, gradient lines can appear as $\hbar$ belongs to certain rays $Arg(\hbar) = const$ (Stokes rays). The number of gradient lines can be also interpreted in terms of the Floer theory as the virtual number of pseudo-holomorphic discs with the boundary on $L_0 \cup L_1$. These numbers (Stokes indices) can be organized into isomorphisms of the cohomology of Morse or Floer complexes on different sides of the Stokes rays. We call them Stokes isomorphisms. Each Stokes isomorphism is equivalent to a certain wall-crossing formula for the Stokes indices. Mathematical structure underlying the wall-crossing formulas was introduced and studied in our papers with Kontsevich arXiv:0811.2435, arXiv:1303.3253, 2005.10651 under the name of wall-crossing structure (WCS for short). This is the structure which I am going to discuss today.

From the point of view of de Rham cohomology theory, introducing the parameter $\hbar$ in the story corresponds to the deformation of the de Rham differential (C^∞ version was introduced by Witten in 1982 in his "Supersymmetry and Morse theory"): instead of the de Rham complex $(\Omega^\bullet(X), d)$ we consider the complex $(\Omega^\bullet(X), d - \frac{df}{\hbar} \wedge (\bullet))$. Its cohomology forms an algebraic vector bundle over $\mathbb{C}^*_\hbar$ endowed with a meromorphic flat connection. The above-mentioned Stokes isomorphisms correspond to the Stokes isomorphisms of the sheaf of solutions of this holonomic D-module.

We see that in the complex framework we have two sides of the story: de Rham side and Betti side. The Betti side is a complex analog of the Morse/Floer theory. It is necessary to introduce the "quantization parameter" $\hbar$ on both sides. Then we see the wall-crossing structure on the Betti side. The isomorphism of de Rham and Betti cohomology corresponds to the generalized Riemann-Hilbert correspondence introduced by Kontsevich and myself as a part of our Holomorphic Floer Theory (HFT for short).

Our HFT project (it is rather a program) is devoted to various aspects of the Floer theory in the framework of complex symplectic manifolds. In the above example we considered two exact Lagrangian submanifolds of an exact symplectic manifold. In general the HFT studies Fukaya categories of complex symplectic manifolds, i.e. one considers all real Lagrangian submanifolds of complex manifold. Their "interaction" via pseudo-holomorphic discs is encoded in the structure of A_∞-category.



We will not need the general HFT today, and we don't need Fukaya categories (although they will be behind the scene). Main geometric objects for today are:

1) A complex symplectic manifold $(M, \omega^{2,0})$.

2) A pair of complex Lagrangian submanifolds $L_0, L_1 \subset M$.

Main algebraic structure which I will rigorously define later is:

3) The wall-crossing structure.

There is another important algebraic structure motivated by HFT which I would like to mention:

4) Quantum wave functions associated with L_0, L_1.

I do not have time today to discuss quantum wave functions in detail, but I will explain their role in one example.

On the next few slides I will give examples of pairs of holomorphic Lagrangian submanifolds and mathematical and physical problems in which they appear. The corresponding wall-crossing structures will be discussed later.

Examples of pairs of complex Lagrangian subvarieties for today

a) Finite-dimensional exponential integrals $I(\hbar) := \int_{C\subset X} e^{f/\hbar} vol$. Here X is a complex n-dimensional manifold, f is a holomorphic function with nice behavior "at infinity", the real n-dimensional integration cycle C belongs to an appropriate class of chains, and $vol \in \Omega^{n,0}(X)$ is a fixed holomorphic volume form on X. The parameter $\hbar$ belongs to $\mathbb{C}^*$. In this example $M = T^*X, L_0 = X, L_1 = graph(df)$.

b) Same as a), but f is multivalued, i.e. instead of the pair (X, f) we have (X, α), where α is a holomorphic closed 1-*form* on X. Then $M = T^*X, L_0 = X, L_1 = graph(\alpha)$.

c) Example related to the Chern-Simons theory. Let M^3 be a compact oriented 3-manifold, $k \in \mathbb{Z}_{\geq 1}$, G_c a compact Lie group, G be its complexification. Fix a knot $K \subset M^3$. Its small tubular neighborhood in M^3 gives rise to a complex symplectic manifold (more precisely, symplectic stack) of flat G-connections on the boundary of this tubular neighborhood. Then we have two holomorphic Lagrangian subvarieties $L_0 = L_{in}$ (resp. $L_1 = L_{out}$) where L_0 (resp. L_1) consists of those connections which can be extended inside (resp. outside) of the tubular neighborhood of K. Then $L_0 \cap L_1$ can be identified with the set of flat G-connections on M^3, which is the same as the set of critical points of the (complexified) multivalued Chern-Simons functional

$$CS(A) = \int_{M^3} Tr(\frac{1}{2} A \wedge dA + \frac{1}{3} A \wedge A \wedge A).$$

Infinite-dimensional exponential integral $Z_{CS} = \int_{G_c - connections} e^{ikS(A)} \mathcal{D}A$ can be given a mathematical meaning either perturbatively as $k \to \infty$ or for a fixed k via quantum groups at roots of 1 (Witten, Reshetikhin, Turaev).

Decomposing the hyperbolic manifold $M^3 - K$ into the union of ideal tetrahedra one can use the results of Hikami and give another mathematical meaning to Z_{CS} in terms of the Faddeev's quantum dilogarithm (for details see e.g. arXiv:0903.2472). This is related to the following pair of complex Lagrangian submanifolds:

d)

$$L_0 := \{(q_1, p_1, q_2, p_2, \dots, q_n, p_n) \in \mathbb{C}^{2n} | e^{q_i} + e^{p_i} = 1, i = 1, \dots, n\} \subset (\mathbb{C}^*)^{2n},$$

where $(\mathbb{C}^*)^{2n}_{x_1,\dots,x_n,y_1,\dots,y_n}$ is endowed with the standard symplectic form $\omega^{2,0} = \sum_{1\le i\le n} \frac{dx_i}{x_i} \wedge \frac{dy_i}{y_i}$. Here $x_i = e^{q_i}, y_i = e^{p_i}, 1 \le i \le n$.

E.g. for $n = 1$ L_0 is a "pair of pants" $\{x_1 + y_1 = 1\} \subset (\mathbb{C}^*)^2$.

To define L_1 we fix an integer symmetric matrix $(a_{ij})_{1\le i,j\le n}$ and consider the abelian subgroup in $\mathbb{Z}^n$.

$$\Lambda = \{(q_1, p_1, ..., q_n, p_n) | p_j = \sum_{1\le i\le n} a_{ij} q_i\}, 1 \le j \le n.$$

Then $L_1 := L_\Lambda \subset (\mathbb{C}^*)^{2n}$ is the corresponding Lagrangian torus. E.g. in the case $n = 1$ for some choice of Λ we get $L_1 = \{x_1 = 1\}$.

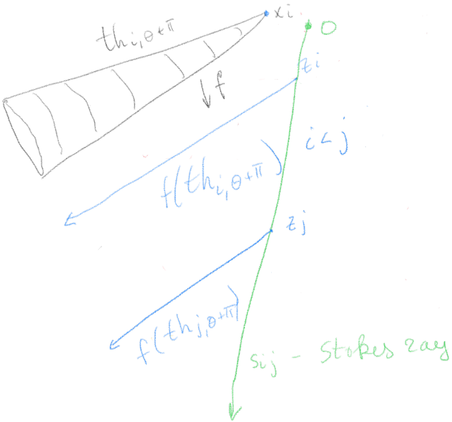
Arbitrary complex Lagrangian submanifolds L_0, L_1 in the standard symplectic $\mathbb{C}^{2n}$ can appear as boundary conditions for the Feynman path integral:
e) Feynman integral $Z(\hbar) = \int e^{S(\varphi)/\hbar} D\varphi$ where $\varphi : [0,1] \to \mathbb{C}^{2n}, \varphi(0) \in L_0, \varphi(1) \in L_1$, where $S(\varphi) = \int_0^1 \sum_i p_i(t) \frac{dq_i(t)}{dt} + \int_0^1 H(\mathbf{q}(t), \mathbf{p}(t), t) dt$ and $H : \mathbb{C}^{2n} \times [0,1] \to \mathbb{C}$ is holomorphic in complex coordinates $(\mathbf{q}, \mathbf{p})$ and C^∞ in the real coordinate $t \in [0,1]$.

About relations between Examples c), d, e).
Probably perturbative expansion of $Z(\hbar), \hbar \to 0$ is related to the perturbative expansion of $Z_{CS}, k \to \infty$ if we take L_0, L_1 as in Example d) and set $\hbar = 2\pi i/k$.
Yet another approach to the perturbative expansion of Z_{CS} is via the above-mentioned theory of quantum way functions. More precisely, one considers the Moyal deformation quantization $\mathcal{O}_{\mathbb{C}^{2n},q}$ of the sheaf of algebras $\mathcal{O}_{\mathbb{C}^{2n}}$. Then one associates to L_0 and L_1 certain cyclic modules over the Weyl algebra $W_q = \Gamma(\mathbb{C}^{2n}, \mathcal{O}_{\mathbb{C}^{2n},q})$. There is a natural choice of quantum wave functions ψ_{L_0} and ψ_{L_1} as generators of these cyclic modules. Then if e.g. $H = 0$ the Feynman integral coincides as a series in $\hbar$ with the properly defined pairing $\langle \psi_{L_0}, \psi_{L_1} \rangle$.

Thimbles

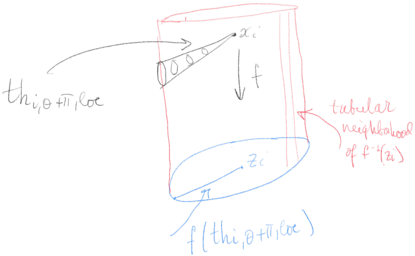
Let us now discuss finite-dimensional exponential integrals, which are related to the geometry of Examples a) and b). We will learn what is the WCS in this case and will see how it is related to the Betti cohomology theory.

Let $X, dim_{\mathbb{C}}X = n$ be algebraic and Kähler, and $f : X \to \mathbb{C}$ be regular Morse function (later I will say what to do when f is not Morse). Let $Crit(f) = \{x_1, ..., x_k\}$ and $S = \{z_1, ..., z_k\}$ be the corresponding set of critical values of f (assume they all distinct). Fix angle $\theta = Arg(\hbar)$. We define a thimble $th_{z_i,\theta+\pi}$ as the union of gradient lines (for the Kähler metric) of the function $Re(e^{-i\theta} f)$ which originate at the critical point $x_i \in X$ such that $f(x_i) = z_i$. It is easy to see that $f(th_{z_i,\theta+\pi})$ is a ray $Arg(z) = \theta + \pi$ emanating from the critical value $z_i \in S$.



$th_{i,\theta+\pi}$
x_i
0
z_i
f
$i<j$
$f(th_{i,\theta+\pi})$
z_j
$f(th_{j,\theta+\pi})$
S_{ij} - Stokes ray

Under some mild conditions the collection of thimbles $th_{z_i,\theta+\pi}, 1 \leq i \leq k$ generates the lattice of relative homology $\Gamma = H_n(X, f^{-1}(z), \mathbb{Z})/tors \simeq \mathbb{Z}^k$. Here $|z| \gg 1$, so we will write $\Gamma = H_n(X, f^{-1}(\infty), \mathbb{Z})/tors$. This global relative homology can be identified with the direct sum of similarly defined local relative homology groups (alternatively one can work with the cohomology groups) generated by local thimbles. This is known as local-to-global Betti isomorphism. Local thimbles are defined in terms of a small tubular neighborhood of the critical fiber, i.e. z is taken close to z_i for each z_i. For the rescaled function $f/\hbar$ we should rescale the critical value $z_i \mapsto z_i/\hbar$ as well.



$th_{i,\theta+\pi,loc}$
x_i
f
tubular neighborhood of $f^{-1}(z_i)$
z_i
$f(th_{i,\theta+\pi,loc})$

Let us consider the following collection of exponential integrals for those $\hbar \in \mathbb{C}^*$ which do not belong to the Stokes rays $Arg(\hbar) = Arg(z_i - z_j), i \neq j$:

$$I_i(\hbar) = \int_{th_{z_i,\theta+\pi}} e^{f/\hbar} vol.$$

Assume that the set of critical values $S = \{z_1, ..., z_k\}$ is in generic position in the sense that no straight line contains three points from S. Then a Stokes ray contains two different critical values which can be ordered by their proximity to the vertex.

As any exponential integral, this one can be interpreted as an exponential period, i.e. the pairing of the closed with respect to the differential $d - \frac{df}{\hbar} \wedge (\bullet)$ de Rham form $e^{f/\hbar} vol$ with the (dual to the) Betti cohomology class. The global Betti-to-de Rham isomorphism corresponds to this pairing.

Wall-crossing formulas

It is easy to see that if in the $\hbar$-plane we cross the Stokes ray $s_{ij} = s_{\theta_{ij}}$ containing critical values $z_i, z_j, i < j, \theta_{ij} = Arg(z_i - z_j)$, then the integral $I_i(\hbar)$ changes such as follows:

$$I_i(\hbar) \mapsto I_i(\hbar) + n_{ij} I_j(\hbar),$$

where $n_{ij} \in \mathbb{Z}$ is the number of gradient trajectories of the function $Re(e^{i(Arg(z_i - z_j)/\hbar)} f)$ joining critical points x_i and x_j.

Let us modify the exponential integrals:

$$I_i^{mod}(\hbar) := \left(\frac{1}{2\pi\hbar}\right)^{n/2} e^{-z_i/\hbar} I_i(\hbar).$$

Then as $\hbar \to 0$ the stationary phase expansion ensures that as a formal series

$$I_i^{mod}(\hbar) = c_{i,0} + c_{i,1}\hbar + ... \in \mathbb{C}[[\hbar]],$$

where $c_{i,0} \neq 0$. The jump of the modified exponential integral when we cross the Stokes ray s_{ij} is given by $\Delta(I_i^{mod}(\hbar)) = n_{ij} I_j^{mod}(\hbar) e^{-(z_i - z_j)/\hbar}$.

RH problem

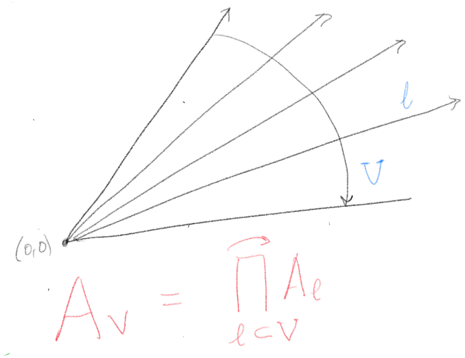
The wall-crossing formulas gives rise to a holomorphic vector bundle coming from a certain Riemann-Hilbert problem. Namely, the vector $\overline{I}^{mod}(\hbar) = (I_1^{mod}(\hbar), ..., I_k^{mod}(\hbar)), k = |S|$ satisfies the Riemann-Hilbert problem on $\mathbb{C}$ with known jumps across the Stokes rays and known asymptotic expansion as $\hbar \to 0$ (notice that because of our ordering of the points in S, the function $e^{-(z_i - z_j)/\hbar}$ has trivial Taylor expansion as $\hbar \to 0$ along the Stokes ray s_{ij}). This determines a pair consisting of a holomorphic vector bundle on $\mathbb{C}_\hbar$ and its section. This pair encodes all properties of our exponential integral.

Let us now discuss the notion of wall-crossing structure in general. The notion was introduced by Kontsevich and myself for the purposes of our theory of Donaldson-Thomas (=BPS) invariants. Later we demonstrated that it was useful in the theory of complex integrable systems. In relation to the resurgence theory we introduced a subclass of the so-called analytic WCS.

On the next few slides I will recall very briefly main facts about WCS and analytic WCS. WCS is defined as a locally constant sheaf of more elementary structures called stability data on a graded Lie algebra. I am going to recall the latter.

Stability data

Let Γ be a free abelian group of finite rank ("charge lattice"). Stability data on a Γ-graded Lie algebra $\mathfrak{g} = \oplus_{\gamma\in\Gamma}\mathfrak{g}_\gamma$ over $\mathbb{Q}$ are given by:
1) homomorphism of abelian groups $Z : \Gamma \to \mathbb{C}$ ("central charge");
2) collection of elements $a(\gamma) \in \mathfrak{g}_\gamma, \gamma \in \Gamma - \{0\}$ (rational enumerative invariants).
These data are required to satisfy one axiom called *Support Property*. It says that there exists a quadratic form Q on $\Gamma_\mathbb{R} = \Gamma \otimes \mathbb{R}$ such that $Q > 0$ on the support of the collection $(a(\gamma))_{\gamma\in\Gamma-\{0\}}$ and $Q < 0$ on $Ker\, Z$. The notion of stability data can be equivalently described via a collection of groups elements
$A_{V_i} = exp(\sum_{Z(\gamma)\in V_i} a(\gamma)) \in exp(\sum_{Z(\gamma)\in V_i, Q(\gamma)>0} \mathfrak{g}_\gamma) := G_{V_i}$, where $\mathbb{R}^2 - \{0\} = \cup_{i\in I} V_i$ is an open (closed, semiclosed) cover by strict sectors (angle less than π) with vertex at $(0, 0)$. For any strict sector V we have $A_V = \prod_{l\subset V} A_l$, the clockwise product over the set of rays in V with the vertex in $(0, 0)$. The product is well-defined since $A_l \neq 1$ only for countably many rays l.


(0,0)
l
V
$A_V = \overrightarrow{\prod_{l \subset V}} A_l$

We can fix $\mathfrak{g}$ and consider the set $Stab(\mathfrak{g})$ of all stability data on $\mathfrak{g}$.

Theorem

There is a Hausdorff topology on the set $Stab(\mathfrak{g})$ such that the natural projection to the central charge $Z \in Hom(\Gamma, \mathbb{C})$ is a local homeomorphism.

The wall-crossing formulas for $a(\gamma)$ show how these elements get changed when we cross a real codimension one walls in $Stab(\mathfrak{g})$. This can be described by the collection of group elements A_l ("Stokes automorphisms") corresponding to certain rays ("Stokes rays"). We will see examples later.

WCS is a local system of stability data. Illustrating example from the "real world": complex integrable systems of Seiberg-Witten type. Then local system of lattices=local system of first integer homology of fibers over the base of integrable system, and the local system of Lie algebras=local system of torus algebras associated with these symplectic lattices, the central charge $Z(\gamma) = \int_\gamma \lambda_{SW}$, where λ_{SW} is the Seiberg-Witten meromorphic 1-form on fibers. In this case $a(\gamma)$ are integers constructed inductively, starting from the neighborhood of the discriminant (key words: split attractor flow). Details in arXiv:1303.3253.

Analytic stability data

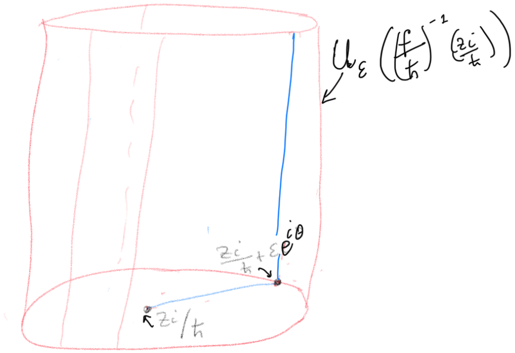
Let $\mathfrak{g} := Vect_\Gamma$ be the graded Lie algebra of algebraic vector fields on the torus $\mathbf{T}_\Gamma = Hom(\Gamma, \mathbb{C}^*)$. Then stability data of $\mathfrak{g}$ can be restated in terms of certain gluing data of a formal scheme over $\mathbb{C}$. If this formal scheme comes from a complex analytic space, the stability data are called analytic. Using the $\mathbb{C}^*$-action $Z \mapsto Z/\hbar$ one can construct an analytic fiber bundle over $\mathbb{C}^*_\hbar$ with the fiber isomorphic to $\mathbf{T}_\Gamma$, roughly, by correcting the trivial fiber bundle by means of the Stokes automorphisms A_l (equivalently, by solving certain Riemann-Hilbert problems).

This bundle can be extended analytically to $\mathbb{C}_\hbar$, and it has a canonical trivialization at $\hbar = 0$. In arXiv:2005.10651 we formulated the resurgence conjecture. It says that the Taylor series $J(\hbar) = \sum_{n\geq 0} a_n \hbar^n$ of an analytic section of the above bundle is resurgent, i.e. it is divergent, but its Borel transform $B(J)(s) = \sum_{n\geq 0} \frac{a_n}{n!} s^n$ defines an analytic function. In all examples a)-e) (and in many others including Feynman integrals) the naturally defined wall-crossing structures are either analytic or expected to be analytic. Therefore the concept of wall-crossing structure gives a new approach to the notion of resurgence.

Finally let me spell out the WCS over $\mathbb{C}^*_\hbar$ which corresponds to exponential integrals, including the case when f is non-Morse.
In order to define it we set $\Gamma = \mathbb{Z}^k$, and $Z(e_i) = z_i, 1 \leq i \leq k$ for the standard basis $e_i, 1 \leq i \leq k$ of $\mathbb{Z}^k$. The fiber of the local system Γ-graded Lie algebras can be written in a way which is appropriate for infinite-dimensional generalizations:

$$\mathfrak{g}_\hbar = \oplus_{i \in S} Hom(H_n((f/\hbar)^{-1}(D_\varepsilon(z_i)), (f/\hbar)^{-1}(z_i - \varepsilon), \mathbb{Z}), H_n((f/\hbar)^{-1}(D_\varepsilon(z_j)), (f/\hbar)^{-1}(z_j - \varepsilon), \mathbb{Z}))$$

where $\varepsilon > 0$ is small. In order to define this local system algebra-geometrically one should replace (dual to) the relative homology by $H^{middle}(Crit(f), \varphi_f(\mathbb{Z}))$, where φ_f is the functor of the sheaf of vanishing cycles of f.
The Stokes automorphisms arise from the comparison of local and global relative (co)homology. This is an incarnation of the Betti local-to-global isomorphism for non-Morse functions. The WCS defined in such a way turns out to be analytic.


$U_\varepsilon\left(\left(\frac{f}{\hbar}\right)^{-1}\left(\frac{z_i}{\hbar}\right)\right)$
$\frac{z_i}{\hbar}+\varepsilon e^{i\theta}$
$z_i/\hbar$

Stokes automorphisms can be also written in a non-linear (or "cluster") form. Namely, in the standard coordinates $(x_1, ..., x_k)$ on $\mathbf{T}_\Gamma$ they can be written as $x_i \mapsto x_i(1 + n_{ij}e^{-Z(\gamma_{ij})/\hbar}x^{\gamma_{ij}})$, where $\gamma_{ij} = e_j - e_i, x^{\gamma_{ij}} = x_j x_i^{-1}$.
Notice that the integer $n_{ij} \in \mathbb{Z}$ can be also interpreted as the intersection index of two "almost opposite" thimbles outcoming from x_i and x_j towards each other. Equivalently, n_{ij} can be interpreted as the virtual number of pseudo-holomorphic discs with the boundary on the union of holomorphic Lagrangian manifolds $X \cup graph(df/\hbar) \subset T^*X$. As I mentioned at the beginning such discs generically are absent, but they appear when $\hbar$ belongs to a Stokes ray.

Theorem

The Taylor expansions of $I^{mod}(\hbar)$ at $\hbar = 0$ is resurgent, in agreement with the fact that the wall-crossing structure is analytic.

Holomorphic 1-forms

In this case instead of a pair (X, f) we have a pair (X, α) where is α a closed holomorphic 1-form on X. We assume that the closed subset $\mathcal{Z}(\alpha) := Zeros(\alpha)$ is a finite disjoint union $\sqcup_{i \in I} \mathcal{Z}_i(\alpha)$ of compact connected components. The corresponding two complex Lagrangian submanifolds in T^*X are $L_0 = X, L_1 = graph(\alpha)$.
The theory for 1-forms is similar to the one for functions but more interesting and complicated. The corresponding WCS can be thought of as holomorphic analog of the Morse-Novikov theory. Main difficulties are on the Betti side, i.e. on the side of integration cycles. In particular, even if all zeroes of α are simple (analog of the Morse function case), the notion of thimble is more complicated because of the possible chaotic dynamics of the corresponding gradient flow. There is a WCS for 1-forms with $\Gamma = H_n(X, \mathcal{Z}(\alpha), \mathbb{Z})/tors$, and central charge $Z(\gamma) = \int_\gamma \alpha$. It is expected to be analytic with Stokes automorphisms given by matrices with rational functions as coefficients. Instead of going into the details I will illustrate the theory in a simple example of Γ-function.

Let $\alpha = (\frac{1}{x} - 1)dx$ as 1-form on $\mathbb{C}^* \subset \mathbb{CP}^1$. Notice that α can be integrated along the thimble $L_+ = (0, +\infty)$ with the volume form on $\mathbb{C}^*$ equal to dx/x. When restricted to L_+ we can write $\alpha = df, f = log(x) + 1 - x$. Then the corresponding version of the modified exponential integral for $Re(\hbar) > 0$ becomes $I^{mod}(\hbar) = \frac{1}{\sqrt{2\pi\hbar}} \int_L e^{\frac{1}{\hbar} f(x)} \frac{dx}{x} = \frac{\Gamma(\lambda)}{\sqrt{2\pi} e^{-\lambda} \lambda^{\lambda - 1/2}}$, where $\lambda = 1/\hbar$. Its Taylor expansion belongs to $\mathbb{C}[[\hbar]]$ and gives rise to a resurgent series.

Similar considerations hold for $Re(\hbar) < 0$ and the opposite thimble $L_- = (-\infty, 0)$. Then $I^{mod}(\hbar) = 1/I^{mod}(-\hbar)$. Let us set $I_R(\hbar) = \frac{1}{\sqrt{2\pi\hbar}} e^{1/\hbar}\hbar^{1/\hbar}\Gamma(1/\hbar)$ for $Re(\hbar) > 0$ and $I_L(\hbar) = 1/I_R(-\hbar)$ for $Re(\hbar) < 0$. The corresponding WCS is equivalent to the following Riemann-Hilbert problem which connects these two functions:

$$I_L(\hbar) = I_R(\hbar)(1 - exp(-2\pi i/\hbar)), \hbar \in i\mathbb{R}_{>0},$$

$$I_R(\hbar) = I_L(\hbar)(1 + exp(-2\pi i/\hbar))^{-1}, \hbar \in i\mathbb{R}_{<0}.$$

In order to construct the WCS and to glue the corresponding bundle one has to know the Stokes indices, i.e. the intersection numbers of thimbles lifted to the universal covering of $\mathbb{C}^*$. There are infinitely many critical points of the antiderivative of α on the universal covering of $\mathbb{C}^*$, namely $x_k = 2\pi k, k \in \mathbb{Z}$. The corresponding Stokes indices (i.e. generalized DT-invariants) are given by the following formulas: $n_{ab} = -1, b = a - 1$, $n_{ab} = +1, b > a$ and $n_{ab} = 0$ otherwise. Here $a, b \in \mathbb{Z}$.
For the WCS we take $\Gamma = \mathbb{Z}$, and the central charge given by the periods of α. Numbers n_{ab} is the remaining piece of data.

Another interesting example is the one of a holomorphic 1-form on a square-tiled surface. Then periods of α belong to $\mathbb{Z} \oplus i\mathbb{Z}$. One can show that Stokes automorphisms are given by rational in parameters matrices in the basis of homology classes of thimbles. This result can be applied to the study of the properties of saddle connections on Riemann surfaces. It is expected that the same is true in the higher-dimensional case as well. Unfortunately I do not have time for the details.

Summary of the finite-dimensional story

Let us summarize our discussion from the perspective of HFT. We are given two holomorphic Lagrangian subvarities L_0 and L_1 in a complex symplectic manifold M (in the above examples L_0 and L_1 intersect transversally). With each intersection point $p \in L_0 \cap L_1$ we associated a formal series $I_p(\hbar) \in \mathbb{C}[[\hbar]] \otimes V_p$, where V_p is a vector space. The collection $\overline{I}^{mod}(\hbar) = (I_p^{mod}(\hbar))_{p \in L_0 \cap L_1}$ of modified exponential integrals is interpreted as a section of a formal vector (or fiber) bundle over the formal disc $Spec(\mathbb{C}[[\hbar]])$. Pseudo-holomorphic discs with the boundary on $L_0 \cup L_1$ give rise to the Stokes automorphism $A(\hbar) = (A_{pq}(\hbar))_{p,q \in L_0 \cap L_1} : \oplus_{p \in L_0 \cap L_1} (\mathbb{C}[[\hbar]] \otimes V_p)$. Using $A_{pq}(\hbar)$ as gluing automorphisms one glue another bundle which is often holomorphic (this happens when the corresponding WCS is analytic). This approach can be generalized to the infinite-dimensional case.

WCS for Chern-Simons theory

In the above example the group of periods of the 1-form is $\mathbb{Z}$. In the case of the CS-functional the group of periods of the corresponding holomorphic 1-form $\alpha_{CS} = dCS$ is $(2\pi i)^2\mathbb{Z} \simeq \mathbb{Z}$. One can speculate about the corresponding wall-crossing structure. It can be described either in terms of the geometry of the infinite-dimensional space $\mathcal{A}_{\mathbb{C}}^{fr}$ of G-connections on M^3 trivialized at a point. This can be done in various ways. E.g. consider the holomorphic function $f_{CS} = exp(CS/2\pi i)$ which has only finitely many critical values. Then the WCS is defined in terms of the following data:

1) $\Gamma = H_1(\mathbb{C}^*, Critval(f_{CS}), \mathbb{Z})$, and $Z : \Gamma \to \mathbb{C}$ given by $Z(\gamma) = \int_\gamma \frac{dw}{w}$.

2) Central charge $Z : \Gamma \to \mathbb{C}, \gamma \mapsto 2\pi i \int_\gamma \frac{dw}{w}$.

3) Local system of Γ-graded Lie algebras $\underline{\mathfrak{g}}$ on $\mathbb{C}^*_\hbar$ with the fiber given by

$$\mathfrak{g}_\hbar := \oplus_{\gamma\in\Gamma}\mathfrak{g}_{\hbar,\gamma} =$$

$$\bigoplus Hom(H^0(\mathcal{Z}_{j_1}(\alpha_{CS}), \varphi_{\frac{CS}{\hbar}}(\mathbb{Z})), H^0(\mathcal{Z}_{j_2}(\alpha_{CS}), \varphi_{\frac{CS}{\hbar}}(\mathbb{Z}))$$

where summation is taken over the set $\{w_1, w_2 \in Critval(f_{CS}), \gamma \in \Gamma\, s.t.\, \partial\gamma = [w_1] - [w_2], j_1, j_2 \in J\, s.t.\, f_{CS|\mathcal{Z}_{j_m}}(\alpha_{CS}) = \hbar w_m, m = 1, 2\}$.

4) For any $\hbar \in \mathbb{C}^*$ we define the pronilpotent completion $\widehat{\mathfrak{g}}_\hbar = \prod_{Z(\gamma)\in \hbar\cdot\mathbb{R}_{>0}} \mathfrak{g}_{\gamma,\hbar}$. Let $G_\hbar = exp(\widehat{\mathfrak{g}}_\hbar)$ be the corresponding pronilpotent group. The pronilpotent Lie algebra $\widehat{\mathfrak{g}}_\hbar$ is well-defined because the corresponding set of γ with $\mathfrak{g}_{\hbar,\gamma} \neq 0$ belongs to a strict convex cone. The Stokes automorphisms $A_\hbar := A_{l_\hbar} \in G_\hbar$ are not equal to 1 for at most countable set of rays $l_\hbar := \{Arg(\hbar) = const\}$. In general the Stokes automorphisms are not well-understood even at the physics level of rigor. One expects that they can be derived from the study of a generalization of Kapustin-Witten equations to the case of non-Morse critical points of the CS functional. For isolated Morse critical points the Stokes automorphisms are derived from a collection of integers (Stokes indices). They can be interpreted as the number of gradient lines of the function $Re(CS/\hbar)$ between the corresponding critical points. This resembles our toy-model example with Γ-function.

It is expected that this Chern-Simons WCS is *analytic*. If we accepted our resurgence conjecture, this would imply resurgence of the perturbative expansions of the CS functional at critical points. There are many interesting results and conjectures about arising structures. In particular, they are related to the theory of quantum wave functions of the mentioned in our Example d) K_2-Lagrangian submanifolds $L_0, L_1 \subset (\mathbb{C}^*)^{2n}$.

I would like to finish with a discussion about the integral over the space of unitary connections, which was the initial purpose of the Chern-Simons theory. For simplicity we consider $G = SL(2, \mathbb{C})$.
Let $\mathcal{A}^{fr}_{unit}$ denote the set of unitary (i.e. $SU(2)$) framed connections on M^3. The intersection $\mathcal{A}^{fr}_{unit} \cap Crit(CS)$ is compact. Assume that it is 0-dimensional and consists of rigid flat connections. By analogy with the finite-dimensional case we can hope that the "integration cycle" $\mathcal{A}^{fr}_{unit}$ is homologically equivalent to the integer linear combination of thimbles, corresponding to the segments joining $0 \in \mathbb{C}$ with the critical values of the CS functional.

Conjecture about the integral over the semi-infinite cycle of unitary connections

Let us fix the level $k \in \mathbb{Z}_{\geq 1}$. Then

$$\int_{\mathcal{A}^{fr}_{unit}} e^{\frac{kCS}{2\pi i}} \mathcal{D}A = \sum_{\rho_j,\, s.t.\, |exp(CS(\rho_j)/2\pi i)| \leq 1} n_{\rho_j} \int_{th_{\rho_j}} e^{\frac{kCS}{2\pi i}} \mathcal{D}A,$$

where $n_\rho \in \mathbb{Z}$ is the virtual number solutions of the Kapustin-Witten equation on $M^3 \times [0, \infty)$ with the unitary boundary conditions at $M^3 \times \{0\}$ and flat boundary conditions $\rho, |exp(CS(\rho)/2\pi i)| \leq 1$ at $M_3 \times \{\infty\}$. Here th_ρ denote the thimble outcoming from the critical point ρ in such a way that $Re(kCS/2\pi i) < 0$ along the thimble.

Analyticity conjecture

Consider the generating function

$$N(w)=\sum_{1\le k\le\infty}\left(\int_{\mathcal{A}^{fr}_{unit}} e^{kCS(A)/2\pi i}\mathcal{D}A\right)w^k.$$

If the above-defined WCS for the CS theory is analytic then the generating series $N(w)$ converges in the disc $|w|<1$ and analytically continues to $\mathbb{C}$ with singularities at $\{0\}\cup Critval(CS)$.

There is a finite-dimensional toy-model example illustrating the conjecture. Namely, let $f = z - log(z)$ and $I_k = \int_{|z|=1} e^{kf(z)} \frac{dz}{z}$. Then $I_k = \frac{1}{2\pi i} \int_{|z|=1} z^{-k} e^{kz} \frac{dz}{z} = \frac{k^k}{k!}$. Consider the generating function $N(w) = \sum_{k\geq 1} w^k k^k / k!$. This function has ramifications at the critical values of f i.e. at $0, e^{-1}, \infty$. Setting $0^0 = 1$ we can rewrite $N(w)$ such as follows

$$N(w) = \sum_{k\geq 0} \frac{1}{2\pi i} w^k \int_{|z|=1} (e^z/z)^k \frac{dz}{z} = \frac{1}{2\pi i} \int_{|z|=1} \frac{1}{w^{-1} - \frac{e^z}{z}} \frac{dz}{z}.$$

The poles $z_n(w), n \geq 1$ of the denominator $w^{-1} - \frac{e^z}{z}$ form a countable subset of $\mathbb{C}$, and the group $H_1(\mathbb{C}^* - \{z_n(w)\}_{n\geq 1}, \mathbb{Z})$ is generated by the cycle $|z| = 1$ and small circles about the poles. One can show that $\int_{|z-z_n(w)|\ll 1} \frac{1}{w^{-1} - \frac{e^z}{z}} \frac{dz}{z} = 1 + \frac{1}{z_n(w)}$. The generic fiber of the homology bundle carries two permutations: one is a permutation of two elements, while the other pne is an infinite cyclic shift. Making the analogy with the CS functional we see that the cycle about 0 corresponds to the trivial local system $\rho = 1$, while other cycles give residues and correspond to rigid local systems $\rho \neq 1$.

THANK YOU!