PART II. Deligne–Lusztig theory for finite reductive groups

Chapter 6. $\mathbb{F}_q$-structures on alg. groups

We fix an algebraic group G.
(linear)

6.A. Definition

An $\mathbb{F}_q$-structure on G is the datum of a Frobenius endomorphism $F : G \to G$ over $\mathbb{F}_q$ which is a morphism of groups.

We set $G^F = \{ g \in G \mid F(g) = g \}$

Examples. (0) If G is finite, this is just the datum of an automorphism of G.

(1) $F : \mathbb{F}^+ \longrightarrow \mathbb{F}^+$, $x \longmapsto x^q$ $\Rightarrow$ $(\mathbb{F}^+)^F = \mathbb{F}_q^+$

(2) $F : \mathbb{F}^\times \longrightarrow \mathbb{F}^\times$, $x \longmapsto x^q$ $\Rightarrow$ $(\mathbb{F}^\times)^F = \mathbb{F}_q^\times$

(3) $F : GL_n(\mathbb{F}) \longrightarrow GL_n(\mathbb{F})$, $(a_{ij}) \longmapsto (a_{ij}^q)$

$GL_n(\mathbb{F})^F = GL_n(\mathbb{F}_q)$

$F_u : GL_n(\mathbb{F}) \longrightarrow GL_n(\mathbb{F})$, $(a_{ij}) \longmapsto {}^t(a_{ij}^q)^{-1}$

Then $F_u^2 = F^2$ so F_u is also a Frob. endo/$\mathbb{F}_q$.

and

$GL_n(\mathbb{F})^{F_u} = U_n(\mathbb{F}_{q^2})$

$= \{ g \in GL_n(\mathbb{F}_{q^2}) \mid g \cdot {}^tF(g) = I_n \}$

(unitary group)

(4) Let F be a Frob. endo. of G over $\mathbb{F}_q$.

Let
$$F' : G^n \longrightarrow G^n$$
$$(g_1, \dots, g_n) \longmapsto (F(g_n), F(g_1), \dots, F(g_{n-1}))$$

Then $F'^n = F^n$ so F' is a Frob. endo. $/\mathbb{F}_q$ of G^n and
$$(G^n)^{F'} \xleftarrow{\sim} G^{F^n}$$
$$(g, F(g), \dots, F^{n-1}(g)) \longleftarrow g$$

(5) If $g \in G$, then
$$gF : G \longrightarrow G$$
$$h \longmapsto gF(h)g^{-1}$$
is a Frob. endo. $/\mathbb{F}_q$ (exercise).

6.B. Lang's Theorem.

We fix a Frob. endo. $F : G \to G$ $/\mathbb{F}_q$.

F-conjugacy. The formula
$$g \cdot_F h = g\, h\, F(g)^{-1}$$
defines an action of G on itself (F-conjugacy). We denote by $H^1(F, G)$ the set of F-conjugacy classes.

Theorem 6.1 (Lang). If G is connected, then $H^1(F, G) = \{G\}$. In other words, the Lang map $\mathcal{L} : G \to G$, $g \mapsto g^{-1}F(g)$ is surjective.

Proof. If $a \in G$, then $G_{a, \cdot_F} = G^{aF}$ is finite. So the orbit $\Omega(a)$ of a has dimension $\dim G$. Since G is irreducible, this means that $\overline{\Omega(a)} = G$ so $\Omega(a)$ is open in G (by 5.13).

Therefore $\Omega(a) \cap \Omega(1) \neq \emptyset$. ∎

Corollary 6.2. $H^1(F, G) = H^1(F, G/G^\circ)$

Proof. The natural map is onto. So let $g, h \in G$ be such that be such that there exists $a \in G$ and $g_\circ \in G^\circ$ such that $a\, h\, F(a)^{-1} = g_\circ g$.

By Lang Theorem applied to gF, there exists $b \in G^\circ$ such that
$$b^{-1} g F(b) g^{-1} = g_\circ$$
$\Rightarrow a\, h\, F(a)^{-1} = b^{-1} g\, F(b)$. ∎

Remark. Assume that G is finite: $H^1(F, G) = \{G\}$
$\Leftrightarrow \mathcal{L}$ is surjective $\Leftrightarrow \mathcal{L}$ is injective
$\Leftrightarrow G^F = \{1\} \overset{\text{Thompson}}{\Longrightarrow} G$ is nilpotent. ∎

6.C. Consequences. Let X be a variety and $F: X \longrightarrow X$ be a Frob. endo. $/\mathbb{F}_q$ and assume that G acts on X and that is action is defined over $\mathbb{F}_q$ ($F(g\cdot x) = F(g)\cdot F(x)$).

Proposition 6.3. Let $\mathcal{O}$ be an F-stable G-orbit and assume that G is connected.

(a) $\mathcal{O}^F \neq \emptyset$.

(b) Let $x_0 \in \mathcal{O}^F$. Then the map

$$\begin{array}{ccc} \mathcal{O}^F/G^F & \longrightarrow & H^1(F, G_{x_0}/G_{x_0}^\circ) \\ G^F\text{-orbit of } g\cdot x_0 & \longmapsto & F\text{-class of } g^{-1}F(g) \end{array}$$

is well-defined and bijective.

Proof. (a) Let $x \in \mathcal{O}$. Then $\exists\, g \in G$ s.t. $F(x) = g\cdot x$. By Lang Theorem, we can write $g^{-1} = a^{-1}F(a)$. Then

$$F(a\cdot x) = F(a)\cdot F(x) = (a g^{-1})\cdot(g\cdot x) = a\cdot x$$

So $a\cdot x \in \mathcal{O}^F$.

(b) $F(g\cdot x_0) = g\cdot x_0 \iff g^{-1}F(g) \in G_{x_0}$.

If $g\cdot x_0 = h\cdot x_0$, then $u = g^{-1}h \in G_{x_0}$

and
$$\begin{aligned} g^{-1}F(g) &= (g^{-1}h)\, h^{-1}F(h)\, F(h^{-1}g) \\ &= u\, h^{-1}F(h)\, F(u)^{-1} \end{aligned}$$

So $g^{-1}F(g) \sim_F h^{-1}F(h)$ in G_{x_0}.

If $\alpha \in G^F$ then $(\alpha g)^{-1}F(\alpha g) = g^{-1}F(g)$

so $\alpha g\cdot x_0$ and $g\cdot x_0$ have the same image.

This proves that the map is well-defined.

Bijectivity is easy (exercise). ∎

Flexibility. Let $g \in G^\circ$. Then the pairs (G, F) and (G, gF) are isomorphic.

Indeed, write $g = a^{-1}F(a)$ and $F' = gF$.

Then the map
$$\begin{array}{cccc} c_a: & G & \xrightarrow{\sim} & G \\ & h & \longmapsto & aha^{-1} \end{array}$$

satisfies
$$c_a(F'(h)) = F(c_a(h))$$

In particular
$$G^{gF} \underset{c_a}{\xrightarrow{\sim}} G^F \qquad (g \in G^\circ\ !!!).\ ∎$$

<u>Example 6.4.</u> If H is an F-stable connected closed subgroup of G, then
$$(G/H)^F \xrightarrow{\sim} G^F/H^F . \blacksquare$$

<u>Galois unramified covering.</u>

The Lang map $\mathcal{L} : G \longrightarrow G$ is an orbit map for the the action of G^F on G by left translation: $\mathcal{L}(g) = \mathcal{L}(h) \Leftrightarrow g h^{-1} \in G^F$.

But $d_1 \mathcal{L} : T_1(G) \longrightarrow T_1(G)$
$$x \longmapsto -x$$

$\underset{(4.5)}{\Rightarrow}$ $G^F \backslash G \xrightarrow[\mathcal{L}]{\sim} G$ (!!)

Since G^F acts freely, it is a Galois unramified covering of G.

6.D. Finite reductive groups (finite groups of Lie type)

Let G be a connected reductive group endowed with a Frob. endo. F / $\mathbb{F}_q$.

Let B' be a Borel subgroup. The set of Borel subgroups $\leftrightarrow G/N_G(B') = G/B'$
(5.20)

F(Borel subgroup) is a Borel subgroup.

The variety of Borel subgroups (flag variety) is defined over $\mathbb{F}_q$.

By Prop. 6.3(a) $\Rightarrow \exists$ F-stable Borel subgroup B_0.

Similarly, there exists an F-stable maximal torus $T_0 \subset B_0$.

<u>Exercise.</u> The pair (T_0, B_0) is unique up to G^F-conjugacy. $\blacksquare$

Let $W = N_G(T_0)/T_0$ be the Weyl group and $S = \{ s \in W \mid \dim B_0 \dot{s} B_0 - \dim B_0 = 1 \}$.

- F-stable Borel subgroup B_0 $\cup$
- F-stable maximal torus T_0
- $W = N_G(T_0)/T_0$ is acted on by F
- S : $F(S) = S$.

If $w \in W^F$, we choose a representative $\dot{w}$ of w in $N_G(T_0)^F$ (example 6.4).

<u>Proposition 6.5</u> (Bruhat decomposition).

$$G^F = \dot{\bigcup_{w \in W^F}} B_0^F \dot{w} B_0^F$$

<u>Proof</u>. Let $g \in G^F \subset G$. So there exists $w \in W$ s.t. $g \in B_0 \dot{w} B_0$

But $F(g) = g \in F(B_0 \dot{w} B_0) = B_0 F(\dot{w}) B_0$

$\Rightarrow F(w) = w$ (5.24(b)).

$\Rightarrow w \in W^F$.

Let $\mathcal{O}(\dot{w}) = B_0 \dot{w} B_0$: it is an F-stable $(B_0 \times B_0)$-orbit. So

$$\underset{\substack{\Psi \\ g}}{B_0^F \backslash \mathcal{O}(\dot{w})^F / B_0^F} \underset{6.3(b)}{\simeq} \underset{\substack{\| \\ \text{singleton}}}{H^1(F,} \underbrace{B_0 \cap {}^{\dot{w}}B_0}_{\text{connected}})$$

■

<u>Example 6.6</u>. Take $G_n = GL_n(\mathbb{F})$ and

$F : (a_{ij}) \longmapsto (a_{ij}^q)$.

Take $B_0 = \mathbb{B}_n$ is F-stable

$T_0 = \mathbb{T}_n$ is F-stable

Now, let $F_u : (a_{ij}) \longmapsto {}^t(a_{ij}^q)^{-1}$

Then $\mathbb{T}_n$ is F_u-stable

BUT $\mathbb{B}_n$ is not (!).

Using the "flexibility", let

$$w_0 = \begin{pmatrix} 0 & & 1 \\ & \diagup & \\ 1 & & 0 \end{pmatrix} \in G_n = G_n^\circ$$

so we can replace F_u by $F' = w_0 F_u$

$(G_n^{F'} \simeq G_n^{F_u})$. Then

$\mathbb{B}_n$ and $\mathbb{T}_n$ are F'-stable.

$$\begin{cases} F \text{ acts trivially on } W = S_n \\ F' \text{ acts on } W \text{ by conjugacy by } w_0. \end{cases}$$

$(w_0 s_i w_0^{-1} = s_{n-i})$. ■

F-stable parabolic subgroups.

If $I \subset S$, then $P_I = \bigcup_{w \in W_I} B_0 w B_0$
$(W_I = \langle I \rangle)$ is a parabolic subgroup.

It is F-stable $\Leftrightarrow$ $F(I) = I$.

Proposition 6.6. $(P_I)_{\substack{I \subset S \\ F(I) = I}}$ is a set of representatives of G^F-conjugacy classes of F-stable parabolic subgroups.

Chapter 7. Harish-Chandra theory.

We fix a connected reductive group G endowed with a Frob. endo. F / $\mathbb{F}_q$.

KG^F-mod?

7.A. HC induction and restriction.

Let P be an F-stable parabolic subgroup:
$\Rightarrow$ P contains an F-stable max. torus T
$\Rightarrow$ so the unique Levi complement L of P containing T is F-stable
Write $V = R_u(P)$

So $P = L \ltimes V \Rightarrow P^F = L^F \ltimes V^F$

So $K[G^F/V^F]$ is a (KG^F, KL^F)-bimodule

$K[G^F/V^F]^* \simeq K[V^F \backslash G^F]$ as a (KL^F, KG^F)-bimodule

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$$R^G_{L\subset P} : KL^F\text{-mod} \longrightarrow KG^F\text{-mod}$$
$$M \longmapsto K[G^F/V^F] \otimes_{KL^F} M = \mathrm{Ind}_{P^F}^{G^F} \tilde{M}$$
$\tilde{M}$: V^F acts trivially

(HC-induction)

$${}^*R^G_{L\subset P} : KG^F\text{-mod} \longrightarrow KL^F\text{-mod}$$
$$M \longmapsto K[V^F\backslash G^F] \otimes_{KG^F} M = M^{V^F}.$$

(HC restriction)

Proposition 7.1 (transitivity). Let P' be an F-stable parabolic subgroup of L and let L' be an F-stable Levi complement. We set $V' = R_u(P')$.

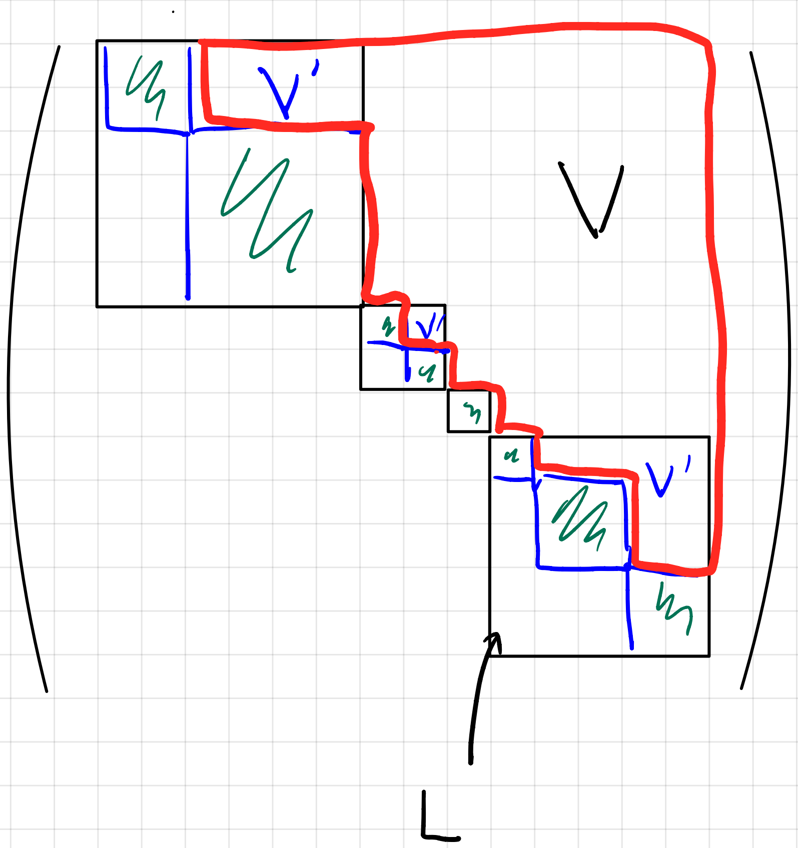
(a) $P' \ltimes V$ is an F-stable parabolic subgroup of G with Levi complement L' and unipotent radical $V'V$.

(b) $R^G_{L\subset P} \circ R^L_{L'\subset P'} = R^G_{L'\subset P'V}$

${}^*R^L_{L'\subset P'} \circ {}^*R^G_{L\subset P} = {}^*R^G_{L'\subset P'V}$

Proof. $K[G^F/V^F] \otimes_{KL^F} K[L^F/V'^F] \xrightarrow{\sim} K[G^F/V'^F V^F]$
$$gV^F \otimes lV'^F \longmapsto gl\, V'^F V^F. \quad \blacksquare$$

Example. $G = GL_n(\mathbb{F})$



$L' =$ 

$V'V =$