1. Introduction

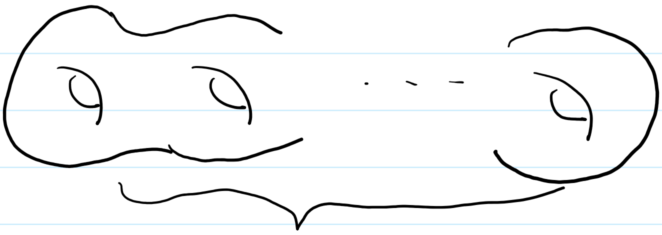
Reference

- [BP] Benedetti-Petronio, "Lectures on hyperbolic geometry"
- [T] Thurston, "Three-Dimensional geometry and topology, Volume 1".
- [A] Abikoff, "The real analytic theory of Teichmüller space"
- [IT] Imayoshi, Taniguchi, "An introduction to Teichmüller spaces."
- [FO09] Farrell, Ontaneda, "The Teich. space of pinched neg. curved metrics on a hyp. mfd is not contractible, 2009.
- [H88] Hamilton, "The Ricci flow on surfaces", 1988
- [TW] Tuschman-Wraith, "Moduli space of Riemannian metrics".
- [E70] Ebin, "The manifold of Riemannian metrics", 1970

1.1 Teichmüller space (classical case)

M_g: closed, smooth, orientable surface of genus g

$M_g \cong$ diffeomorphic



g "holes"

When $g>1$, M_g admits a hyperbolic (hyp.) metric, i.e. a Riemmanian metric of constant sectional curvature -1
(cf. [Prop. B. 3.1, BP], [T])

Q: "How many" hyp. metrics are there on M_g?

$\mathcal{H}_g \triangleq \{$all hyp. metrics on $M_g\}$

$\mathrm{Diff}(M_g) \triangleq \{f: M_g \xrightarrow[\text{diffeom}]{\cong} M_g\}$

$\underset{f}{\mathrm{Diff}(M_g)} \curvearrowright \underset{h}{\mathcal{H}_g} \quad : \quad f \cdot h = (f^{-1})^* h$

Def: For $g \geq 2$, moduli set

$\mathcal{M}_g \triangleq \mathcal{H}_g / \mathrm{Diff}(M_g) \longrightarrow$ set of orbits

$\mathcal{H}_g$, $\mathrm{Diff}(M_g)$ with C^∞-topology introduce later

moduli space : $\mathcal{M}_g$ with quotient top.

$\downarrow$ difficult

Require: a preliminary study of Teichmüller space (Teich. sp.)

Def: For $g \geq 2$. Teich. sp.

$\mathcal{T} \triangleq \mathcal{H}_g /$

$$\mathcal{T}_g \triangleq \mathcal{H}_g / \mathrm{Diff}_0(M_g)$$

$$\mathrm{Diff}_0(M_g) \triangleq \{ f : M_g \xrightarrow{\cong} M_g \mid f \overset{\text{homotopic}}{\sim} \mathrm{id} \} \triangleleft \mathrm{Diff}(M_g).$$

<u>Th 1.1</u> $\mathcal{T}_g \underset{\text{homeomorphic}}{\approx} \mathbb{R}^{6g-6}$ for $g \geq 2$

(cf. [Th. B.4.21., BP], [T])

<u>Rk</u>:

1. $\mathrm{Mod}_g \triangleq \mathrm{Diff}(M_g) / \mathrm{Diff}_0(M_g)$.

   $\Rightarrow \mathcal{M}_g = \mathcal{T}_g / \mathrm{Mod}_g$.

   $\mathrm{Mod}_g \curvearrowright \mathcal{T}_g$ not free

   $\Rightarrow \mathcal{M}_g$ has singularites.

2. Other def. for $\mathcal{T}_g$

   cf. [A]
   [IT].

1.2 Teichmüller space for manifold (mfd.) of dimension (dim.) ≥ 3

We call M a <u>hyp. mfd</u> if M is a smooth, closed mfd of dim ≥ 3 admitting a hyp. metric

Def: Standard Teich. space for M:

$$T^H(M) \triangleq H(M) / \mathrm{Diff}_0(M)$$

$H(M) \triangleq \{$all hyp. metrics on $M\}$ with C^∞-top.

Th 1.2 (Mostow's Rigidity) Let M_i, $i=1,2$ be smooth mfd of dim ≥ 3 with complete finite volume hyp. metric h_i,

$$f: (M_1, h_1) \xrightarrow{\simeq} (M_2, h_2)$$
homotopy equivalence

$\Rightarrow$ $f \sim$ an isometry: $(M_1, h_1) \longrightarrow (M_2, h_2)$

$\Downarrow$

Cor. $T^H(M)$ is a pt.

Pf: h_1, h_2 : hyp. metrics on M

Th 1.2 $\Rightarrow$ $id: (M, h_1) \longrightarrow (M, h_2)$
$\wr$
an isometry $i: (M, h_1) \longrightarrow (M, h_2)$

$\Rightarrow$ $i \in \mathrm{Diff}_0(M)$ and $(i^{-1})^* h_1 = h_2$

$\Rightarrow$ $[h_1] = [h_2] \in T^H(M)$. $\square$

[FO09] gives another def. for Teich. space based on

Th 1.3 $g \geq 2$, $\mathcal{T}_g = H_g / \mathrm{Diff}_0(M_g)$ is a deformation retraction of

$$T^{<0}(M_g) \triangleq R^{<0}(M_g) / \mathrm{Diff}_0(M_g)$$

where

$R^{<0}(M_g) \triangleq \{$negatively (neg.) curved metrics on $M_g\}$
with C^∞-top.

complete Riemannian metrics of neg. sectional curvature.

Rk: $\mathcal{H}_g \subset R^{<0}(M_g)$

sketch pf of Th 1.3:
Hamilton's Ricci flow [H88] shows

- Every neg. curved metric on M_g for $g \geq 2$ can be canonically deformed (through neg. curved metrics) to a hyp. metric.

  $\Rightarrow$ $\mathcal{H}_g$ is canonically a def. ret. of $R^{<0}(M_g)$

- This deformation commutes with the action of $\mathrm{Diff}_0(M_g)$.

$\square$

Def ([FO09]): Teich. space (of neg. curved metrics) for M:

$$T^{<0}(M) \triangleq R^{<0}(M) / \mathrm{Diff}_0(M)$$

where M is a neg. curved mfd,

i.e., a smooth, closed mfd admitting neg. curved metric

Q: Is $T^{<0}(M)$ contractible?

Rk: $T^{<0}(M_g) \approx \mathbb{R}^{6g-6}$ is contractible.

Th 1.4 ([FO09]) $T^{<0}(M)$ is not contractible

Th 1.4 ([FO09]) $\mathcal{T}^{<0}(M)$ is not contractible for many hyp. mfd M.

Def Moduli space (of neg. curved metrics) for M:

$$\mathcal{M}^{<0}(M) \triangleq \mathcal{R}^{<0}(M) / \mathrm{Diff}(M)$$

Rk: $\mathcal{M}^{<0}(M) = \mathcal{T}^{<0}(M) \Big/ \mathrm{Diff}(M) / \mathrm{Diff}_0(M)$

Rk: $\mathcal{T}^{<0}(M)$ is the classifying space of "neg. curved bundles with homotopy trivializations" (introduce later)

This course will concentrate on

- $\pi_* \mathcal{T}^{<0}(M) = ?$ $\quad \pi_* \mathcal{T}^{<0}(M) \otimes \mathbb{Q} = ?$
- neg. curved bundles

1.3 About C^∞-top.

M, N: smooth mfd

$$C^\infty(M, N) \triangleq \{\text{all smooth maps } f: M \to N\}.$$

Endow $C^\infty(M, N)$ with C^∞-top.
(C^∞ compact-open top. or the weak top)

Roughly speaking, in the weak top. a neighborhood of f is $\{g \in C^\infty(M,N) \mid D^k g \text{ is "close" to } D^k f \text{ on a compact set} \subseteq M \text{ for all } k \geq 0\}$

Rk: In the strong top., a neighborhood of f is $\{g \in C^\infty(M,N) \mid D^k g$ is "close" to $D^k g$ on

(M) for all $k \geq 0$ }

cf. [GTM 33, chap. 2].

$$C^\infty(M, M) \text{ with } C^\infty\text{-top.}$$
$$\cup$$
$$\mathrm{Diff}_0(M) \subset \mathrm{Diff}(M) \quad \text{— subspace top.}$$

$S^2T^*M \triangleq$ the second symmetric power of T^*M (cotangent bundle of M)

endow $C^\infty(M, S^2T^*M)$ with C^∞-top.

$$\cup$$
$$R(M) \triangleq \{\text{all complete Riem. metrics on } M\}$$
$$\cup$$
$$R^{<0}(M)$$
$$\cup$$
$$H(M)$$

subspace top.

Rk: We work with the weak top. instead of the strong top. because when M is not compact, $R(M)$ is contractible in the weak top. but has infinitely many path component in the strong top. (cf. [P.5, TW])

2. $T(M)$, $T^{<0}(M)$ as classifying spaces

2.1 $T(M)$

M: smooth, closed mfd

Def: $T(M) \triangleq R(M)/\mathrm{Diff}_0(M)$ with C^∞-top.

Th 2.1. If M is a neg. curved mfd, then $T(M)$ is the classifying space for smooth M-bundles with a fiber homotopy trivialization i.e. there is a bijection

$$[B, T(M)] \longrightarrow \{\text{equivalence classes of smooth } M\text{-bundles over } B \text{ with fiber homotopy trivializations}\}$$

$[B,T(M)] \cong$ {homotopy classes of continuous (cont.) maps $B \longrightarrow T(M)$}

provided that B is a paracompact top. space.

<u>Cor 2.2</u> If M is a neg. curved mfd, then $\exists$ a bijection

$$[B, T(M)] \longrightarrow \{\text{equivalence classes of smooth } M\text{-bundles over } B\}$$

provided that B is a simply-connected top. space $\simeq$ a finite simplicial complex.

M: smooth, closed mfd

Def: Let $\pi: E \to B$ be a cont. map with $\pi^{-1}(b)$ a smooth mfd for $\forall b \in B$. The triple (E, B, π) defines a <u>smooth M-bundle</u> if (1) (2) are satisfied:

(1) locally triviality: $\exists$ open cover $\mathcal{U} = \{U_i\}_{i \in I}$ for B s.t. for each $i \in I$, $\exists \varphi_i : U_i \times M \xrightarrow{\approx} \pi^{-1}[U_i]$ s.t.

$$\begin{tikzcd}
U_i \times M \arrow[rr, "\approx", "\varphi_i"'] \arrow[dr] & & \pi^{-1}[U_i] \arrow[dl] \\
& U_i \arrow[u, phantom, "\circlearrowleft"] &
\end{tikzcd}$$

$(x, m) \mapsto x$

commutes

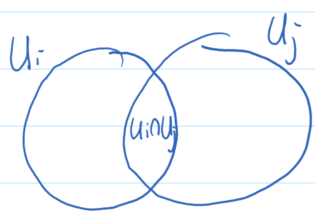
notation: $\varphi_{ib}: M \longrightarrow b \times M \xrightarrow{\varphi_i(b,\cdot)} \pi^{-1}(b)$

$m \longmapsto (b, m)$

(2) consistency: For any $U_i, U_j \in \mathcal{U}$ and each $b \in U_i \cap U_j$,

$\varphi_{ij}(b): M \xrightarrow[\varphi_{jb}]{} \pi^{-1}(b) \xrightarrow[\varphi_{ib}^{-1}]{} M$ is $\cong$ diffeom.

and $\varphi_{ij}: U_i \cap U_j \longrightarrow \mathrm{Diff}(M)$ is cont.



In this case, φ_{ij} is called a <u>transition</u>

function and $\mathrm{Diff}(M)$ is the structure group for the bundle ξ

$E_i \xrightarrow{\pi_i} B$, $i = 0, 1$: smooth M-bundles are equivalent ($\equiv$) if $\exists f : E_0 \xrightarrow{\cong} E_1$ fiberwise diffeom.

$$\begin{tikzcd} E_0 \arrow[rr, "\cong"] \arrow[dr, "\pi_0"'] & \circlearrowleft & E_1 \arrow[dl, "\pi_1"] \\ & B & \end{tikzcd}$$

Fact:

fiberwise diffeom.

$$\begin{tikzcd} E \arrow[rr, "\cong"] \arrow[dr, "\pi"'] & \circlearrowleft & \bigsqcup_{i\in I} U_i \times M / \sim \arrow[dl] \\ & B = \bigcup_{i \in I} U_i & \end{tikzcd}$$

$\varphi_{ij}(b) : M \xrightarrow{\varphi_{jb}} \pi^{-1}(b) \xrightarrow{\varphi_{ib}^{-1}} M$

$\forall U_i, U_j \in \mathcal{U}$, $(b, x) \in (U_i \cap U_j) \times M$

$U_j \times M \ni (b, x) \sim (b, \varphi_{ij}(b)(x)) \in U_i \times M$

Construct a Principal $\mathrm{Diff}(M)$-bundle out of the smooth M-bundle:

$$P = \bigsqcup_{i \in I} U_i \times \mathrm{Diff}(M) / \sim$$

$\pi_0 \downarrow$ → the projection to the 1st factor

$$B = \bigcup_{i \in I} U_i$$

$\forall U_i, U_j \in \mathcal{U}$, $(b, f) \in (U_i \cap U_j) \times \mathrm{Diff}(M)$

$U_j \times \mathrm{Diff}(M) \ni (b, f) \sim (b, \varphi_{ij}(b) \circ f) \in U_i \times \mathrm{Diff}(M)$

$P \overset{\text{free}}{\curvearrowleft} \mathrm{Diff}(M)$: $(b, f) \in U_i \times \mathrm{Diff}(M)$

$$P \times \mathrm{Diff}(M) \longrightarrow P$$
$$([b, f], g) \longmapsto [b, f \circ g]$$

The orbit map of this action is $P \xrightarrow{\pi_0} B$ exactly.

$P \longrightarrow B$ is the principal $\mathrm{Diff}(M)$-bundle associated to the smooth M-bundle $\xi : E \to B$

Both of the bundles have the same transition functions

Both of the bundles have the same transition functions

Fact: $E \xrightarrow{\cong}$ $\not\sim$

$\searrow$

$\forall (e,x), (e',x') \in P \times M$:

$(e,x) \sim (e',x') \Leftrightarrow \exists g \in \mathrm{Diff}(M)$ s.t. $e' = eg^{-1}$, $x' = gx$.

i.e. $P \times_{\mathrm{Diff}(M)} M$ is the orbit space of

$\mathrm{Diff}(M) \curvearrowright P \times M$:

$$\mathrm{Diff}(M) \times (P \times M) \longrightarrow P \times M$$
$$(g, (e,x)) \longmapsto (eg^{-1}, gx).$$

called Borel construction / balance product.

Def A <u>fiber homotopy trivialization</u> for a smooth M-bundle E over B is a cont. map $f: E \to B \times M$ s.t.

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr] & \arrow[d, phantom, "\circlearrowleft"] & B \times M \arrow[dl] \\ & B & \end{tikzcd}$$

and for each $b \in B$, $f|_{\pi^{-1}(b)}: \pi^{-1}(b) \xrightarrow{\simeq} b \times M$

$(E_0, f_0), (E_1, f_1)$: smooth M-bundles over B with fiber homotopy trivializations

$$f_i : E_i \longrightarrow B \times M, \quad i = 0, 1$$

$(E_0, f_0), (E_1, f_1)$ are $\equiv$ if

$$\exists\ f: \begin{tikzcd} E_0 \arrow[rr, "\cong"] \arrow[dr] & & E_1 \arrow[dl] \\ & B & \end{tikzcd}$$ fiberwize diffeom.

s.t. $$\begin{tikzcd} E_0 \arrow[dr, "f_0"] \arrow[d, "f"'] & \\ E_1 \arrow[r, "f_1"'] & B \times M \end{tikzcd}$$ commutes up to homotopy

(i.e. $f_1 \circ f \simeq f_0$)

Pf Th2.1:

$\Uparrow$

$\Uparrow$

Lemma 2.3, 2.4 $\square$

<u>Lemma 2.3</u>: If M is a neg. curved mfd, then $\exists$ bijection

$$[B, T(M)] \longrightarrow \{\equiv \text{ classes of principal } \mathrm{Diff}_0(M)\text{-bundles over } B\}$$

for a paracompact top. space B.

<u>Lemma 2.4</u>: If M is a neg. curved mfd, then $\exists$ bijection

$$\{\equiv \text{ classes of principal } \mathrm{Diff}_0(M)\text{-bundles over } B\}$$
$$\longrightarrow \{\equiv \text{ classes of smooth } M\text{-bundles over } B \text{ with fiber homotopy trivialization}\}$$

for a paracompact top. space B.

<u>Pf Lemma 2.3</u>:

Recall

<u>Th</u> Let G be a top. group (grp), $\eta: P \to BG$ be a principal G-bundle and P is contractible. Then $\exists$ bijection

$$\begin{aligned} [B, BG] &\longrightarrow \{\equiv \text{ classes of principal } G\text{-bundles over } B\} \\ f &\longmapsto f^*\eta \end{aligned}$$

for a paracompact top. space B.

BG is called the <u>classifying space</u> for pricipal G-bundles, unique up to homotopy

So to prove Lemma 2.3, it suffices to show

$$\boxed{\begin{gathered} R(M) \longrightarrow R(M)/\mathrm{Diff}_0(M) = T(M) \\ \text{is a } \underline{\text{principal}} \ \mathrm{Diff}_0(M)\text{-bundle} \end{gathered}}$$

is a principal $\mathrm{Diff}_0(M)$-bundle
and $R(M)$ is contractible. ✓

(1) $\mathrm{Diff}_0(M) \curvearrowright R(M)$ free

(2) $R(M) \xrightarrow{\pi} T(M)$ has local cross-sections,
i.e. $\forall x \in T(M)$, $\exists\, x \in U \underset{open}{\subseteq} T(M)$ and
cont. map $s: U \longrightarrow R(M)$
s.t. $\pi \circ s = id_U$.

pf (1): It follows from Lemma 2.5, 2.6 $\square$

Lemma 2.5 If M is a smooth aspherical mfd and $\mathrm{Center}(\pi_1 M) = \{1\}$,
then $\mathrm{Diff}_0(M) \curvearrowright R(M)$ free.

Rk: M is a aspherical mfd if M is a closed mfd and $\pi_i M = 0$ for all $i \neq 1$.

Lemma 2.6 If M is a neg. curved mfd,
then M is aspherical and $\mathrm{Center}(\pi_1 M) = \{1\}$.

pf Lemma 2.5: For path-conn. space X,

$\mathrm{Top}(X) \triangleq \{ f: X \xrightarrow{\approx} X \}$
with compact-open top.

$\downarrow$
top. grp

$\forall f \in \mathrm{Top}(X) \rightsquigarrow f_*: \pi_1(X, x_0) \longrightarrow \pi_1(X, f(x_0)) \triangleq \pi_1(X, x_0)$ not canonical

~~$\rightsquigarrow$~~ $\mathrm{Top}(X) \longrightarrow \underbrace{\mathrm{Aut}(\pi_1(X, x_0))}$ automorphism grp

$\rightsquigarrow$ (ex.) $\Psi_X: \mathrm{Top}(X) \longrightarrow \underbrace{\mathrm{Out}(\pi_1(X, x_0))}$ outer automorphism grp

$\mathrm{Out}(\pi_1(X, x_0)) = \mathrm{Aut}(\pi_1(X, x_0))/$

(ex.) Φ_X : $\mathrm{Top}(X) \longrightarrow$ Out ...

$\mathrm{Aut}(\pi_1(X, x_0)) / \mathrm{Inn}(\pi_1(X, x_0))$ grp

write as $\mathrm{Out}(\pi_1 X)$

Th (Borel's Th) M : aspherical mfd with $\mathrm{Center}(\pi_1 M) = \{1\}$
G : compact lie grp $\underset{\text{closed}}{\subset} \mathrm{Top}(M)$

$\Rightarrow$

$$\Phi_M|_G : G \longrightarrow \mathrm{Out}(\pi_1 M) \text{ is monic.}$$

Rk: This th. is proved by Borel, extended to non-orientable case by Conner-Raymond.