Plan of talk

1. Hamiltonian reduction in math and physics (classical theory)
2. Hamiltonian reduction in phys. (quantum theory - examples)
3. CS theory on $\Sigma \times S^1$ in $d=3$ as a Quantum Hamiltonian reduction → → exact answers in QFT!

We study CS theory since 3d gravity is a close analogue to CS theory, moreover 3d gravity with cosm. constant is a pair of 2 CS theories! → we will see it later

1. Hamiltonian reduction

Math. origin: Example:

consider $\mathbb{C}P^1 = \dfrac{\mathbb{C}^2 \setminus 0}{\mathbb{C}^*}$

Remark: we cannot study just $\mathbb{C}^2/\mathbb{C}^*$, really it has two types of orbits

— $\mathbb{C}^*$

↑ - bad orbit - close to all orbits - we have to eliminate it. In general, it is studied in the theory of stable quotients

a) classify orbits

b) Eliminate bad orbits

There is another way to study $\mathbb{C}^2 \setminus 0 / \mathbb{C}^*$

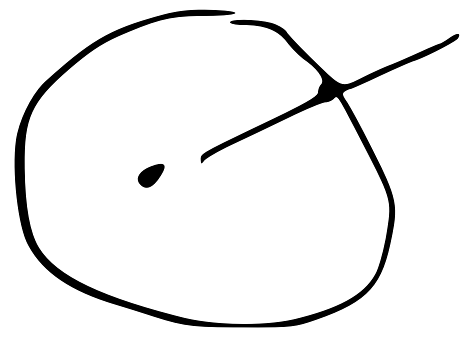
a) Decompose $\mathbb{C}^*$ as $U(1) \times \mathbb{R}_+$

there is no problem in taking quotient w. respect to $U(1)$ - since it is compact

problem is in $\mathbb{R}_+$ - noncompact

Idea (how to eliminate bad orbits):

Impose condition $|z_1|^2+|z_2|^2 = R^2$ (E)
Then good orbits intersect (E) only once, while bad orbits do not intersect at all.



Consider U(1) action:

$$z_a \to e^{i\varphi} z_a, \quad a = 1,2;$$

consider simplectic structure $\sum_a dz_a \wedge d\bar{z}_a$ i

Rotation $z_a \to e^{i\varphi} z_a$ preserve simplectic structure, so they are simpl. vector fields, and have Hamiltonians.

Inter., H for U(1) action is just

$$|z_1|^2 + |z_2|^2 + \text{const}$$

cannot be determined from the vector field $\sum_a z_a \frac{\partial}{\partial z_a} - \bar{z}_a \frac{\partial}{\partial \bar{z}_a}$. So, good orbits correspond to putting Ham. to zero, and then taking quobient by U(2)

This procedure is known as hamiltonian reduction.

Theorem – good orbits intersect surface of $H=0$ only once.

prove it in general setting: $X, \omega,$

be kahler form for metric g

(we also assume that X is complex)

Hamiltonian flow preserves hamiltonian, but we will consider another flow:

Ham. flow is $v^a(z)$, $\bar{v}^a(\bar{z})$ (in example they were z and $-\bar{z}$)

But we will consider modified flow due to noncompact part of the group $(v^a(z), -\bar{v}^{\bar{a}}(\bar{z}))$ - do not preserve Hamiltonian

$$\frac{d}{dt}\Big| H = \frac{\partial H}{\partial z^a} v^a - \frac{\partial H}{\partial \bar{z}^{\bar{a}}} \bar{v}^{\bar{a}} =$$

due to modified flow.

$$= \frac{\partial H}{\partial z^a}\frac{\partial H}{\partial \bar{z}^{\bar{a}}} g^{a\bar{a}} + \frac{\partial H}{\partial \bar{z}^{\bar{a}}}\frac{\partial H}{\partial z^a} g^{a\bar{a}}$$

relative sign is +

note that we have $g^{a\bar{a}}$ → related to

$g_{a\bar{a}} = \omega_{a\bar{a}}$ $\quad \frac{dH}{dt}\Big|_{\text{mod. flow}} > 0$ →

→ so we have only one intersection.

General case - consider complex space X, consider a complex group G acting on X. Let X be Kahler such that G_c preserve ω.

↑ compact part of the complex group G

Then it clear that there is a map (Hamiltonians)

$\mathcal{Y}_c \longrightarrow$ functions on X

↑ moment map

$\mathbb{C}^n$, there is $GL(n, \mathbb{C})$ acting on $\mathbb{C}^n$.. Compact form of $GL(n, \mathbb{C})$ $U(n)$, preserving metric $dz^a \otimes d\bar{z}^{\bar{a}} + d\bar{z}^{\bar{a}} \otimes dz^a$ and kahler form $dz^a \wedge d\bar{z}^{\bar{a}}$

$$v_{ab}(z) = z^a \frac{\partial}{\partial z^b} - z^b \frac{\partial}{\partial z^a} \quad \text{and also}$$

$$v(z) = z^a \frac{\partial}{\partial z^a} \text{ - for } U(1) \text{ part.}$$

$H_{ab} = z^a \bar{z}^b - z^b \bar{z}^a$

You may see that for $n=3$ they are just rotational moments

$M_i = \epsilon_{ijk} x^i p^k$ - it is the same.

for each $i = 1,2,3$ we have Hamiltonians M_i (they are conv. called moments), and a map

$\mathcal{G}_c \to Fun(X)$ is called a moment map.

$$\{M_A, M_B\} = f_{AB}^C M_C \quad (M)$$

where A is an index in $\mathcal{G}_c$.

Hamiltonian reduction is taking $M_A = 0$ and then taking a quotient w.r.t. G_c.

Another important example of H.R. construction - toric manifolds

$\mathbb{C}^n$ and consider action of $(\mathbb{C}^*)^m$ $m<n$ on $\mathbb{C}^n$

In some coord. system on $\mathbb{C}^n$ we have hamiltonians

$$H_A = \sum_a q_A^a |z_a|^2 - \underline{\underline{C_A}}$$

$A = 1, \dots, m$, here q_A^a are called charges

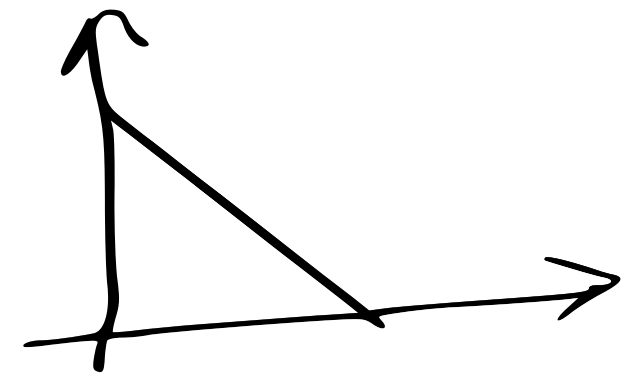
(this is a nice generalization of CP^1 example; q_A^a may be negative or positive)

A lot of interesting manifolds may be constructed this way.

Easy tric – how to solve moment map condition: Go from coord z_a to new coord $|z_a|^2 = r_a$ ⊕ $r_a > 0$, and momentum map condition is linear (hyperplane) plus positivity condition ⊕.

I will have momentum polytopes

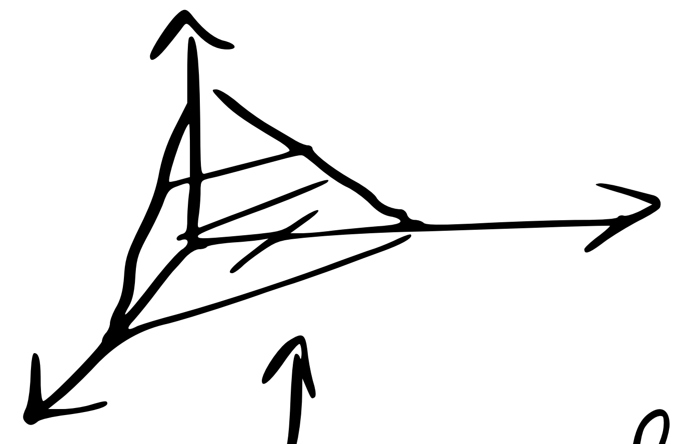
$\mathbb{C}P^2$:



$r_1 > 0$
$r_2 > 0$
$r_1 + r_2 = R^2$

its an interval

$\mathbb{C}P^3$



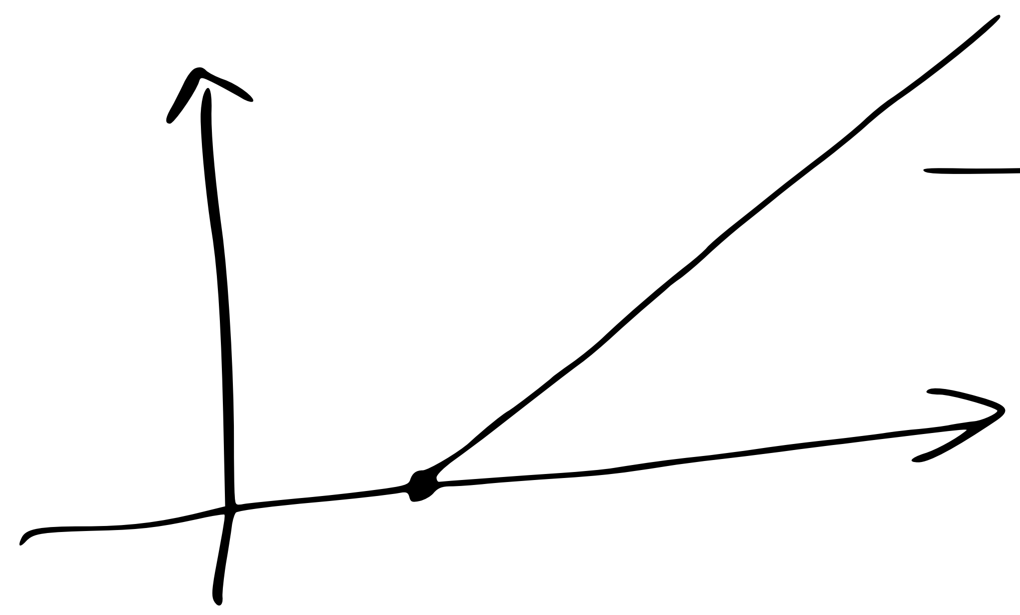
triangle

$r_1 > 0$
$r_2 > 0$
$r_3 > 0$

$r_1 + r_2 + r_3 = R^2$

$\mathbb{C}^2$ $q^1 > 0$ $q^2 \geq 0$

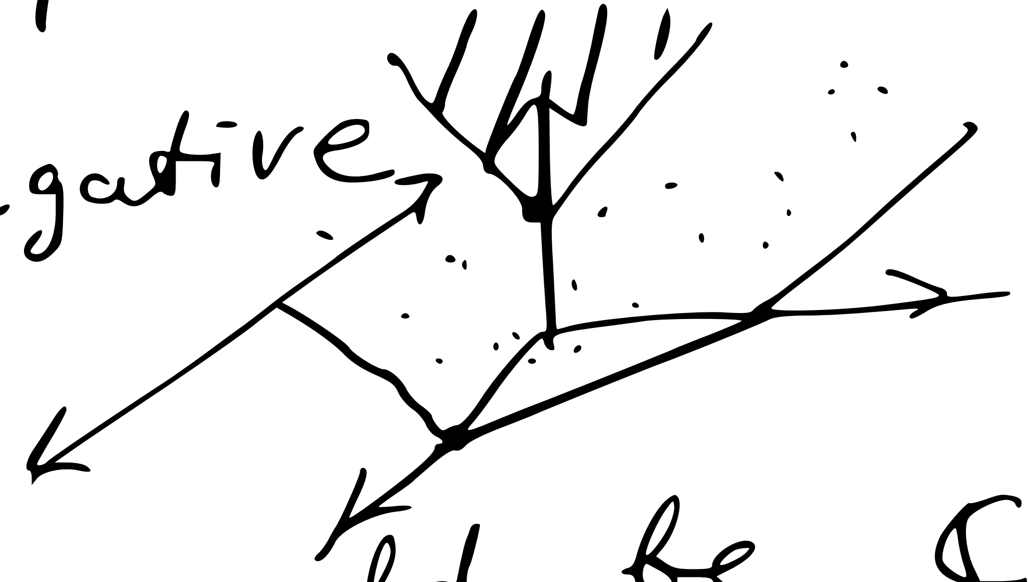
$r_1 > 0, r_2 > 0$
$r_1 - r_2 = R^2$



$\longrightarrow$ a ray (quotient will be noncompact $\mathbb{C}$)

$\mathbb{C}^3$ $r_1 + r_2 - r_3 = C$ (I study it for dif. signs of C)

C is negative



One polytope would be $\mathbb{C}^2$

Another polytope is a plane with the triangle that is cut out!
It corresponds to $\mathbb{C}^2$ with a point (origin) blown up! $\longrightarrow$ ....

Appearence in supersymmetric physics.
When we study SYM (just Maxwell) coupled to matter (say charged matter described by chiral fields, then due to interaction with gauge field a momentum map potential is generated.

$$\int \bar{\Phi}\Box\Phi + \dots + g_0^2 |q_i \Phi\bar{\Phi} - c|^2$$

momentum map

Hyperkahler reduction

G_c acting on a Hyperkahler manifold: there are 3 symplectic forms invariant under G_c.
There are 3 moment maps!
Each for corresponding simpl. form.

<u>Example</u>! $\mathbb{C}^2$, $z_1 \to e^{i\varphi} z_1$
$z_2 \to e^{-i\varphi} z_2$

$dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2$,
$dz_1 \wedge dz_2$, $d\bar{z}_1 \wedge d\bar{z}_2$ all invariant

$M_1 = 0,\ M_2 = 0,\ M_3 = 0 \big/ G_c$ – hyperkahler reduction

Applications

1. Extended SUSY gauge theories (!)
   (higher SUSY)
2. Instantons in d=4 YM theory are also hyperkahler reduction

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Hyperkahler manifold – manifold with 3 complex structures (actually, $\mathbb{C}P^1$ of them), such that metric is kahler with respect to all of them.

$$\mathbb{C}^2 \qquad z_1, z_2$$

$$\tilde{z} = a z_1 + b \bar{z}_2$$

Complex st. on $\mathbb{C}^2$ are

$$SO(4) \big/ U(2) = \mathbb{C}P^1$$

Arnold – math methods of classical mech.

Gauged mechanics is missing.

Example: oscillator

$S = \int \dot{x}^2 - \omega^2 x^2$, in Hamiltonian language

$\omega = 1$

$S = \int (p\dot{x} + (p^2 + x^2))dt$ $\quad z = x + ip$

$\bar{z} = x - ip$

$S = i\int (\bar{z}\dot{z} + z\bar{z})dt \qquad \dot{z} = \frac{dz}{dt}$

Gauge it: global symmetry $\quad z \to e^{i\varphi} z$, $\bar{z} \to e^{-i\varphi}\bar{z}$

Let's make it local $\quad z \to e^{i\varphi(t)} z$

$\bar{z} \to e^{-i\varphi(t)} \bar{z}$ by

adding a gauge field A

$$S_0 = i\int \left(\bar{z}^a\left(\frac{d}{dt} + iA\right)z^a + z^a\bar{z}^a\right)dt$$

$a = 1, \dots, n+1 \qquad A \to A + d\varphi$ action is invariant

Modify this action by adding a Wilson line

$$S \to S_0 + K\int A$$

Now, K has to be quantized if the gauge group is $U(1)$.

Study this th. classically:

1) Equation of motion for A

$z^a\bar{z}^a - K$ — Here we see our momentum map for $\mathbb{C}P^n$.

On the classical level I have $\bar{z}^a z^a - K = 0$

what about quantum level

Two ways to treat it:

1) Operator way: study first F.I. over z fields, and then integrate A

2) Functional integral over z-fields (doable, since theory is quadratic in $z, \bar{z}$)

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way #1

we have $n+1$ oscillators

Hilb. space is given by

$$\mathcal{H} = P(a_1^+, \dots, a_{n+1}^+)\,|0\rangle$$

vacuum of oscillators

$(\bar{z}^a z^a + K)$ is an operator of total energy K

we extract states from $\mathcal{H}$ that have energy that equals exactly K.

There are finitely many states like this.

For example, for $K=0$ we have the only state - vacuum,

for $K=1$ we have $n+1$ states

for $K=2$ we have $\frac{(n+1)(n+2)}{2}$ states

# of states is equal to the number of symmetric pol. in $n+1$ variables of degree K.

Another interpretation - states are holom. sections of the line bundle $\mathcal{O}(K)$ over $\mathbb{C}P^n$ $\longleftrightarrow$ why?

$$\hat{H}_M |state\rangle = 0 \qquad (z^a \bar{z}^a - K)\,|state\rangle = 0$$

$$\bar{z}^a \Rightarrow \frac{\partial}{\partial z^a}$$

$$\Big(z^a \frac{\partial}{\partial z^a} - K\Big)|state\rangle = 0$$

Exactly condition of being section of $\mathcal{O}(K)$

In particular, we may study S^1 of the length $T \to 0$, then the part. function in this theory would be

$$\mathrm{Tr}\,(e^{TH}) \overset{T\to 0}{=} \#\text{ states with energy } K.$$

states with the energy K

Another way: Integrate $z, \bar{z}$ fields out and make a gauge fixing

$$\bar{z}\dot{z} + A\,\bar{z}z - A\cdot K, \qquad A = A_0 - \text{constant}$$

I am studying theory on S^1 for $T\to 0$

$$\int dA_0 \frac{e^{KA_0}}{\det\left(\frac{d}{dt} + A_0\right)^{n+1}}$$

$$\int dA_0 \frac{1}{(1-e^{A_0})^{n+1}}\, e^{-KA_0} \longrightarrow \text{it is an integral representation for the}$$

number of sym. polynomials

This construction has a generalization for nonabelian group acting on the oscillator

$$\bar{z}^a \dot{z}^a + \bar{z}^a \quad \ldots$$

QFT with and without plank constant.

physics of Q.M and how plank constant appears.

1. QM does not need plank constant in its definition!
2. "plank constant" is the measure of how close QM is to classical mechanics
3. Examples

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later we will generalize this point of view to QFT, but today and on Friday — QM.

1930 by Dirac –
QM is a new physics, everything we know before should be reconsidered, such that in some limit we would get classical physics.

In relativity → space × time by space-time

In QM the structure of physics was completely changed.

1. State of the physical system
previously it was a point on a phase space, in examples: coordinate and momentum of the particle
In QM the state is a projective line in a Hilbert space.
It could be a finite dim vector space with Herm. inner product, or ∞-dim. vector space with such a product.
Example of such Hilb. space is a space of complex functions on a compact manifold with inner product given by $\int_X \bar{f}_1 f_2 \cdot \mu_X = \langle f_1, f_2 \rangle$
measure

Finite dim space is simpler –!
Projective line is a class of equivalence of vectors (traditionally called $|\psi\rangle$) under $|\psi\rangle \sim \lambda|\psi\rangle$, where $\lambda \in \mathbb{C}^*$
In original formulation Dirac said:

state is just a vector $|\psi\rangle \in \mathcal{H}ilb$, with $\underline{\langle\psi|\psi\rangle = 1}$, such that $|\psi\rangle \sim e^{i\varphi}|\psi\rangle$ - but it is the same as to consider projective line:

Example: $\mathbb{C}P^1 = \frac{\mathbb{C}^2 \setminus \{0\}}{\mathbb{C}^*} = \frac{S^3}{U(1)}$

projective lines

$S^3 : |z_0|^2 + |z_1|^2 = 1 \iff$ condition that norm of the vector $\begin{pmatrix} z_1 \\ z_0 \end{pmatrix}$ is 1.

Then we take $e^{i\varphi}$ identification

Then we need to define the following

1) How state is changing in time (dynamics)

2) Was not obvious before $\to$ how do we see state in experiments.

1) + 2) $\to$ we may predict results of experiments in future from result at initial time.

In classical physics 2) was not considered as a problem $\to$ we said that we just observed coordinate and momentum of the particle.

In general classical mechanics we may say that we observed functions on the phase space.

In classical mechanics state $\to$ point observable $\to$ what we observed $\to$ $\to$ Function

Result of observation — evaluation of the function at a point - it was so obvious that it was never formulated explicitely!

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statistical mechanics had <u>no foundations</u> before QM.

QM <u>cannot be</u> derived from statistical mechanics → Bell experiments

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(2) Observables in QM are Hermitean operators in Hilbert space.

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(1) Dynamics of the state is just a unitary rotation in Hilbert space that leads to the motion of the space of projective lines.

In particular, if Hilbert space is $\mathbb{C}^N$, the space of projective lines is $\mathbb{C}P^{N-1}$, and unitary rotations ~~are is~~ corresp. to the action of $U(N)$ on $\mathbb{C}P^{N-1}$.

In particular, $N=2$ we have an action of $U(2)$ on $\mathbb{C}P^1 = S^2$

$U(2) = U(1) \times SU(2)$

$U(1)$ acts trivially;

$SU(2)$ acts nontrivially

$SU(2)$ is a double cover of $SO(3)$.

we know how $SO(3)$ is acting on S^2 (just ordinary Eucledian rotations)

In general; for $\mathbb{C}P^{N-1}$
$U(N) = U(1) \times SU(N-1)$, $U(1)$ act trivially while $SU(N-1)$ acts nontrivially.

It is surprizing that dynamics is so simple.

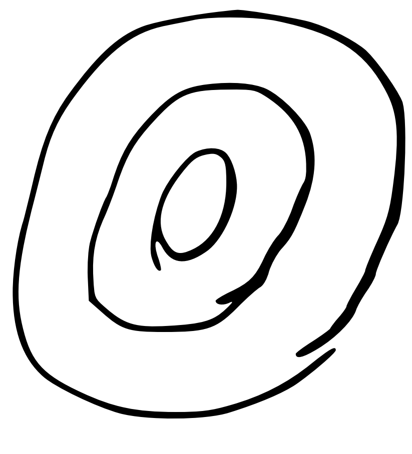
In classical theory we should solve equations of motion: (cl)

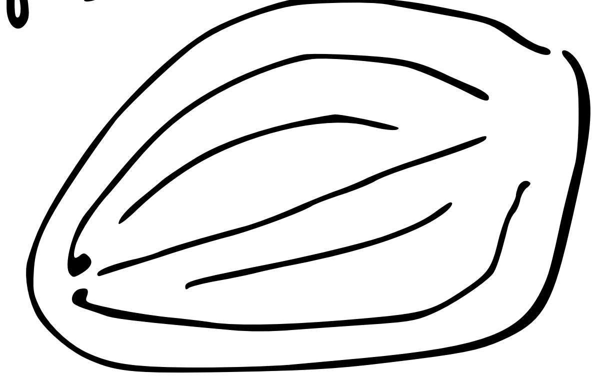
$$\frac{d\varphi^i}{dt} = (\omega^{-1})^{ij} \frac{\partial H}{\partial \varphi^j}, \text{ where}$$

φ^i is a coordinate on the phase space, ω_{ij} is a symplectic form.

Solutions to (cl) can be very different. It could be:

$\leftarrow$ regular





$\leftarrow$ stochastic

$\uparrow$ cannot intersect

depending on the type of classical Hamiltonian

In the regular case we said we have integrable system, in stoch. case - dynamical chaos $\to$ it was a

Classical picture of dynamics

In QM there is <u>no</u> stochastics in

dynamics $\to$ all motions are just unitary rotations.

It is a second time in physics when going to fundamentals simplify basic equations.

1.) From gas dynamics given in terms of PDE to kinetic theory of particles
ordinary dif. equations like (cl)

2) From cl. physics to Q. physics
(cl) - ODE $\to$ just rotations

What is the infinitesimal version of unitary rotation
it easy to write it in terms of $|\psi\rangle$ representatives of the projective lines:
(QM)

$$|\psi\rangle \to e^{itH}|\psi\rangle, \text{ where}$$

H is the Hermitean operator.

H is the Lie algebra of $U(N)$

Really, the Lie algebra of $U(N)$ is the space of antiherm. operators
Antihermitean operator is i times Hermitean operator.

What is implied in (QM) is a Evolution in time t_1+t_2 is a composition of evolutions in time t_1

and time t_2.
Note, that it is just functoriality of an abstract notion of QFT in 1-dimensional case.
Differential equation

(S) $\frac{d|\psi\rangle}{dt} = iH|\psi\rangle$ – Sch. equation for evolution of a state

Please note, that no $\hbar$ appears here.
Moreover, $|\psi\rangle$ is just a representative of a projective line
take (S) and then induce equations on the space of lines.
In particular, change $H \to H + c$
constant
does not change the motion of the lines (it just changes the representatives)

Observables is what we observe in a physical system by doing experiments.

Observable is a Hermitean operator (any Herm. operator $\mathcal{O}$).
$\mathcal{O}$ - for observable
In observations in the state $[|\psi\rangle]$
line passing through $|\psi\rangle$
of an observable $\mathcal{O}$
we get a probabilistic distribution, i.e.

sometimes experiment shows λ_1,
sometimes ... shows λ_2, etc...
with probability ω_1, ω_2, etc.
So there is a concept of the average

$$\langle \mathcal{O} \rangle_{[|\psi\rangle]} = \sum_i \omega_i \lambda_i$$ (Prob. expectations)

And there is a formula for $\underline{\underline{\langle \mathcal{O} \rangle_{[|\psi\rangle]}}}$

$$= Tr_{Hilb} (\mathcal{O} \cdot \underline{\underline{Proj_{|\psi\rangle}}})$$ (<u>Observation axiom</u>)

Projection operator on the line
which representative is $|\psi\rangle$.

(OA) is very simple.

(OA) $\Rightarrow$ Let $\mathcal{O} = \sum_i \lambda_i \, Proj_i$

Then OA:

$$\langle \mathcal{O} \rangle = \sum_i \lambda_i \underbrace{Tr_{Hilb} (Proj_i \cdot Proj_{|\psi\rangle})}_{\text{probability to get eigenvalue } \lambda_i \text{ in experiment}}$$

In particular, it is clear that $\sum_i \omega_i = 1$
as it should be for probabilities.

Really, $1 = \sum_i Proj_i$

$$1 = Tr_{Hilb} (Proj_{|\psi\rangle}) = \sum_i Tr_{Hilb} (Proj_i \cdot Proj_{|\psi\rangle}) =$$

$$= \sum_i \omega_i$$ — it works.

Equation for changing of the $\langle \mathcal{O} \rangle_{[|\psi\rangle]}$ in time

$\frac{d}{dt} \langle \mathcal{O} \rangle_{[|\psi\rangle(t)]}$ – ? Dynamics not of the state but of the probability distributions

It is easier to go to global picture

$$\langle \mathcal{O} \rangle_{[|\psi\rangle(t)]} = \mathrm{Tr}_{Hilb} \Big(\mathcal{O} \underbrace{e^{itH} \mathrm{Proj}_\psi e^{-itH}}_{\text{rotation of a physical state}} \Big) =$$

$$= \mathrm{Tr}_{Hilb} \Big(\underbrace{e^{-itH} \mathcal{O} e^{itH}}_{\mathcal{O}_t \text{ in Heisenberg picture}} \mathrm{Proj}_\psi \Big)$$

Here we see that we may replace rotation of state by rotation of the operator (computed at the same state)

Go to differential form for change of the ~~...~~ distribution

$$\frac{d}{dt} \langle \mathcal{O} \rangle_{[|\psi\rangle_t]} = -i\, \mathrm{Tr}_{Hilb} \big([H, \mathcal{O}]\, \mathrm{Proj}_\psi \big)$$

It is instructive to compare it with cl. mechanics

In classical mechanics

$\mathcal{O}$ may be just some function f on the phase space.

In cl. mech. we expect

$$\frac{d}{dt} f = \frac{\partial f}{\partial \varphi^i} \frac{d\varphi^i}{dt} = \frac{\partial f}{\partial \varphi^i} (\omega^{-1})^{ij} \frac{\partial H}{\partial \varphi^j} =$$

$$= \{ f, H \}$$ Poisson bracket between f and H

Note that

if we have a family of associative algebras $A_\hbar$: $\dfrac{a_\hbar \circ b_\hbar = c^c_{ab}(\hbar) c_\hbar}{c^c_{ab}(0) = c^c_{ba}(0)}$ on Friday - examples

such that for $\hbar = 0$

then $c^c_{ab}(\hbar) = c^c_{ab}(0) + i\hbar P^c_{ab} + O(\hbar^2)$ (!)

with P^c_{ab} being a Poissonian bracket (intesting math. theorem, that can be proved algebraically)

It is the place where $\hbar$ appears (not in formulation of QM but in relation to cl. mechanics)

<u>Idea:</u> Consider a family of QM depending on $\hbar$!

$O_\hbar$, and $H_\hbar$. At $\hbar = 0$

$[O_0, H_0] = 0$ so it seems we

have no dynamics.

we will consider $H_\hbar^{phys} = \frac{1}{\hbar} H_\hbar$ and study the following family:

observables would be $O_\hbar$, while Hamiltonians would be $H_\hbar^{phys} = \frac{1}{\hbar} H_\hbar$

Then the QM evolution of the average would be

$H_\hbar = H_0 + \hbar(*)$

not cont. in cl. mech

$$\frac{d}{dt}\langle O\rangle_{[|\psi_t\rangle]} = \frac{-i}{\hbar}\langle [H_\hbar, O]\rangle_{[|\psi_t\rangle]} =$$

cancel $\hbar$ P.B. $(H_\hbar, O)$

$$= \langle PB(H_0, O)\rangle_{[|\psi_t\rangle]} + O(\hbar)$$

Correspondence between QM and Cl. Mech.