Panorama of Dynamics and Geometry of Moduli Spaces and Applications

Lecture 10. Masur-Veech volume of the moduli space of quadratic differentials, random square-tiled surfaces of large genus and random multicurves of surfaces of large genus. Part 1

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(after a joint work with V. Delecroix, E. Goujard and P. Zograf
Duke Math. J. 170:12 (2021), 2633–2718,
and arXiv.2007.04740, to appear in Inventiones Math.)

YMSC, Tsinghua University, May 19, 2022

Formula for the Masur–Veech volume of the moduli space of quadratic differentials

Intersection numbers (Witten–Kontsevich correlators)

The Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of the moduli space of smooth complex curves of genus g with n labeled marked points $P_1, \ldots, P_n \in C$ is a complex orbifold of complex dimension $3g-3+n$.
Choose index i in $\{1, \ldots, n\}$. The family of complex lines cotangent to C at the point P_i forms a holomorphic line bundle $\mathcal{L}_i$ over $\mathcal{M}_{g,n}$ which extends to $\overline{\mathcal{M}}_{g,n}$.
The first Chern class of this *tautological bundle* is denoted by $\psi_i = c_1(\mathcal{L}_i)$.

Any collection of nonnegative integers satisfying $d_1 + \cdots + d_n = 3g-3+n$ determines a positive rational "*intersection number*" (or the "*correlator*" in the physical context):

$$\langle \tau_{d_1} \ldots \tau_{d_n} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \ldots \psi_n^{d_n} \,.$$

The famous Witten's conjecture claims that these numbers satisfy certain recurrence relations which are equivalent to certain differential equations on the associated generating function ("*partition function in 2-dimensional quantum gravity*"). Witten's conjecture was proved by M. Kontsevich; alternative proofs belong to A. Okounkov and R. Pandharipande, to M. Mirzakhani, to M. Kazarian and S. Lando (and there are more).

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Recursive relations

Initial data: $\langle \tau_0^3 \rangle = 1, \qquad \langle \tau_1 \rangle = \frac{1}{24}\,.$

String equation:

$$\langle \tau_0 \tau_{d_1} \ldots \tau_{d_n} \rangle_{g,n+1} = \langle \tau_{d_1-1} \ldots \tau_{d_n} \rangle_{g,n} + \cdots + \langle \tau_{d_1} \ldots \tau_{d_n-1} \rangle_{g,n}\,.$$

Dilaton equation:

$$\langle \tau_1 \tau_{d_1} \ldots \tau_{d_n} \rangle_{g,n+1} = (2g-2+n) \langle \tau_{d_1} \ldots \tau_{d_n} \rangle_{g,n}\,.$$

Virasoro constraints (in Dijkgraaf–Verlinde–Verlinde form; $k \geq 1$):

$$\langle \tau_{k+1} \tau_{d_1} \cdots \tau_{d_n} \rangle_g = \frac{1}{(2k+3)!!} \left[\sum_{j=1}^{n} \frac{(2k+2d_j+1)!!}{(2d_j-1)!!} \langle \tau_{d_1} \cdots \tau_{d_j+k} \cdots \tau_{d_n} \rangle_g \right.$$

$$+ \frac{1}{2} \sum_{\substack{r+s=k-1 \\ r,s \geq 0}} (2r+1)!!(2s+1)!! \langle \tau_r \tau_s \tau_{d_1} \cdots \tau_{d_n} \rangle_{g-1}$$

$$\left. + \frac{1}{2} \sum_{\substack{r+s=k-1 \\ r,s \geq 0}} (2r+1)!!(2s+1)!! \sum_{\{1,\ldots,n\} = I \coprod J} \langle \tau_r \prod_{i \in I} \tau_{d_i} \rangle_{g'} \langle \tau_s \prod_{i \in J} \tau_{d_i} \rangle_{g-g'} \right].$$

Uniform large genus asymptotics

We stated in August 2019 a conjecture which was proved by Amol Aggarwal already in April 2020.

Theorem (Aggarwal'21). *The following **uniform** asymptotic formula is valid:*

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \dots \psi_n^{d_n} =$$

$$= \frac{1}{24^g} \cdot \frac{(6g-5+2n)!}{g!\,(3g-3+n)!} \cdot \frac{d_1! \dots d_n!}{(2d_1+1)! \cdots (2d_n+1)!} \cdot \big(1+\varepsilon(\boldsymbol{d})\big)\,,$$

*where $\varepsilon(\boldsymbol{d}) = O\left(1 + \frac{(n+\log g)^2}{g}\right)$ **uniformly** for all $n = o(\sqrt{g})$ and all partitions $\boldsymbol{d}$, $d_1 + \cdots + d_n = 3g-3+n$, as $g \to +\infty$.*

Volume polynomials

Consider the moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces of genus g with n marked points. Let $d_1, \ldots, d_n$ be an ordered partition of $3g-3+n$ into the sum of nonnegative numbers, $d_1 + \cdots + d_n = 3g-3+n$, let $\mathbf{d}$ be the multiindex $(d_1, \ldots, d_n)$ and let $b^{2\mathbf{d}}$ denote $b_1^{2d_1} \cdot \cdots \cdot b_n^{2d_n}$.
Define the homogeneous polynomial $N_{g,n}(b_1, \ldots, b_n)$ of degree $6g-6+2n$ in variables $b_1, \ldots, b_n$:

$$N_{g,n}(b_1, \ldots, b_n) := \sum_{|d|=3g-3+n} c_{\mathbf{d}} b^{2\mathbf{d}}\,,$$

where

$$c_{\mathbf{d}} := \frac{1}{2^{5g-6+2n}\,\mathbf{d}!} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \ldots \psi_n^{d_n}$$

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Up to a numerical factor, the polynomial $N_{g,n}(b_1, \ldots, b_n)$ coincides with the top homogeneous part of the Mirzakhani's volume polynomial $V_{g,n}(b_1, \ldots, b_n)$ providing the Weil–Petersson volume of the moduli space of bordered Riemann surfaces:

$$V_{g,n}^{top}(b) = 2^{2g-3+n} \cdot N_{g,n}(b) \,.$$

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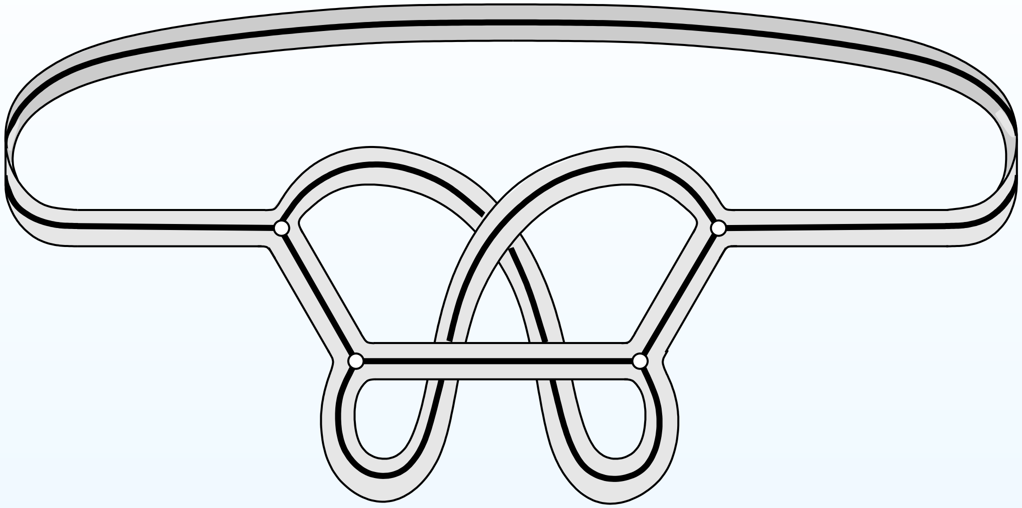
Define the formal operation $\mathcal{Z}$ on monomials as

$$\mathcal{Z} \;:\; \prod_{i=1}^{n} b_i^{m_i} \longmapsto \prod_{i=1}^{n} \big(m_i! \cdot \zeta(m_i+1)\big)\,,$$

and extend it to symmetric polynomials in b_i by linearity.

Trivalent ribbon graphs



This trivalent ribbon graph defines an orientable surface of genus $g = 1$ with $n = 2$ boundary components. If we assigned lengths to all edges of the core graph, each boundary component gets induced length, namely, the sum of the lengths of the edges which it follow.

Note, however, that in general, fixing a genus g, a number n of boundary components and integer lengths $b_1, \ldots, b_n$ of boundary components, we get plenty of trivalent integral metric ribbon graphs associated to such data. The Theorem of Kontsevich counts them.

Kontsevich's count of metric ribbon graphs

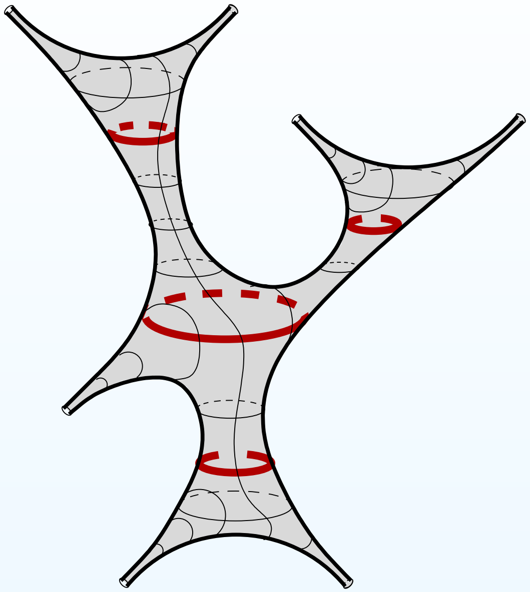
Theorem (Kontsevich'92; in this form — Norbury'10). *Consider a collection of positive integers* $b_1, \ldots, b_n$ *such that* $\sum_{i=1}^n b_i$ *is even. The weighted count of genus* g *connected trivalent metric ribbon graphs* Γ *with integer edges and with* n *labeled boundary components of lengths* $b_1, \ldots, b_n$ *is equal to* $N_{g,n}(b_1, \ldots, b_n)$ *up to the lower order terms:*

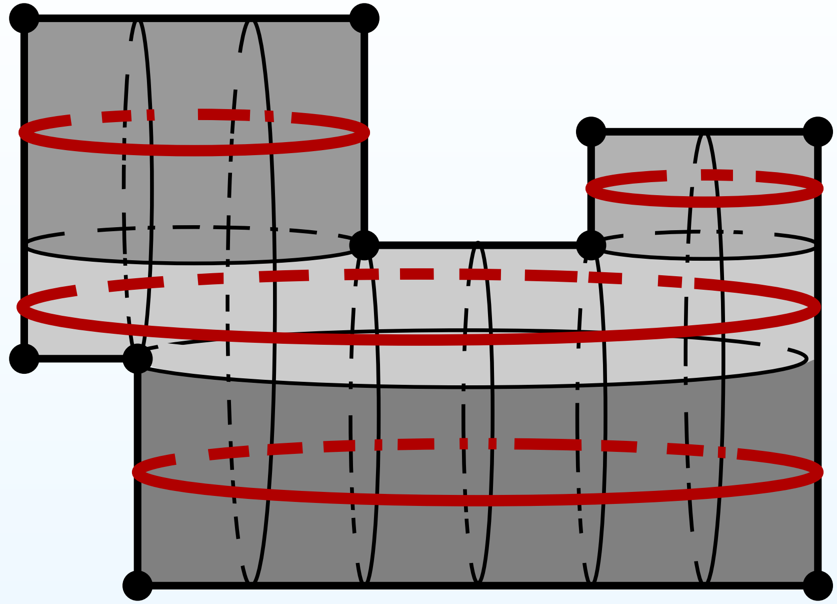
$$\sum_{\Gamma \in \mathcal{R}_{g,n}} \frac{1}{|\operatorname{Aut}(\Gamma)|} N_\Gamma(b_1, \ldots, b_n) = N_{g,n}(b_1, \ldots, b_n) + \textit{lower order terms}\,,$$

where $\mathcal{R}_{g,n}$ *denote the set of (nonisomorphic) trivalent ribbon graphs* Γ *of genus* g *and with* n *boundary components.*

This Theorem is an important part of Kontsevich's proof of Witten's conjecture.

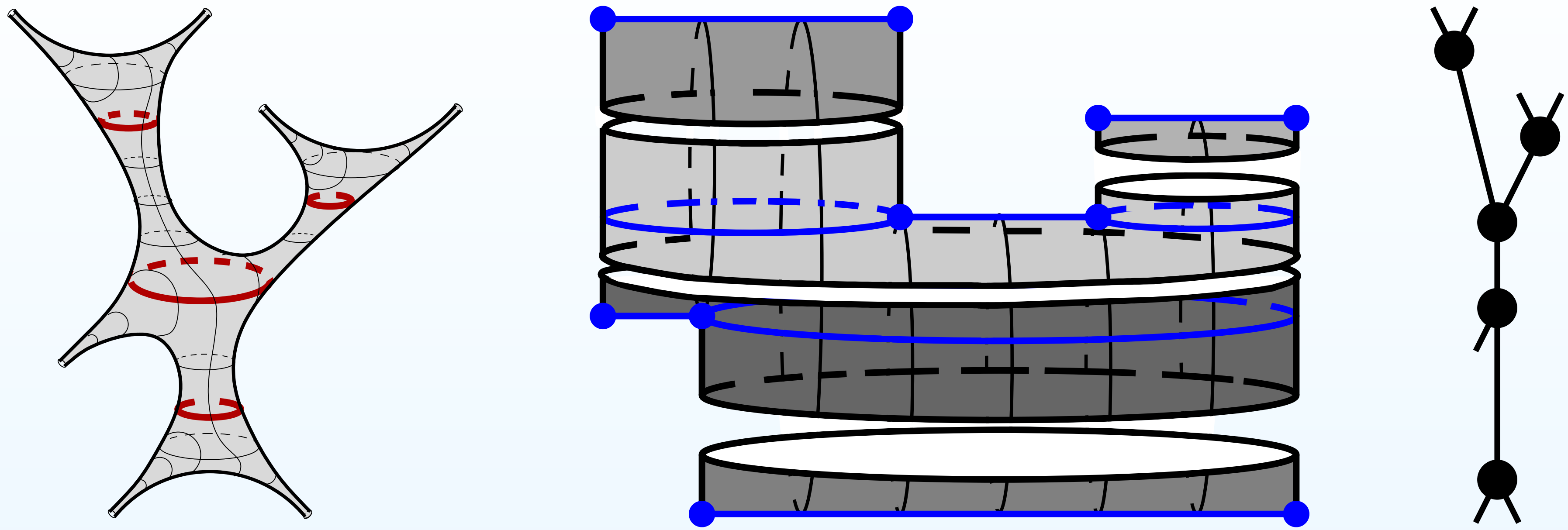
Stable graph associated to a square-tiled surface





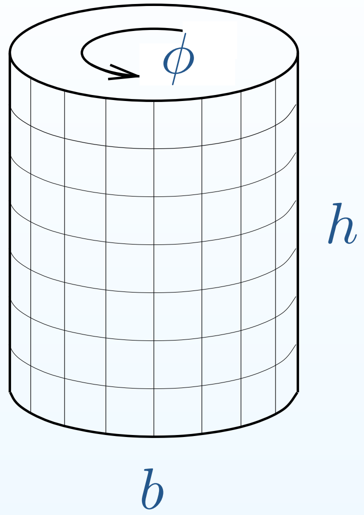
Having a square-tiled surface we associate to it a topological surface S on which we mark all "corners" with cone angle π (i.e. vertices with exactly two adjacent squares). By convention the associated hyperbolic metric has cusps at the marked points. We also consider a multicurve γ on the resulting surface composed of the waist curves γ_j of all maximal horizontal cylinders.

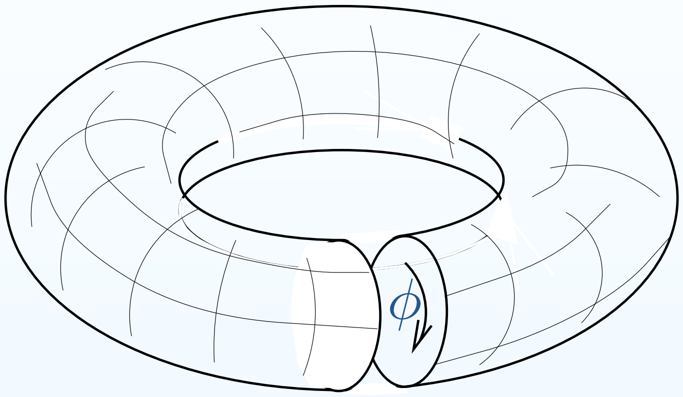
Stable graph associated to a square-tiled surface



Having a square-tiled surface we associate to it a topological surface S on which we mark all “corners” with cone angle π (i.e. vertices with exactly two adjacent squares). By convention the associated hyperbolic metric has cusps at the marked points. We also consider a multicurve γ on the resulting surface composed of the waist curves γ_j of all maximal horizontal cylinders. The associated *stable graph* Γ is the dual graph to the multicurve. The vertices of Γ are in the natural bijection with metric ribbon graphs given by components of $S \setminus \gamma$. The edges are in the bijection with the waist curves γ_i of the cylinders. The marked points are encoded by “legs” — half-edges of the dual graph.

Number of square-tiled tori

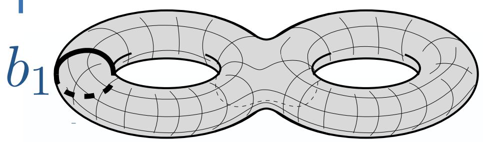




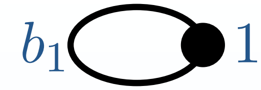
The number of square-tiled tori tiled with at most N squares has asymptotics

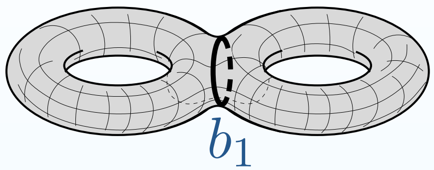
$$\sum_{\substack{b,h\in\mathbb{N}\\ b\cdot h\le N}} b = \sum_{\substack{b,h\in\mathbb{N}\\ b\le \frac{N}{h}}} b \sim \sum_{h\in\mathbb{N}} \frac{1}{2}\cdot\left(\frac{N}{h}\right)^2 = \frac{N^2}{2}\sum_{h\in\mathbb{N}}\frac{1}{h^2} = \frac{N^2}{2}\zeta(2) =$$

$$= \frac{N^2}{2}\mathcal{Z}(b) = \frac{N^2}{2}\cdot\frac{\pi^2}{6}.$$

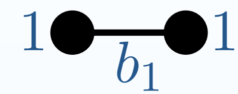


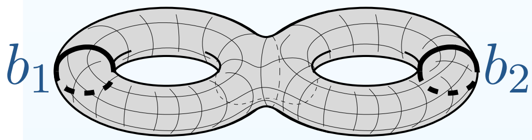
$$\frac{1}{2} \cdot 1 \cdot b_1 \cdot N_{1,2}(b_1, b_1)$$



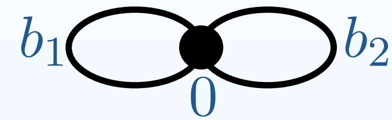


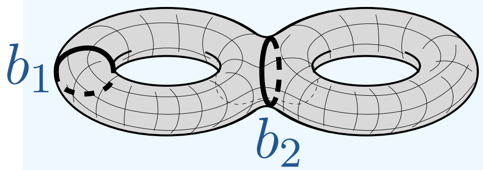
$$\frac{1}{2} \cdot \frac{1}{2} \cdot b_1 \cdot N_{1,1}(b_1) \cdot N_{1,1}(b_1)$$



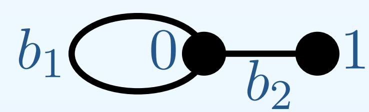


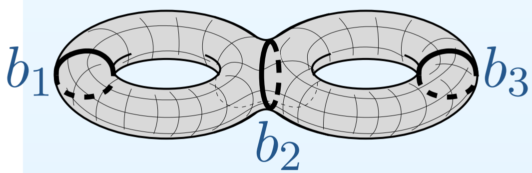
$$\frac{1}{8} \cdot 1 \cdot b_1 b_2 \cdot N_{0,4}(b_1, b_1, b_2, b_2)$$



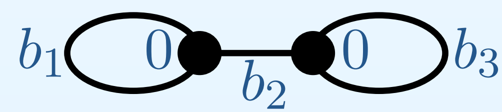


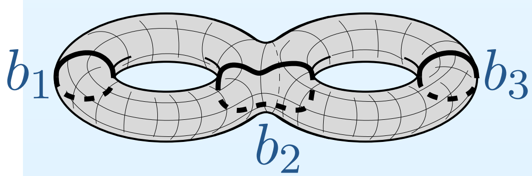
$$\frac{1}{2} \cdot \frac{1}{2} \cdot b_1 b_2 \cdot N_{0,3}(b_1, b_1, b_2) \cdot$$
$$\cdot N_{1,1}(b_2)$$



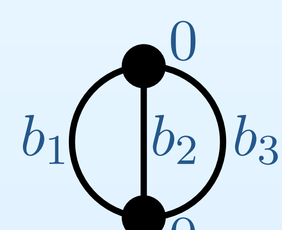


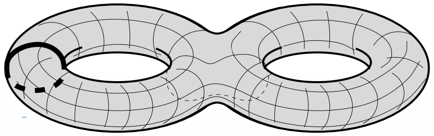
$$\frac{1}{8} \cdot \frac{1}{2} \cdot b_1 b_2 b_3 \cdot N_{0,3}(b_1, b_1, b_2) \cdot$$
$$\cdot N_{0,3}(b_2, b_3, b_3)$$



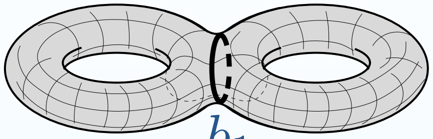


$$\frac{1}{12} \cdot \frac{1}{2} \cdot b_1 b_2 b_3 \cdot N_{0,3}(b_1, b_2, b_3) \cdot$$
$$\cdot N_{0,3}(b_1, b_2, b_3)$$

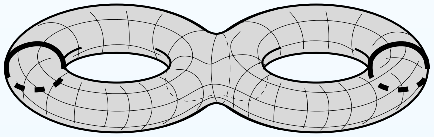




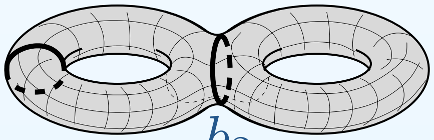
$$\frac{1}{2}\cdot 1\cdot b_1\cdot N_{1,2}(b_1,b_1) = \frac{1}{2}\cdot b_1\left(\frac{1}{384}(2b_1^2)(2b_1^2)\right)$$



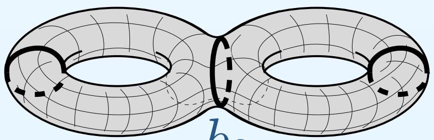
$$\frac{1}{2}\cdot \frac{1}{2}\cdot b_1\cdot N_{1,1}(b_1)\cdot N_{1,1}(b_1) = \frac{1}{4}\cdot b_1\left(\frac{1}{48}b_1^2\right)\left(\frac{1}{48}b_1^2\right)$$



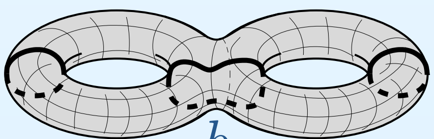
$$\frac{1}{8}\cdot 1\cdot b_1b_2\cdot N_{0,4}(b_1,b_1,b_2,b_2) = \frac{1}{8}\cdot b_1b_2\cdot\left(\frac{1}{4}(2b_1^2+2b_2^2)\right)$$



$$\begin{aligned}\frac{1}{2}\cdot \frac{1}{2}\cdot b_1b_2\cdot N_{0,3}(b_1,b_1,b_2)\cdot\\ \cdot N_{1,1}(b_2) &= \frac{1}{4}\cdot b_1b_2\cdot(1)\cdot\left(\frac{1}{48}b_2^2\right)\end{aligned}$$



$$\begin{aligned}\frac{1}{8}\cdot \frac{1}{2}\cdot b_1b_2b_3\cdot N_{0,3}(b_1,b_1,b_2)\cdot\\ \cdot N_{0,3}(b_2,b_3,b_3) &= \frac{1}{16}\cdot b_1b_2b_3\cdot(1)\cdot(1)\end{aligned}$$



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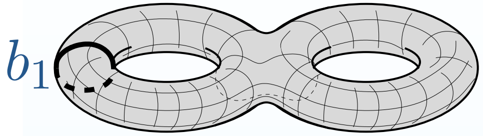
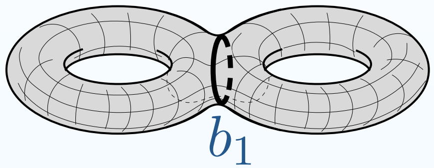
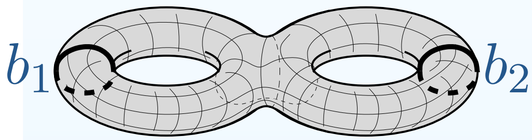
Volume of $\mathcal{Q}_2$



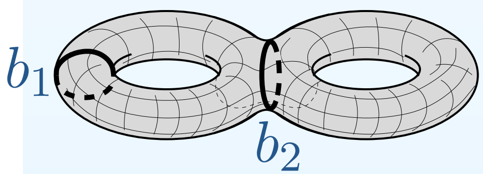
$$\frac{1}{192}\cdot b_1^5 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{192}\cdot\big(5!\cdot\zeta(6)\big) = \frac{1}{1512}\cdot\pi^6$$



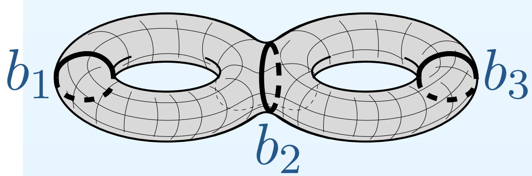
$$\frac{1}{9216}\cdot b_1^5 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{9216}\cdot\big(5!\cdot\zeta(6)\big) = \frac{1}{72576}\cdot\pi^6$$



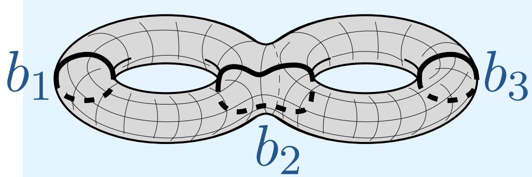
$$\frac{1}{16}(b_1^3 b_2 + b_1 b_2^3) \overset{\mathcal{Z}}{\longmapsto} \frac{1}{16}\cdot 2\big(1!\cdot\zeta(2)\big)\cdot\big(3!\cdot\zeta(4)\big) = \frac{1}{720}\cdot\pi^6$$



$$\frac{1}{192}\cdot b_1 b_2^3 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{192}\cdot\big(1!\cdot\zeta(2)\big)\cdot\big(3!\cdot\zeta(4)\big) = \frac{1}{17280}\cdot\pi^6$$



$$\frac{1}{16} b_1 b_2 b_3 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{16}\cdot\big(1!\cdot\zeta(2)\big)^3 = \frac{1}{3456}\cdot\pi^6$$

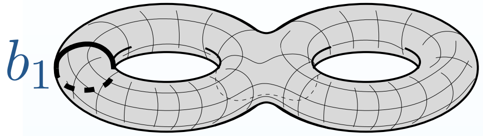


$$\frac{1}{24} b_1 b_2 b_3 \overset{\mathcal{Z}}{\longmapsto} \frac{1}{24}\cdot\big(1!\cdot\zeta(2)\big)^3 = \frac{1}{5184}\cdot\pi^6$$

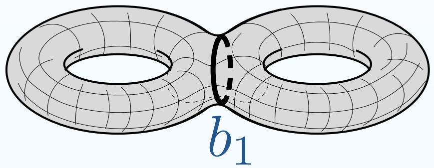
$$\operatorname{Vol}\mathcal{Q}_2 = \frac{128}{5}\cdot\left(\frac{1}{1512}+\frac{1}{72576}+\frac{1}{720}+\frac{1}{17280}+\frac{1}{3456}+\frac{1}{5184}\right)\cdot\pi^6 = \frac{1}{15}\pi^6\,.$$

These contributions to $\operatorname{Vol}\mathcal{Q}_2$ are proportional to Mirzakhani's frequencies of corresponding multicurves.

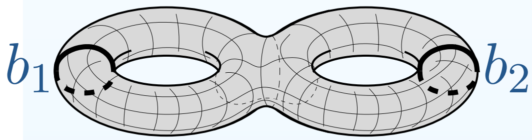
Volume of $\mathcal{Q}_2$



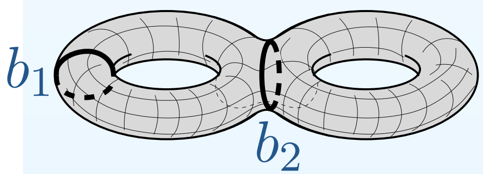
$$\frac{1}{192}\cdot b_1^5 \quad \overset{\mathcal{Z}}{\longmapsto} \quad \frac{1}{192}\cdot\big(5!\cdot\zeta(6)\big) \quad = \frac{1}{1512}\cdot\pi^6$$



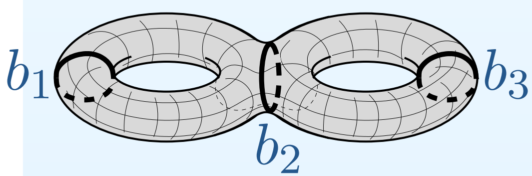
$$\frac{1}{9216}\cdot b_1^5 \quad \overset{\mathcal{Z}}{\longmapsto} \quad \frac{1}{9216}\cdot\big(5!\cdot\zeta(6)\big) \quad = \frac{1}{72576}\cdot\pi^6$$



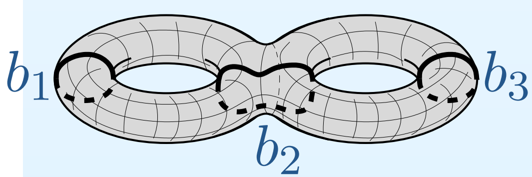
$$\frac{1}{16}\big(b_1^3 b_2 + {} + b_1 b_2^3\big) \quad \overset{\mathcal{Z}}{\longmapsto} \quad \frac{1}{16}\cdot 2\big(1!\cdot\zeta(2)\big)\cdot\big(3!\cdot\zeta(4)\big) \quad = \frac{1}{720}\cdot\pi^6$$



$$\frac{1}{192}\cdot b_1 b_2^3 \quad \overset{\mathcal{Z}}{\longmapsto} \quad \frac{1}{192}\cdot\big(1!\cdot\zeta(2)\big)\cdot\big(3!\cdot\zeta(4)\big) \quad = \frac{1}{17280}\cdot\pi^6$$



$$\frac{1}{16} b_1 b_2 b_3 \quad \overset{\mathcal{Z}}{\longmapsto} \quad \frac{1}{16}\cdot\big(1!\cdot\zeta(2)\big)^3 \quad = \frac{1}{3456}\cdot\pi^6$$



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Volume of $\mathcal{Q}_{g,n}$

Theorem (Delecroix–Goujard–Zograf–Zorich'21). *The Masur–Veech volume $\operatorname{Vol}\mathcal{Q}_{g,n}$ of the moduli space of meromorphic quadratic differentials with n simple poles has the following value:*

$$\operatorname{Vol}\mathcal{Q}_{g,n} = \frac{2^{6g-5+2n}\cdot(4g-4+n)!}{(6g-7+2n)!}\cdot \sum_{\substack{\text{Weighted graphs } \Gamma \\ \text{with } n \text{ legs}}} \frac{1}{2^{\text{Number of vertices of } \Gamma - 1}}\cdot\frac{1}{|\operatorname{Aut}\Gamma|}\cdot$$

$$\cdot\,\mathcal{Z}\left(\prod_{\text{Edges } e \text{ of } \Gamma} b_e \cdot \prod_{\text{Vertices of } \Gamma} N_{g_v,n_v+p_v}(\boldsymbol{b}_v^2,\underbrace{0,\dots,0}_{p_v})\right)\,,$$

The partial sum for fixed number k of edges gives the contribution of k-cylinder square-tiled surfaces.

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$$\cdot\,\mathcal{Z}\left(\prod_{\text{Edges } e \text{ of } \Gamma} b_e \cdot \prod_{\text{Vertices of }\Gamma} N_{g_v,n_v+p_v}(\boldsymbol{b}_v^2,\underbrace{0,\dots,0}_{p_v})\right),$$

Remark. The Weil–Petersson volume of $\mathcal{M}_{g,n}$ corresponds to the *constant term* of the volume polynomial $N_{g,n}(L)$ when the lengths of all boundary components are contracted to zero. To compute the Masur–Veech volume we use the *top homogeneous parts* of volume polynomials; i.e. we use them in the opposite regime when the lengths of all boundary components tend to infinity.

Mirzakhani's volume polynomials

Theorem (M. Mirzakhani, 2008). *Weil–Petersson volume of the moduli space of boarded hyperbolic surfaces is a polynomial in even powers of lengths of boundary components. Its term of top degree* $6g-6+2n$ *has the form:*

$$\mathrm{Vol}_{\mathrm{WP}}\left(\mathcal{M}_{g,n}(b_1,\dots,b_n)\right) = \frac{1}{2^{3g-3+n}} \sum_{|d|=3g-3+n} \frac{\langle \psi_1^{d_1}\dots\psi_n^{d_n}\rangle}{d_1!\dots d_n!} b^{2d_1}\dots b^{2d_n} + (terms\ of\ lower\ degree)\,.$$

Example: $\boldsymbol{\mathcal{M}_{1,1}(b_1)}$. Here $3g-3+n=1$; $\langle\psi_1^1\rangle=\frac{1}{24}$.

$$\mathrm{Vol}_{\mathrm{WP}}\left(\mathcal{M}_{1,1}(b_1)\right) = \frac{2}{2^1}\frac{\langle\psi_1\rangle}{1!}b_1^{2\cdot 1}+4\pi^2 = \frac{1}{24}b_1^2+4\pi^2\,.$$

Example: $\boldsymbol{\mathcal{M}_{1,2}(b_1,b_2)}$.
Here $3g-3+n=2$; $\langle\psi_1^2\rangle=\langle\psi_1\psi_2\rangle=\langle\psi_2^2\rangle=\frac{1}{24}$.

$$\mathrm{Vol}_{\mathrm{WP}}\left(\mathcal{M}_{1,2}(b_1,b_2)\right) = \frac{1}{2^2}\left(\frac{\langle\psi_1^2\rangle}{2!\,0!}b_1^{2\cdot 2}+\frac{\langle\psi_1\psi_2\rangle}{1!\,1!}b_1^{2\cdot 1}b_2^{2\cdot 1}+\frac{\langle\psi_2^2\rangle}{0!\,2!}b_2^{2\cdot 2}\right)+\cdots$$
$$=\frac{1}{192}\left(b_1^2+b_2^2+4\pi^2\right)\left(b_1^2+b_2^2+12\pi^2\right)\,.$$

Alternative formula for the Masur–Veech volume of $\mathcal{Q}_{g,n}$

Theorem (D. Chen, M. Möller, A. Sauvaget'19)

$$\operatorname{Vol}\mathcal{Q}(1^{4g-4+n},-1^n) =$$

$$= \frac{2^{2g+1}\pi^{6g-6+2n}}{(6g-7+2n)!}\sum_{i=0}^{g}\frac{(4g-4+n)!}{(2g-3+n+i)!}\int_{\overline{\mathcal{M}}_{g,2g-3+2n+i}}\psi_{n+1}^2\dots\psi_{2g-3+2n+i}^2\lambda_{g-i} =$$

$$= \frac{2^{2g+1}\pi^{6g-6+2n}}{(6g-7+2n)!}\sum_{i=0}^{g}\frac{(4g-4+n)!(4g-7+2n+i)!!}{(2g-3+n+i)!(4g-7+i)!!}\cdot$$

$$\cdot\int_{\overline{\mathcal{M}}_{g,2g-3+2n+i}}\psi_1^2\dots\psi_{2g-3+i}^2\lambda_{g-i}$$

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$$\cdot\int_{\overline{\mathcal{M}}_{g,2g-3+2n+i}}\psi_1^2\dots\psi_{2g-3+i}^2\lambda_{g-i}$$

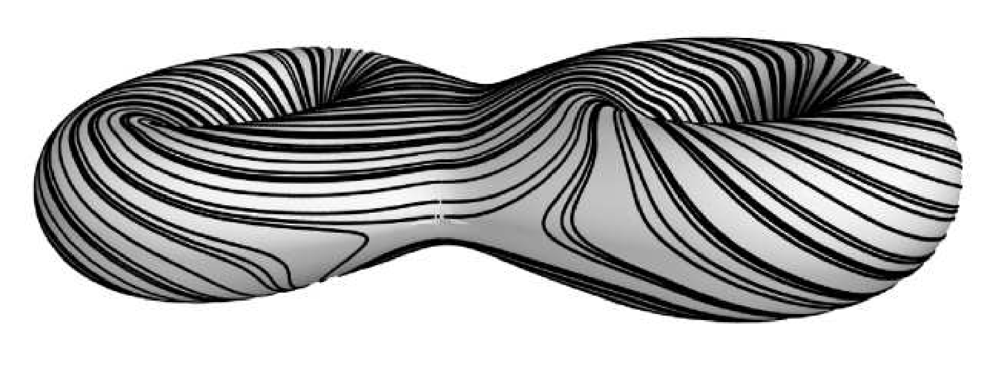
Theorem (M. Kazarian'19; D. Yang–D. Zagier–Y. Zhang'20). *Linear Hodge integrals as above admit simple and very explicit recursion in the spirit of the Virasoro constraints.*

Mirzakhani's count of simple closed geodesics

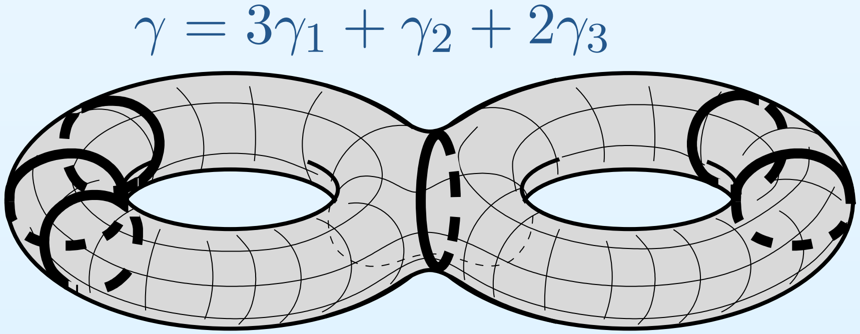
Simple closed multicurve, its topological type and underlying primitive multicurve

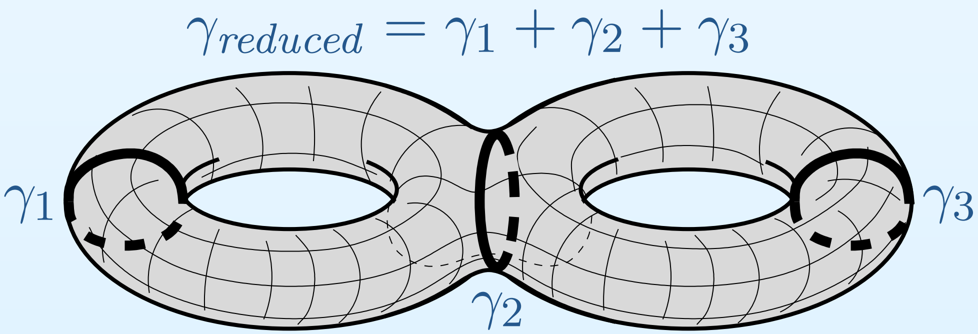
The first homology $H_1(M^2;\mathbb{Z})$ of the surface is great to study closed curves, but it ignores some interesting curves. The fundamental group $\pi_1(M^2)$ is also wonderful, but it is mainly designed to work with self-intersecting cycles. Thurston invented yet another structure to work with simple closed multicurves; in many aspects it resembles the first homology, but there is no group structure.

A general multicurve ρ:



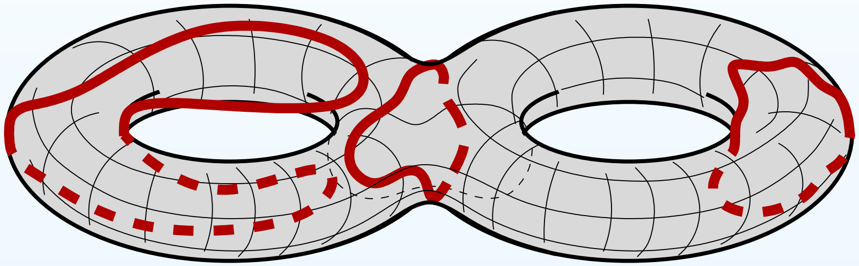
the canonical representative $\gamma = 3\gamma_1 + \gamma_2 + 2\gamma_3$ in its orbit $\mathrm{Mod}_2 \cdot \rho$ under the action of the mapping class group and the associated *reduced* multicurve.

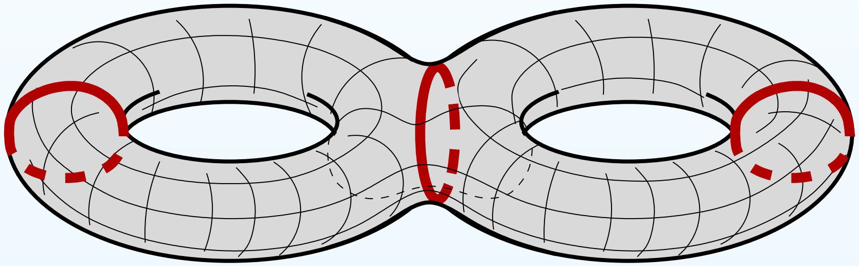




Geodesic representatives of multicurves

Consider several pairwise nonintersecting essential simple closed curves $\gamma_1, \ldots, \gamma_k$ on a smooth surface $S_{g,n}$ of genus g with n punctures. In the presence of a hyperbolic metric X on $S_{g,n}$ the simple closed curves $\gamma_1, \ldots, \gamma_k$ contract to simple closed geodesics.





Fact. *For any hyperbolic metric X the simple closed geodesics representing $\gamma_1, \ldots, \gamma_k$ do not have pairwise intersections.*

We define the hyperbolic length of a multicurve $\gamma := \sum_{i=1}^{k} a_i \gamma_i$ as $\ell_\gamma(X) := \sum_{i=1}^{k} a_i \ell_X(\gamma_i)$, where $\ell_X(\gamma_i)$ is the hyperbolic length of the simple closed geodesic in the free homotopy class of γ_i.

Denote by $s_X(L, \gamma)$ the number of simple closed geodesic multicurves on X of topological type $[\gamma]$ and of hyperbolic length at most L.

Frequencies of multicurves

Theorem (Mirzakhani'08). *For any integral multi-curve γ and any hyperbolic surface X in $\mathcal{M}_{g,n}$ the number $s_X(L,\gamma)$ of simple closed geodesic multicurves on X of topological type $[\gamma]$ and of hyperbolic length at most L has the following asymptotics:*

$$s_X(L,\gamma) \sim \mu_{\mathrm{Th}}(B_X) \cdot \frac{c(\gamma)}{b_{g,n}} \cdot L^{6g-6+2n} \quad \text{as } L \to +\infty\,.$$

Here $\mu_{\mathrm{Th}}(B_X)$ depends only on the hyperbolic metric X; the constant $b_{g,n}$ depends only on g and n; $c(\gamma)$ depends only on the topological type of γ and admits a closed formula (in terms of the intersection numbers of ψ-classes).

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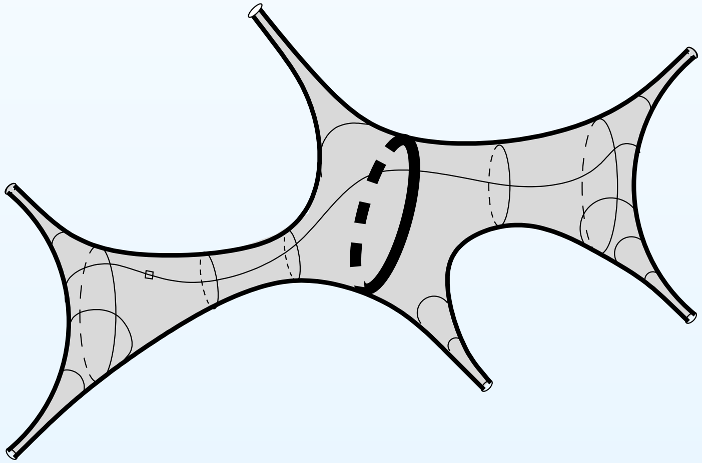
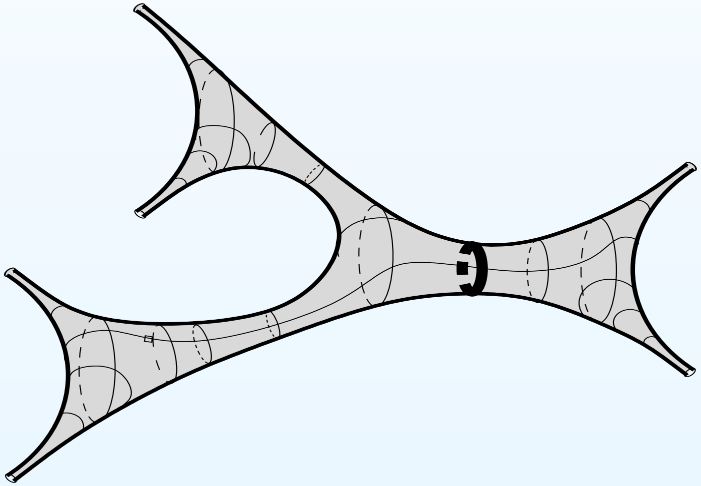
Corollary (Mirzakhani'08). *For any hyperbolic surface X in $\mathcal{M}_{g,n}$, and any two rational multicurves γ_1, γ_2 on a smooth surface $S_{g,n}$ considered up to the action of the mapping class group one obtains*

$$\lim_{L\to+\infty} \frac{s_X(L,\gamma_1)}{s_X(L,\gamma_2)} = \frac{c(\gamma_1)}{c(\gamma_2)}\,.$$

Example

A simple closed geodesic on a hyperbolic sphere with six cusps separates the sphere into two components. We either get three cusps on each of these components (as on the left picture) or two cusps on one component and four cusps on the complementary component (as on the right picture). Hyperbolic geometry excludes other partitions.





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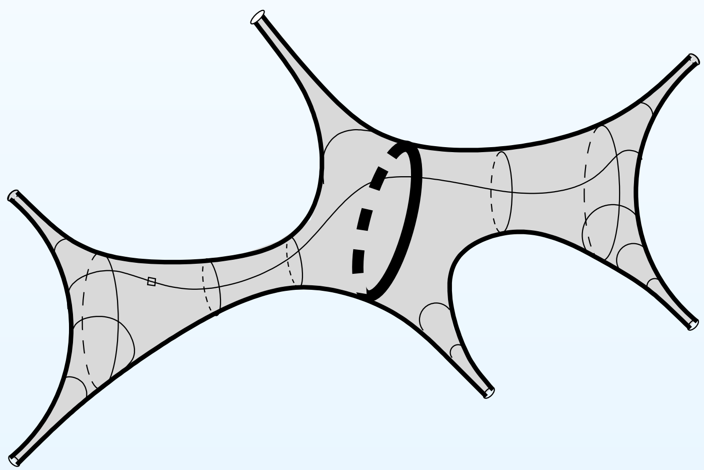
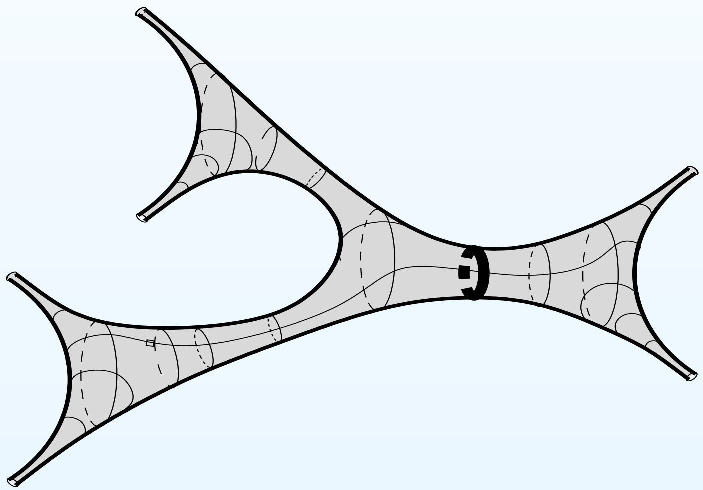
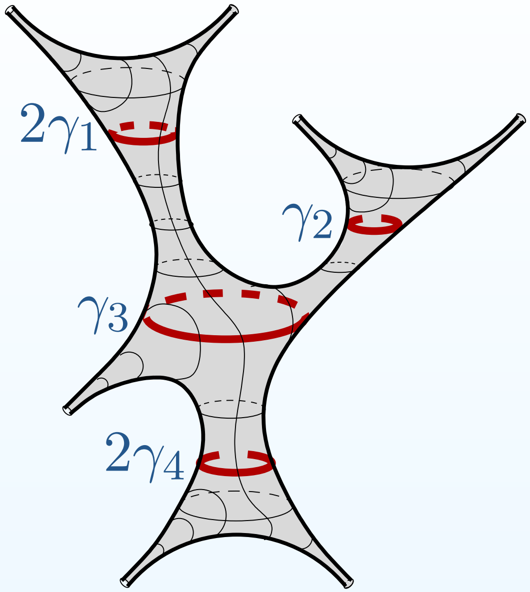


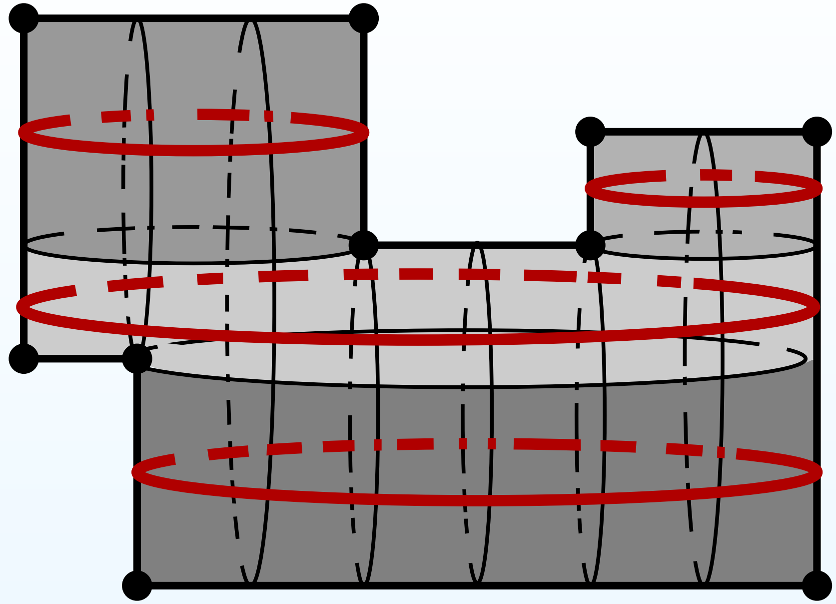


Example (Mirzakhani'08); confirmed experimentally in 2017 by M. Bell; confirmed in 2017 by more implicit computer experiment of V. Delecroix and by relating it to Masur–Veech volume.

$$\lim_{L\to+\infty} \frac{\text{Number of } (3+3)\text{-simple closed geodesics of length at most } L}{\text{Number of } (2+4)\text{- simple closed geodesics of length at most } L} = \frac{4}{3}.$$

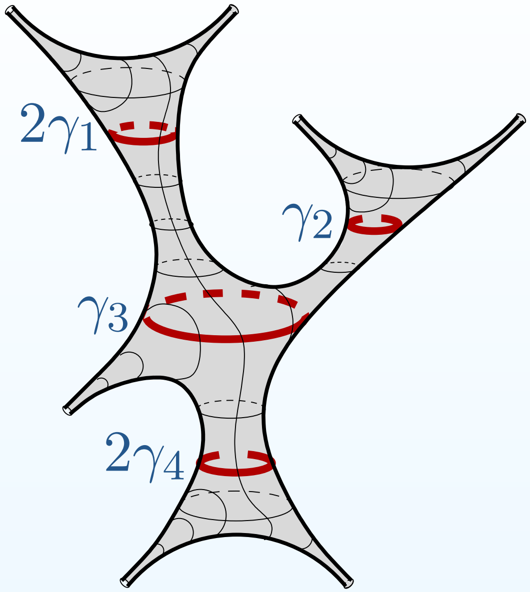
Hyperbolic and flat geodesic multicurves

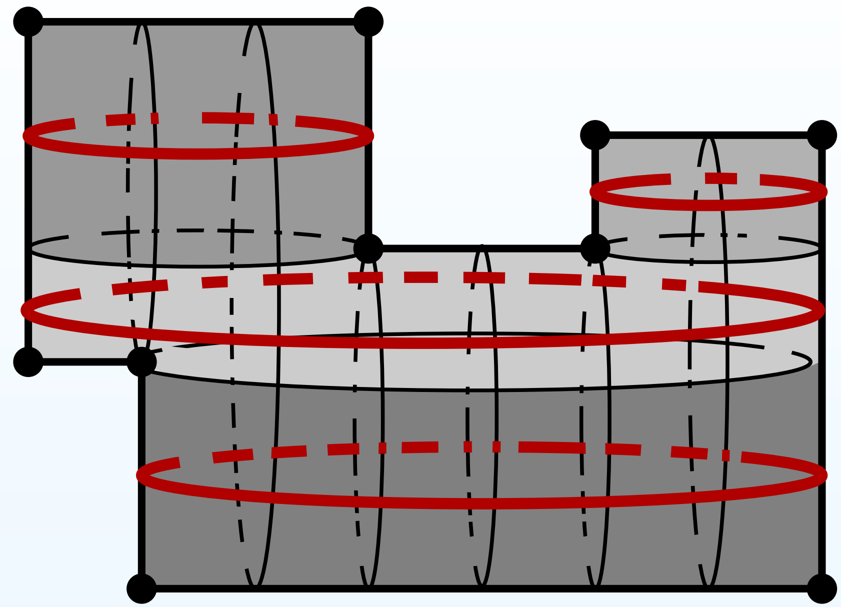




Left picture represents a geodesic multicurve $\gamma = 2\gamma_1 + \gamma_2 + \gamma_3 + 2\gamma_4$ on a hyperbolic surface in $\mathcal{M}_{0,7}$. Right picture represents the same multicurve this time realized as the union of the waist curves of horizontal cylinders of a square-tiled surface of the same genus, where cusps of the hyperbolic surface are in the one-to-one correspondence with the conical points having cone angle π (i.e. with the simple poles of the corresponding quadratic differential). The weights of individual connected components γ_i are recorded by the heights of the cylinders. Clearly, there are plenty of square-tiled surface realizing this multicurve.

Hyperbolic and flat geodesic multicurves



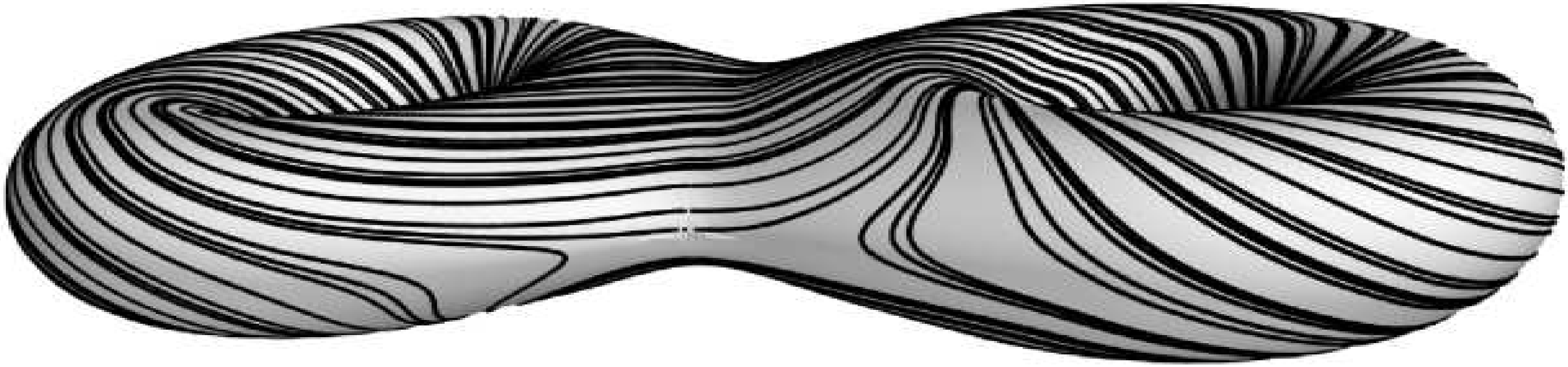


Theorem (Delecroix–Goujard–Zograf–Zorich'21). *For any topological class γ of simple closed multicurves considered up to homeomorphisms of a surface $S_{g,n}$, the associated Mirzakhani's asymptotic frequency $c(\gamma)$ of **hyperbolic** multicurves coincides with the asymptotic frequency of simple closed **flat** geodesic multicurves of type γ represented by associated square-tiled surfaces.*

Remark. Francisco Arana Herrera recently found an alternative proof of this result. His proof uses more geometric approach.

Shape of a random multicurve on a surface of genus two

What shape has a random simple closed multicurve?



Picture from a book of Danny Calegari

Questions.

• *Which simple closed geodesics are more frequent: separating or non-separating?*

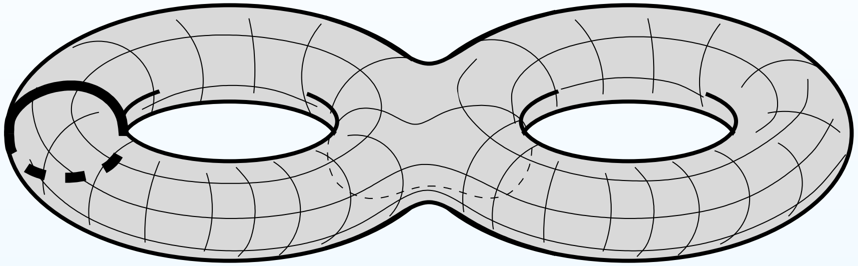
Take a random (non-primitive) multicurve $\gamma = m_1\gamma_1 + \cdots + m_k\gamma_k$. Consider the associated reduced multicurve $\gamma_{reduced} = \gamma_1 + \cdots + \gamma_k$.

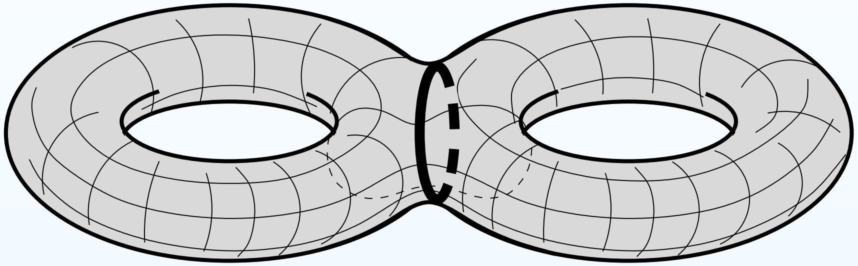
• *What is the probability that $\gamma_{reduced}$ separates S into distinct connected components?*

• *What are the probabilities that $\gamma_{reduced}$ has $k = 1, 2, 3$ primitive connected components $\gamma_1, \ldots, \gamma_k$?*

Separating versus non-separating simple closed curves in $g = 2$

Ratio of asymptotic frequencies (Mirzakhani'08). *Genus* $g = 2$





$$\lim_{L \to +\infty} \frac{\text{Number of } \textbf{separating} \text{ simple closed geodesics of length at most } L}{\text{Number of } \textbf{non-separating} \text{ simple closed geodesics of length at most } L} = \frac{1}{48}$$

Random simple closed curve rarely separates

Theorem (V. Delecroix, E. Goujard, A. Zorich'19). *A random simple closed curve on a surface of large genus separates the surface very rarely. Namely:*

$$\frac{c(\gamma_{sep})}{c(\gamma_{nonsep})} \sim \sqrt{\frac{2}{3\pi g}} \cdot \frac{1}{4^g} \quad \text{as } g \to +\infty\,,$$

An integer multiple $m\gamma$ of a simple closed curve γ has weight m with probability $\frac{1}{m^{6g-6}} \cdot \frac{1}{\zeta(6g-6)}$. Thus, a random one-cylinder square-tiled surface of large genus has height 1 with probability very close to 1.

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Idea of the proof. Frequencies of separating simple closed curves are expressed in terms of the intersection numbers which admit closed expression:

$$\int_{\overline{\mathcal{M}}_{g,1}} \psi_1^{3g-2} = \frac{1}{24^g\, g!}\,.$$

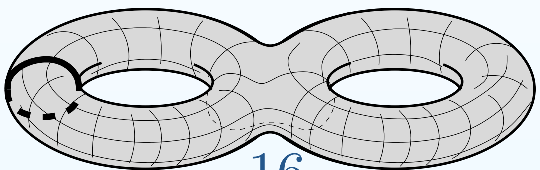
Frequencies of non-separating simple closed curves are expressed in terms of

$$\int_{\overline{\mathcal{M}}_{g,2}} \psi_1^k \psi_2^{3g-1-k}$$

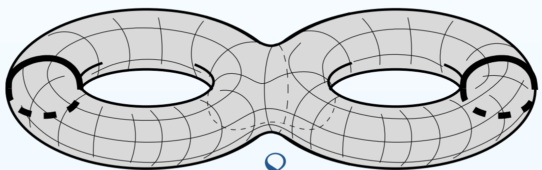
for which we obtain large genus asymptotics uniform for all k in fixed genus g.

Multicurves on a surface of genus two and their frequencies

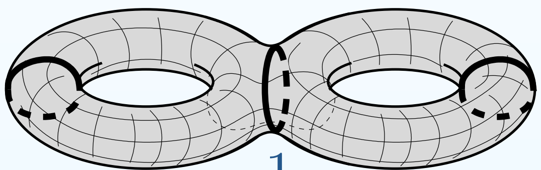
The picture below illustrates all topological types of primitive multicurves on a surface of genus two without punctures; the fractions give frequencies of non-primitive multicurves γ having a reduced multicurve $\gamma_{reduced}$ of the corresponding type.



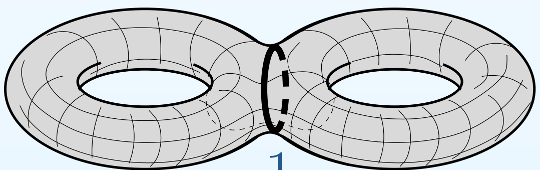
$\frac{16}{63}$



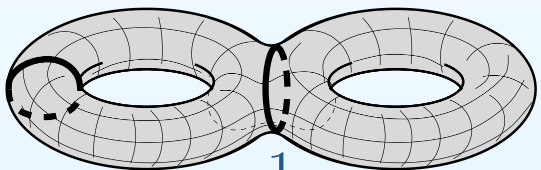
$\frac{8}{15}$



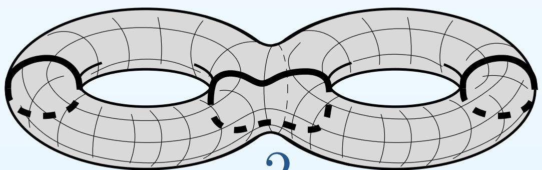
$\frac{1}{9}$



$\frac{1}{189}$



$\frac{1}{45}$



$\frac{2}{27}$

In genus 3 there are already 41 types of multicurves, in genus 4 there are 378 types, in genus 5 there are 4554 types and this number grows faster than exponentially when genus g grows. It becomes pointless to produce tables: we need to extract a reasonable sub-collection of most common types which ideally, carry all Thurston's measure when $g \to +\infty$.

Shape of a random multicurve on a surface of large genus. Shape of a random square-tiled surface of large genus.

Statistics of prime decompositions: random integer numbers

The Prime Number Theorem states that an integer number n taken randomly in a large interval $[1, N]$ is prime with asymptotic probability $\frac{\log N}{N}$.

Actually, one can tell much more about prime decomposition of a large random integer. Denote by $\omega(n)$ the number of prime divisors of an integer n counted without multiplicities. In other words, if n has prime decomposition $n = p_1^{m_1} \dots p_k^{m_k}$, let $\omega(n) = k$. By the Erdős–Kac theorem, the centered and rescaled distribution prescribed by the counting function $\omega(n)$ tends to the normal distribution:

Erdős–Kac Theorem (1939)

$$\lim_{N \to +\infty} \frac{1}{N} \operatorname{card} \left\{ n \le N \,\middle|\, \frac{\omega(n) - \log\log N}{\sqrt{\log\log N}} \le x \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} \, dt \,.$$

The subsequent results of A. Selberg (1954) and of A. Rényi and P. Turán (1958) describe the rate of convergence.

Statistics of prime decompositions: random permutations

Denote by $\mathrm{K}_n(\sigma)$ the number of disjoint cycles in the cycle decomposition of a permutation σ in the symmetric group S_n. Consider the uniform probability measure on S_n. A random permutation σ of n elements has exactly k cycles in its cyclic decomposition with probability $\mathbb{P}\big(\mathrm{K}_n(\sigma) = k\big) = \frac{s(n,k)}{n!}$, where $s(n,k)$ is the unsigned Stirling number of the first kind. It is immediate to see that $\mathbb{P}\big(\mathrm{K}_n(\sigma) = 1\big) = \frac{1}{n}$. V. L. Goncharov computed the expected value and the variance of K_n as $n \to +\infty$:

$$\mathbb{E}(\mathrm{K}_n) = \log n + \gamma + o(1)\,, \qquad \mathbb{V}(\mathrm{K}_n) = \log n + \gamma - \zeta(2) + o(1)\,,$$

and proved the following central limit theorem:

Theorem (V. L. Goncharov, 1944)

$$\lim_{n\to+\infty} \frac{1}{n!} \operatorname{card} \left\{ \sigma \in S_n \;\middle|\; \frac{\mathrm{K}_n(\sigma) - \log n}{\sqrt{\log n}} \leq x \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\, dt\,.$$