TENTATIVE NOTE FOR PART 1

Contents

Part 1. Symplectic manifold

1. Symplectic manifold and Hamiltonian Vector Fields

1.1. **Definition.** Let X be a (real) $2n$ dimensional manifold and ω a differential 2-form on it.

Definition 1. ω is said to be a symplectic structure of X if and only if:
(1) $d\omega = 0$.
(2) The $2n$-form ω^n never vanishes.

A pair (X, ω) is said to be a symplectic manifold.

We remark that at each point $x \in X$ a 2-form ω on X defines an anti-symmetric bilinear map

$$T_xX \otimes T_xX \to \mathbb{R}. \tag{1}$$

Excercise 2. Show that ω^n is non-zero at x if and only if (1) is non-degenerate.

Using local coordinate, ω is written as

$$\omega = \sum \omega_{ij} dx^i \wedge dx^j.$$

Here ω_{ij} is anti-symmetric.

A Riemannian metric g on X can be written

$$g = \sum g_{ij} dx^i dx^j.$$

Here g_{ij} is symmetric. These two notions look similar but there are various serious difference. Here we emphasize the next fact. Suppose X is a compact manifold.

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(I) For a Riemannian metric g the group

$$\mathrm{Aut}(X, g) = \{\varphi : X \to X \mid \varphi^* g = g\}$$

of automorphisms (isometries) of (X, g) is a *finite dimensional* Lie group.

(II) For a symplectic form ω the group

$$\mathrm{Aut}(X, \omega) = \{\varphi : X \to X \mid \varphi^* \omega = \omega\}$$

of automorphisms (symplectic diffeomorphisms) of (X, ω) is infinite dimensional.

Proof of (I) *(Sketch).* We may assume X is connected. We put

$$SX = \{(x, V) \mid x \in X, V \in T_x X, g(V, V) = 0\}.$$

We fix $x_0 \in X$ and an orthonormal basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ of $T_{x_0}X$. We define

$$I : \mathrm{Aut}(X, g) \to (SX)^n$$

by

$$I(\varphi) = (\varphi(x_0), (D_{x_0}\varphi)(\mathbf{e}_i))_{i=1}^n$$

Lemma 3. *I is injective.*[1]

The lemma implies that $\mathrm{Aut}(X, g)$ is finite dimensional. We omit the proof that it is a Lie group.

Sketch of the proof of Lemma 3. □

Suppose $I(\varphi_1) = I(\varphi_2)$. Put $x = \varphi_1(x_0) = \varphi_2(x_0)$. Let $\varphi = \varphi_2 \circ \varphi_1^{-1}$. We have $\varphi(x) = x$. Moreover $(D_x\varphi) : T_x X \to T_x X$ is the identity map.

Let y is an arbitrary point of X. There exists a geodesic $\gamma : [0, T] \to X$ such that $\gamma(0) = x$, $\gamma(T) = y$. Then $\varphi \circ \gamma$ is a geodesic joining x and $\varphi(y)$. Geodesic satisfies a second order ODE (See Theorem 31). Since

$$\frac{D}{dt}\gamma(0) = \frac{D}{dt}(\varphi \circ \gamma)(0),$$

uniqueness of the solution of ODE implies $\gamma = \varphi \circ \gamma$. Hence $y = \varphi(y)$. □

Proof of (II). For the proof we need to find a lot of automorphisms of (X, ω). We use Hamiltonian diffeomorphism for this purpose. The proof is completed at the end of Subsection 4.3. □

To define and study Hamiltonian diffeomorphisms we review some elementary facts on calculus of manifolds.

[1] It is actually a diffeormorphism onto a smooth submanifold.

1.2. **Vector fields on manifolds: Review.**

Definition 4. Let X be a manifold and V a vector field on it. We define one parameter group of transformations $\exp_V^t : X \to X$ ($t \in \mathbb{R}$) associated to V by the following two equalities.

(1) $\exp_V^0(x) = x$.
(2) $\frac{d}{dt}\exp_V^t(x)|_{t=t_0} = V(\exp_V^{t_0}(x))$.

It satisfies the next equality also:

(3) $\exp_V^t \circ \exp_V^{t'} = \exp_V^{t+t'}$.

Remark 5. In the case X is compact without boundary such $\exp_V^t$ exists for all $t \in \mathbb{R}$. In the case X is non-compact, there exists T_K, such $(t,x) \mapsto \exp_V^t(x)$ for $t \in [-T_K, T_K]$ and $x \in K$ for any compact subset K.

We next review Lie derivative of a tensor field. We remark that a tensor $\mathcal{T}$ (called $\ell - k$ tensor) is a section of the tensor product vector bundle

$$TX^{k\otimes} \otimes (TX^*)^{\ell\otimes}.$$

Using a local coordinate x_i it is written as

$$\mathcal{T} = \sum T^{i_1\dots i_k}_{j_1\dots j_\ell} \frac{\partial}{\partial x^{j_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{j_\ell}} \otimes dx^{i_1} \otimes \dots \otimes dx^{i_k},$$

where $T^{i_1\dots i_k}_{j_1\dots j_\ell}$ are smooth functions. If $\varphi : X \to X$ is a diffeomorphism it induces isomorphism $D_x\varphi : T_xX \to T_{\varphi(x)}X$. The inverse of its dual induces $(D_x\varphi^*)^{-1} : T^*_xX \to T^*_{\varphi(x)}X$. They induce

$$T_xX^{k\otimes} \otimes (T_xX^*)^{\ell\otimes} \to T_{\varphi(x)}X^{k\otimes} \otimes (T_{\varphi(x)}X^*)^{\ell\otimes},$$

which we denote by $(D\varphi)_*$ for simplicity. We denote by $(D\varphi)^*$ its inverse.

Definition 6. The Lie derivative $L_V\mathcal{T}$ of a tensor $\mathcal{T}$ by a vector field V is defined by

$$L_V\mathcal{T} = \lim_{t\to 0} \frac{(D\exp_V^t)^*\mathcal{T} - \mathcal{T}}{t}$$

Lemma 7. *For a $(0,0)$ tensor f we have $L_Vf = V(f)$.*

For a vector field W (that is a $(1,0)$ tensor) we have

$$L_VW = [V,W],$$

where the right hand side is defined by

$$[V,W](f) = V(W(f)) - W(V(f)).$$

See ??? for the proof.

Excercise 8. Prove the following identity. (the Jacobi Identity)

$$[[U,V],W] + [[W,U],V] + [[V,W],U] = 0.$$

We next review inner product. The differential k-form is a $(0,k)$ tensor u such that
$$u_x : (T_xX)^{k\otimes} \to \mathbb{R}$$
is anti-symmetric:
$$u_x(V_1,\dots,V_k) = -u_x(\dots V_{i-1}, V_j, V_i, \dots, V_{j-1}, V_i, V_j, \dots).$$

Definition 9. Let u be a differential k form and V a vector. We define differential $(k-1)$ form $i_V u$ by
$$(i_V u)(W_1,\dots,W_{k-1}) = u(V, W_1, \dots, W_{k-1}).$$

The next lemma is important to study Hamiltonian vector field.

Lemma 10. (Cartan's formula) *For a differentia form u and a vector field V we have*
$$(i_V \circ d + d \circ i_V)u = L_V u. \tag{2}$$

See Subsection 1.5 for the proof.

1.3. **Hamiltonian Vector field.** Let (X,ω) be a symplectic manifold and $H : X \to R$ be a (smooth) function.

Definition 11. The Hamiltonian vector field $\mathfrak{X}_H$ generated by H is defined by the next formula.
$$\omega(\mathfrak{X}_H, V) = (dH)(V). \tag{3}$$
Here V is an arbitrary vector field.

The non-degeneracy of ω implies the unique existence of $\mathfrak{X}_H$.

Let $V_* = \{V_t\}$ be a $t \in \mathbb{R}$ dependent family of vector fields. (We require $t \to V_t(f)$ is smooth on t for any function f.) One parameter group of transformations $\exp^t_{V_*}$ is defined in a similar way as Definition 4 by the next formula:

(1) $\exp^0_{V_*}(x) = x$.
(2) $\frac{d}{dt}\exp^t_{V_*}(x)|_{t=t_0} = V(\exp^{t_0}_{V_{t_0}}(x))$.

Definition 12. A diffeomorphism $\varphi : X \to X$ is said to be a *Hamiltonian diffeomorphism* if there exists H_t a $t \in \mathbb{R}$ dependent family of smooth functions such that
$$\varphi = \exp^1_{\mathfrak{X}_{H_t}}$$
Here $\mathfrak{X}_{H_t}$ is the Hamiltonian vector field generated by H_t and we regard it as a $t \in \mathbb{R}$ dependent family of vector fields.

Excercise 13. Show that if φ, φ' are Hamiltonian diffeomorphisms then their composition $\varphi \circ \varphi'$ is a Hamiltonian diffeomorphism.

Proposition 14. *If φ is a Hamiltonian diffeomorphism then*
$$\varphi^*\omega = \omega.$$

For the proof we first prove its 'infinitesimal version'.

Lemma 15. *The Hamiltonian vector field $\mathfrak{X}_H$ satisfies:*

$$L_{\mathfrak{X}_H}\omega = 0.$$

Proof. We remark that Formula (3) can be written as

$$i_{\mathfrak{X}_H}\omega = dH.$$

We now use Cartan's formula (2) and $d\omega = 0$ to find:

$$L_{\mathfrak{X}_H}\omega = i_{\mathfrak{X}_H}(d\omega) + d(i_{\mathfrak{X}_H}\omega) = ddH = 0.$$

□

Proof of Proposition 14. We consider

$$\omega_t := (\exp^t_{\mathfrak{X}_{H_t}})^*\omega.$$

It satisfies the differential equation

$$\frac{d}{dt}\omega_t = L_{\mathfrak{X}_{H_t}}\omega_t. \tag{4}$$

together with initial condition : $\omega_0 = \omega$. Note (4) is an ordinary differential equation of first order on an appropriate Banach space, so its solution with given initial condition is unique. Lemma 15 implies that $\omega_t \equiv \omega$ is a solution. Therefore $\omega_t = \omega$ as required. □

We remark that $\mathfrak{X}_H = \mathfrak{X}_G$ if and only if $d(H - G) = 0$. Therefore Proposition 14 implies that $\mathrm{Aut}(X, \omega)$ is infinite dimensional.

1.4. **Poisson's Bracket.**

Definition 16. For functions $f, g : X \to \mathbb{R}$ on a symplectic manifold, we define ifs Poisson Braket $\{f, g\}$ by:

$$\{f, g\} = df(\mathfrak{X}_g) = \mathfrak{X}_g(f).$$

By definition $df(\mathfrak{X}_g) = \omega(\mathfrak{X}_f, \mathfrak{X}_g)$. Therefore

$$\{f, g\} = -\{g, f\}. \tag{5}$$

Proposition 17.

$$\mathfrak{X}_{\{f,g\}} = -[\mathfrak{X}_f, \mathfrak{X}_g]. \tag{6}$$

For the proof we use:

Lemma 18.

$$[L_V, i_W] = i_{L_V W}. \tag{7}$$

Proof of Lemma 18.

$$
\begin{aligned}
&([L_V, i_W](u))(V_1,\dots,V_k)\\
=&(L_V(i_W u))(V_1,\dots,V_k) - (L_V u)(W,V_1,\dots,V_k)\\
=&V(u(W,V_1,\dots,V_k)) - \sum_{i=1}^k (-1)^i u(W,\dots,[V,V_i],\dots)\\
&- V(u(W,V_1,\dots,V_k) - u([V,W],\dots) + \sum_{i=1}^k (-1)^i u(W,\dots,[V,V_i],\dots)\\
&= (i_{L_V W} u)(V_1,\dots,V_k).
\end{aligned}
$$

□

Proof of Proposition 17.

$$L_{\mathfrak{X}_f} i_{\mathfrak{X}_g}\omega = i_{\mathfrak{X}_g} L_{\mathfrak{X}_f}\omega + i_{L_{\mathfrak{X}_f}\mathfrak{X}_g}\omega = i_{[\mathfrak{X}_f,\mathfrak{X}_g]}\omega.$$

On the other hand,

$$L_{\mathfrak{X}_f} i_{\mathfrak{X}_g}\omega = L_{\mathfrak{X}_f} dg = d\mathfrak{X}_f(g) = -d\{f,g\}.$$

□

Proposition 19. *Poisson bracket satisfies*

$$\{\{f,g\},h\} + \{\{h,f\},g\} + \{\{g,h\},f\} = 0. \tag{8}$$

The fact the left hand side is constant follows from Proposition 17 and Excersise 8. We will prove the fact that this constant is 0 later.

Proposition 19 and (5) implies that $(C^\infty(X),\{\})$ is a Lie algebra. Proposition 17 imply that $f \mapsto \mathfrak{X}_f$ is an anti Lie algebra homomorphism from $(C^\infty(X),\{\})$ to the Lie algebra of vector fields.

We remark that if $\{f,g\} = 0$ then g is constant on the orbit of $\mathfrak{X}_f$. In particular f is constant on the orbit of $\mathfrak{X}_f$.

1.5. **Proof of Cartan's Formula.** We prove Lemma 10. For differential k-form u is exterior differential du is given by

$$
\begin{aligned}
&(du)(V_0,\dots,V_k)\\
&= \sum (-1)^i V_i(u(\dots,\overset{\vee}{V}_i,\dots)) + \sum_{i<j} (-1)^{i+1} u([V_i,V_j],\dots,\overset{\vee}{V}_i,\dots,\overset{\vee}{V}_j,\dots)).
\end{aligned}
\tag{9}
$$

Excercise 20. Show (9). Here use the definition

$$d(f dx^{i_1}\wedge\cdots\wedge dx^{i_k}) = df \wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}, \quad df = \sum \frac{\partial f}{\partial x^i} dx^i$$

for the left hand side.

Now we calculate

$$
\begin{aligned}
((i_X \circ d)u)(V_1,\dots,V_k) =& (du)(X,V_1,\dots,V_k) \\
=& X(u(V_1,\dots,V_k)) + (-1)^i V_i(X,\dots,\check{V}_i,\dots) \\
&+ \sum (-1)^{i+1} u([X,V_i],\dots,\check{V}_i,\dots) \\
&+ \sum_{i<j} (-1)^{i+j} u([V_i,V_j],X,\dots,\check{V}_i,\dots,\check{V}_j,\dots)
\end{aligned}
$$

$$
\begin{aligned}
((d \circ i_X)u)(V_1,\dots,V_k) =& \sum (-1)^{i-1} V_i(X,\dots,\check{V}_i,\dots) \\
&+ \sum_{i<j} (-1)^{i+j} u(X,[V_i,V_j],\dots,\check{V}_i,\dots,\check{V}_j,\dots).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
((i_X \circ d + i_X \circ d)u)(V_1,\dots,V_k) &= X(u(V_1,\dots,V_k)) \\
&\quad + \sum (-1)^{i+1} u([X,V_i],\dots,\check{V}_i,\dots) \\
&= (L_X u)(V_1,\dots,V_k).
\end{aligned}
$$

□

2. Example of Symplectic manifolds 1: Cotangent bundle and brief review of Hamiltonian mechanics.

2.1. **Symplectic structure on the cotangent bundle.** The most important example of symplectic manifold is the cotangent bundle T^*M of a manifold M. An element of T^*M is a pair (x,v) where $x \in M$ and $v : T_xM \to \mathbb{R}$ is a linear map from the tangent space T_xM. (Namely $v \in T_x^*M$.)

Definition 21. The canonical one form? θ on T^*M is defined as follows. Wen consider the projection $\pi : T^*M \to M$. Let $(x,v) \in T^*M$. π induces $D\pi : T_{(x,v)}T^*M \to T_xM$. We put

$$\theta(V) = v(D\pi(V))$$

for $V \in T_{(x,v)}T^*M$.

Lemma 22. *$\omega = d\theta$ is a symplectic form on T^*M.*

Proof. $d\omega = 0$ is obvious. We prove ω^n never vanishes where $n = \dim M$. The problem is local. We take a local coordinate $q^1,\dots,q^n$ of M. An element $v \in T^*xM$ then is written as $\sum p_i dq^i$ where $p^i \in \mathbb{R}$. In fact $p_i = v(\partial/\partial q^i)$. If the coordinate of x is $(q^1,\dots,q^n)$ then we associate $(q^1,\dots,q^n,p_1,\dots,p_n)$ to $(x,v) \in T^*M$. Thus $(q^1,\dots,q^n,p_1,\dots,p_n)$ is a coordinate of T^*M. By definition it is easy to see

$$\theta = \sum_{i=1}^n p_i dq^i$$

in this coordinate. Therefore

$$\omega^n = (\sum dp_i \wedge dq^i)^n = n! dp_1 \wedge dq^1 \wedge \cdots \wedge dp_n \wedge dq^n,$$

which never vanishes. □

Excercise 23. Let $F : M \to N$ be a diffeomorphism. It induces a diffeomorphism $F_* : T^*M \to T^*N$. Prove $(F_*)^*\theta = \theta$.

During the proof of Lemma 22 we showed

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq^i. \tag{10}$$

Suppose $H : T^*M \to \mathbb{R}$ be a function. Using (10) and the definition, we find that the Hamiltonian vector field generated by H is

$$\mathfrak{X}_H = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i} - \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} \right). \tag{11}$$

Therefore the Poisson bracket is given by:

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} \right). \tag{12}$$

The equation that $t \mapsto (q^1(t), \dots, q^n(t), p_1(t), \dots, p_n(t))$ is the integral curve of $\mathfrak{X}_H$ is:

$$\begin{cases} \dfrac{dq^i}{dt} = \dfrac{\partial H}{\partial p_i} \\ \dfrac{dp_i}{dt} = -\dfrac{\partial H}{\partial q^i} \end{cases} \tag{13}$$

(13) is called Hamilton equation.

2.2. **Euler-Lagrange equation.** In this and the next subsection we briefly review how Hamilton equation (13) appeared in mechanics.

Let $L : TM \to \mathbb{R}$ be a smooth function on the tangent bundle M. We call L the Lagrangian function. For a curve $\gamma : [0, T] \to M$ in M we define the Lagrangian functional $\mathcal{L}(\gamma)$ by

$$\mathcal{L}(\gamma) = \int_0^T L(\gamma(t), \dot{\gamma}(t)) dt. \tag{14}$$

Here $\dot{\gamma}(t)$ is the tangent of the curve $\in T_{\gamma(t)}M$.

We consider a family of curves $\gamma_s : [0, T] \to M$ for $s \in [-\epsilon, \epsilon]$ such that:

(1) $\gamma_0 = \gamma$.
(2) $\gamma_s(0)$ and $\gamma_s(T)$ are independent of s.

Let x^i be a local coordinate of M. An element of T_xM is written as $\sum y^i \partial/\partial x^i$. Thus $x^1, \dots, x^n, y^1, \dots, y^n$ are coordinates of TM.

Theorem 24. *The following two conditions for γ are equivalent.*

(I) *For any γ_s as in (1)(2) above*
$$\frac{d}{ds}\mathcal{L}(\gamma_s)|_{s=0} = 0.$$

(II) *We use local coordinate to write $\gamma(t) = (x^1(t), \dots, x^n(t), y^1(t), \dots, y^n(t))$. Then*
$$\frac{\partial L}{\partial x^i}(\gamma(t), \dot{\gamma}(t)) - \frac{d}{dt}\frac{\partial L}{\partial y^i}(\gamma(t), \dot{\gamma}(t)) = 0. \tag{15}$$

(15) is called the Euler-Lagrange equation.

Proof. (I) $\Rightarrow$ (II) Put $(\gamma_s(t), \dot{\gamma_s}(t) = (x_s^1(t), \dots, y_s^n(t))$ We have $y_s^i(t) = dx_s^i(t)/dt$. We calculate:
$$\begin{aligned}\frac{d}{ds}\mathcal{L}(\gamma_s) &= \int_0^T \sum \left(\frac{\partial L}{\partial x^i}\frac{dx_s^i(t)}{ds} + \frac{\partial L}{\partial y^i}\frac{d^2 x_s^i(t)}{dtds}\right) dt \\ &= \int_0^T \sum \left(\frac{\partial L}{\partial x^i} - \frac{\partial^2 L}{\partial t \partial y^i}\right)\frac{dx_s^i(t)}{ds} dt\end{aligned}$$

Here we use (2) and integration by parts. Since this vanish for all $\frac{dx_s^i(t)}{ds}$ at $s = 0$ we have (15).

(II) $\Rightarrow$ (I) can be proved by looking the formula in the opposite direction. $\square$

Example 25. Let g be a Riemannian metric on M. It induces a function $v \mapsto g(v, v)$ on TM. Writing $g = \sum g_{ij} dx^i dx^j$ $g(v, v) = \sum g_i j v^i v^j$. Let $V : M \to \mathbb{R}$ be a function. We put
$$L(x, v) = \frac{1}{2} g(v, v) - V(x). \tag{16}$$
In case $M = \mathbb{R}^n$ and g is the standard metric,
$$L(x, v) = \frac{1}{2}(\sum y^i)^2 - V(x).$$
In this case (15) is
$$\frac{d^2 x^i}{dt^2} = -\frac{\partial V}{\partial x^i}.$$
This is the equation of the motion of particle under the field of force with potential V.

2.3. **Hamilton's formalism.** We discuss the relation between (15) and (13). Let $L : TM \to \mathbb{R}$ be the function. We define
$$\mathrm{Leg}_L : TM \to T^*M$$
the Legendre transformation, as follows. Let $(x, v) \in TM$. We restrict L to $T_x M$ and differentiate at (x, v). Note $T_v T_x M$ is canonically isomorphic to $T_x M$. Therefore a linear map $D_v L|_{T_x M} : T_x M \to \mathbb{R}$. We put
$$\mathrm{Leg}_L(x, v) = (x, D_v L|_{T_x M}) \in T_x^* M.$$

In local coordinate Legendre transformation is written as follows. Let x^i be a coordinate of M. We put $v = \sum y^i \partial/\partial x^i$. Thus x^i and y^i is an coordinate of TM. We define

$$p_i = \frac{\partial L}{\partial y^i}.$$

Then

$$\mathrm{Leg}_L(x^1, \ldots, x^n, y^1, \ldots, y^n) = (x^1, \ldots, x^n, p_1, \ldots, p_n).$$

Remark 26. p_i is called the momentum conjugate to $q^i = x^i$.

For the Lagrangian (16) Legendrian transformaition is given: $(x^i, y^i) \mapsto (q^i, p_i)$

$$q^i = x^i, \qquad p_i = \sum_j g_{ij} x^j. \tag{17}$$

To discuss relationship between (15) and (13) we assume:

Assumption 27. The Legendre transformation $\mathrm{Leg}_L : TM \to T^*M$ is a diffeomorphism.

In various situations, it suffices Assumption 27 locally, that is, Leg_L is a diffeomorphism between open subsets. For simplicity we require Assumption 27 globally.

Suppose L satisfies Assumption 27. We define $H : T^*M \to \mathbb{R}$ by the next formula:

$$H(x, v) = v((\mathrm{Leg}_L)^{-1}(x, v)) - L(\mathrm{Leg}_L)^{-1}(x, v)). \tag{18}$$

We remark that $(\mathrm{Leg}_L)^{-1}(x, v) \in T_x(M)$. Therefore the first term makes sense.

When we use coordinate x^i, y^i, q^i, p^i then

$$H(q^1, \ldots, p_n) = \sum p_i q^i - L(x^1, \ldots, y^n). \tag{19}$$

Theorem 28. *Under Assumption 27 suppose H and L are related as in (18). Then the next two condition for $\gamma : [0, T] \to M$ are equivalent.*

(I) *γ satisfies (15).*

(II) *$(q^1, \ldots, p_n) := \mathrm{Leg}_L \circ \gamma$ satisfies (13).*

Proof. (II) $\Rightarrow$ (I). Let $(q^1(t), \ldots, p_n(t))$ be a path of T^*M and $(x_1(t), \ldots, y_n(t)) := \mathrm{Leg}_L^{-1}(L(\gamma(t)))$. Suppose (II). We calculate:

$$\frac{\partial H}{\partial p_i} = y_i + \sum_j p_j \frac{\partial y^j}{\partial p_i} - \frac{\partial L}{\partial p_i} = y^i + \sum_j \left(p_j \frac{\partial y^j}{\partial p_i} - \frac{\partial L}{\partial y^i}\frac{\partial y^i}{\partial p_j} \right) = y^i.$$

Here we use $p_i = \frac{\partial L}{\partial y^i}$, $\frac{\partial x^j}{\partial p_i} = 0$.

Therefore the second equation of (13) implies

$$y^(t) = \frac{dx^i(t)}{dt}.$$

Namely $(x^1(t), \ldots, y^n(t)) = (\gamma(t), \dot\gamma(t))$.

We next calculate
$$-\frac{\partial H}{\partial q^i} = \sum_j \left(-p_j \frac{\partial y^j}{\partial q^i} + \frac{\partial L}{\partial x^j}\frac{\partial x^j}{\partial q^i} + \frac{\partial L}{\partial y^j}\frac{\partial y^j}{\partial q^i} \right) = \frac{\partial L}{\partial x^i}.$$
Here we use $\frac{\partial x^j}{\partial q^i} = \delta^{ij}$. On the other hand:
$$\frac{dp_i}{dt} = \frac{\partial^2 L}{\partial t \partial y^i}.$$
Therefore the first equation of (13) is
$$\frac{\partial^2 L}{\partial t \partial y^i} - \frac{\partial L}{\partial x^i} = 0.$$
This is nothing but equation (15).

The proof of (I) $\Rightarrow$ (II) is similar. $\square$

2.4. **Equation of geodesic: an example.** We fix a Riemannian metric g on a manifold M. Let $\gamma : [0,1] \to M$. We consider the following two functional:
$$\mathcal{L}(\gamma) = \int_0^1 \sqrt{g(\dot\gamma(t), \dot\gamma(t))}dt \tag{20}$$
$$E(\gamma) = \frac{1}{2}\int_0^1 g(\dot\gamma(t), \dot\gamma(t))dt. \tag{21}$$
$\mathcal{L}$ is the length of the curve γ and E is called the energy of γ.

Lemma 29. *We have*
$$2E(\gamma) \geqslant \mathcal{L}(\gamma)^2. \tag{22}$$
The equality holds if and only if $t \mapsto g(\dot\gamma(t), \dot\gamma(t))$ is constant.

Proof. We put $f(t) = \sqrt{g(\dot\gamma(t), \dot\gamma(t))}$, $\alpha = \mathcal{L}(\gamma) = \int_0^1 f(t)$ and calculate
$$0 \leqslant \int_0^1 (f(t) - \alpha)^2 dt = 2E(\gamma) - 2\alpha\mathcal{L}(\gamma) + \alpha^2 = 2E(\gamma) - \mathcal{L}(\gamma)^2.$$
The lemma follows. $\square$

Excercise 30.
(1) Show for each $\gamma : [0,1] \to M$ there exists a diffeomorphism $s(t) : [0,1] \to [0,1]$ such that $\widetilde\gamma = \gamma(s(t)) : [0,1] \to [0,1]$ satisfies the condition that $\dot{\widetilde\gamma}$ is constant.
(2) Show $\mathcal{L}(\gamma) = \mathcal{L}(\widetilde\gamma)$.

Lemma 29 and Excercise 30 implies that to obtain a critical point of $\mathcal{L}(\gamma)$ (which is called a geodesic) it suffices to study a critical point of $E(\gamma)$. Let us study the latter by Hamiltonian formalism. We regard
$$L(x^1, \dots, y^n) = \frac{1}{2}\sum g_{ij} y^i y^j$$

as a Lagrangian function. Then Legendre transformation is obtained as (17). Therefore (19) becomes

$$H(q^1,\dots,p_n)=\sum g_{ij}y^iy^j-\frac{1}{2}\sum g_{ij}y^iy^j=L(x^1,\dots,y^n).$$

Namely

$$H(q^1,\dots,p_n)=\frac{1}{2}\sum g^{ij}p_ip_j$$

where (g^{ij}) is the inverse matrix of (g_{ij}). (13) becomes

$$\begin{aligned}\frac{dq^i(t)}{dt}&=\frac{\partial H}{\partial p_i}=\sum_j g^{ij}p_j\\ \frac{dp_i(t)}{dt}&=-\frac{\partial H}{\partial q^i}.\end{aligned}\tag{23}$$

Let us rewrite this equation to the equation of $x^i(t)$. Since $\frac{\partial A^{-1}}{\partial x}=-A^{-1}\frac{\partial A}{\partial x}A^{-1}$ we have

$$\frac{\partial g^{jk}}{\partial q^i}=-\sum_{\ell,m}g^{j\ell}\frac{\partial g_{\ell m}}{\partial q^i}g^{mk}\tag{24}$$

Therefore

$$\frac{dp_i}{dt}=-\frac{1}{2}\sum_{j,k}\frac{\partial g^{jk}}{\partial q^i}p_jp_k=\frac{1}{2}\sum_{j,k,\ell,m}\frac{\partial g_{\ell m}}{\partial q^i}p_jp_kg^{j\ell}g^{mk}\tag{25}$$

We take t derivative of the first formula of (23) and use (24), (25) and the second formula of (23) to obtain:

$$\begin{aligned}\frac{d^2q^i}{dt^2}&=\sum_j\frac{dg^{ij}}{dt}p_j+\sum_j g^{ij}\frac{dp_j}{dt}\\ &=\sum_j\frac{\partial g^{ij}}{\partial q^k}\frac{dq_k}{dt}p_j+\sum_j g^{ij}\frac{dp_j}{dt}\\ &=-\sum_{j,k,\ell,m,n}g^{im}\frac{\partial g_{mn}}{\partial q^k}g^{nj}g^{k\ell}p_\ell p_j+\frac{1}{2}\sum_{j,k,\ell,m,n}g^{ij}\frac{\partial g_{mn}}{\partial q^j}g^{mk}g^{\ell n}p_kp_\ell\\ &=-\sum_{k,m,n}g^{im}\frac{\partial g_{mn}}{\partial q^k}y^ny^k+\frac{1}{2}\sum_{j,m,n}g^{ij}\frac{\partial g_{mn}}{\partial q^j}y^my^n\\ &=\frac{1}{2}\sum_{j,m,n}g^{ij}\left(-2\frac{\partial g_{jm}}{\partial q^n}+\frac{\partial g_{mn}}{\partial q^j}\right)\frac{dq^m}{dt}\frac{dq^n}{dt}.\end{aligned}$$

We define Christoffel's symbol Γ^i_{nm} by:

$$\Gamma^i_{nm}=\frac{1}{2}\sum_{j,m,n}g^{ij}\left(\frac{\partial g_{jm}}{\partial q^n}+\frac{\partial g_{jn}}{\partial q^m}-\frac{\partial g_{mn}}{\partial q^j}\right).\tag{26}$$

We have proved the following:

Theorem 31. *A curve $t \mapsto \gamma(t) = (x^1(t), \ldots, x^n(t))$ is a geodesic with $g(\dot\gamma(t), \dot\gamma(t))$ beging constant if and only if it satisfies*

$$\frac{d^2x^i(t)}{dt^2} + \sum_{m,n} \Gamma^i_{nm} \frac{dx^m}{dt}\frac{dx^n}{dt} = 0. \tag{27}$$

2.5. **Variational principle for Hamilton equation.** Let $x, y \in M$ we consider the set of path $\gamma(t) = (q^1(t), \ldots, p_n(t)) : [0, T] \to T^*M$ such that $\gamma(0) \in T^*_x(M)$, $\gamma(T) \in T^*_y(M)$ which we denote by $\mathcal{P}(T^*M; x, y)$.

Remark 32. In the case of general symplectic manifold X (not necessary cotangent bundle) we consider Lagrangian submanifolds L_1, L_2 of X (see ???) and the set of path γ in X such that $\gamma(0) \in L_1$, $\gamma(T) \in L_2$.

Definition 33. Let H be a smooth function on $\mathbb{R} \times T^*M$ we define action functional $\mathcal{A}_H$ by the next formula.

$$\mathcal{A}_H(\gamma) = \int_0^T \gamma^*\theta - \int_0^T H(t, \gamma(t))dt. \tag{28}$$

Theorem 34. *The following two conditions are equivalent.*

(1) For any one parameter family γ_s of elements of $\mathcal{P}(T^*M; x, y)$ with $\gamma_0 = \gamma$ we have

$$\frac{\partial}{\partial s}\mathcal{A}_H(\gamma_s)|_{s=0} = 0.$$

(2)

$$\begin{cases} \dfrac{dq^i}{dt} = \dfrac{\partial H_t}{\partial p_i} \\ \dfrac{dp_i}{dt} = -\dfrac{\partial H_t}{\partial q^i} \end{cases} \tag{29}$$

Note that (13) is a special case of (29) where H is t-independent.

Proof. We write $\gamma_s(t) = (q^1(s, t), \ldots, p_n(s, t))$. Then

$$\begin{aligned} \frac{d}{ds}\int_0^T \gamma_s^*\theta|_{s=0} &= \int_0^T \frac{\partial p_i}{\partial s}\frac{\partial q^i}{\partial t}dt + \int_0^T p_i \frac{\partial^2 q^i}{\partial t \partial s}dt \\ &= \int_0^T \frac{\partial p_i}{\partial s}\frac{\partial q^i}{\partial t}dt - \int_0^T \frac{\partial p_i}{\partial t}\frac{\partial q^i}{\partial s}dt \end{aligned}$$

Here we use $\frac{\partial q^i}{\partial s}(t) = 0$ for $t = 0, T$ to use integration by parts. On the other hand,

$$\frac{d}{ds}\int_0^T H(t, \gamma_s(t))dt|_{s=0} = \int_0^T \frac{\partial H}{\partial p_i}\frac{\partial p_i}{\partial s}dt + \int_0^T \frac{\partial H}{\partial q^i}\frac{\partial q^i}{\partial s}dt$$

Theorem follows from these calculations immediately. Namely $\frac{d}{ds}\int_0^T \gamma_s^*\theta|_{s=0} = \frac{d}{ds}\int_0^T H(t, \gamma_s(t))dt|_{s=0}$ for all γ_s if and only if (29) holds. □

However it is difficult to use this variational principle to show an existence of a solution of Hamilton equation. For example:

Excercise 35. Prove that $\mathcal{A}_H$ never has local minimum unless M is a point.

On the other hand there are many cases where Lagrangian functional has local minimum, such as the case of geodesic. Only after Floer's invention of Floer homology we can use $\mathcal{A}_H$ to prove the existence of a solution of Hamilton equation.

3. Example of Symplectic manifolds 2: Kähler manifolds.

3.1. **Almost complex structure.** The other type of important example of symplectic manifolds are Kähler manifolds. It is a special type of complex manifolds. We first review almost complex and complex manifolds.

Definition 36. Let X be a manifold. An almost complex structure of M is a family of linear maps $J_x : T_xX \to T_xX$ depending smoothly on $x \in X$ such that:

(*) $J_x \circ J_x = -1$.

An open subset of $\mathbb{C}^n$ has a canonical almost complex structure. In fact $T_x(\mathbb{C})^n$ is canonically isomorphic to $\mathbb{C}^n$ and we define $J_x(v) = \sqrt{-1}v$. We call it the standard almost complex structure.

A diffeomorphism $\varphi : X \to Y$ between almost complex manifold is said to be isomorphism if $D_x\varphi \circ J_x = J_{\varphi(x)} \circ D_x$ for all $x \in X$.

An almost complex structure of X induces one on an open submanifold of X in an obvious way.

An almost complex manifold (X, J) is said to be integrable if for each $x \in X$ there exists its neighborhood U_x and an open set V_x of $\mathbb{C}^n$ such that (U_x, J) is isomorphic to (V_x, J). Here J on V_x is the standard one.

An almost complex manifold (X, J) is said to be a complex manifold if J is integrable.

If (X, J) is an almost complex manifold and $x \in X$ then T_xX has a unique structure of complex vector space such that $J_x = \sqrt{-1}$.

A submanifold Y of almost complex manifold (X, J) is said to be a complex submanifold, if for each $x \in Y$, the subspace J_xT_xY is in contained in T_xY. This is equivalent to the condition that J_xT_xY is a complex linear subspace.

Lemma 37. *A complex submanifold Y of an integrable almost complex manifold (X, J) is a complex manifold.*

Proof. Since the problem is local we may assume $X = \mathbb{C}^n$. Let $p \in Y$. By a complex linear transformation we may assume $T_pY = \mathbb{C}^m \times \{0\}$. By implicit function theorem we may assume $\pi : Y \to \mathbb{C}^m$ $(y, z) \mapsto y$ is a diffemorphism (by replacing Y its open subset). Then $D_x\pi : T_xY \to \mathbb{C}^m$ is complex linear for each $x \in X$. Therefore $J \circ D\varphi = D\varphi \circ J$. □

Our concern in this book on almost complex structure is its relation to symplectic structure.

Definition 38. Let (X,ω) is a symplectic manifold. An almost complex structure J of X is said to be compatible with ω if:

$$g(v,w) := \omega(V, J(W)) \tag{30}$$

is a Riemannian metric.

We elaborate on (30). We recall that g is a Riemannian metric if and only if $g(v,w) = g(w,v)$, $g(v,v) \geqslant 0$ and $g(v,v) = 0$ implies $v = 0$.

Lemma 39. *(30) is a Riemannian metric if and only if the following two conditions are satisfied.*

(1) $\omega(JV, JW) = \omega(V,W)$.
(2) $\omega(V, JV) \geqslant 0$. $\omega(V,JV) = 0$ if and only if $V = 0$.

Proof. Define g by (30). Then $g(v,w) = \omega(v,Jw)$. $g(w,v) = \omega(w,Jv) = -\omega(Jv,w)$. We write $Jw = w'$. Then $g(v,w) = g(w,v)$ is $\omega(v,w') = \omega(Jv,Jw')$. (Here we use $JJ = -1$). Thus (1) is equivalent to $g(v,w) = g(w,v)$. It is easy to see that '$g(v,v) \geqslant 0$ and $g(v,v) = 0$ implies $v = 0$' is equivalent to (2). □

Remark 40. There is a slightly weaker notion that an almost complex structure is tamed by a symplectic structure ω, that is, $g(v,w) : \omega(v,Jw) + \omega(w,Jv)$ is a Riemannian metric. We do not discuss it here.

The next lemma is sometimes useful.

Lemma 41. *Let (X,ω) is a symplectic manifold and J is a compatible almost complex structure. Let Y be a complex submanifold of (X,J). The $\omega|_Y$ is a symplectic structure of Y.*

Proof. Put $\omega_Y = \omega|_Y$. $d\omega_Y = 0$ is obvious. Note that for $v \ne 0$, $v \in T_xY$ we have

$$\omega_Y(v,Jv) \ne 0.$$

This implies that ω_Y at x is non-degenerate. Therefore ω_Y^n never vanish by Excercise 2. □

Solution of Excersice 2

Let $\Omega : V \otimes V \to \mathbb{R}$ be an anti-symmetric bi-linear form. We prove the next lemma.

Lemma 42. *Let $\mathbf{e} \in V$ such that $v \mapsto \Omega(\mathbf{e},v)$ is non-zero. Then there exists $\mathbf{f} \in V$ and $V^\perp$ such that*

$$(V,\Omega) = (\mathbb{R}\mathbf{e} \oplus \mathbb{R}\mathbf{f}, \Omega_0) \oplus (V^\perp, \Omega|_{V^\perp}).$$

Here $\Omega_0(\mathbf{e},\mathbf{f}) = 1$ and two summands are orthogonal.

Proof. We can find $\mathbf{f}$ with $\Omega(\mathbf{e},\mathbf{f}) = 1$ easily. We put

$$V^\perp = \{v \in V \mid \Omega(\mathbf{e},v) = \Omega(\mathbf{f},v) = 0\}.$$

Since $V \to \mathbb{R}^2$, $v \mapsto (\Omega(\mathbf{e},v), \Omega(\mathbf{f},v))$ is surjective, $\dim V^\perp = \dim V - 2$. The lemma follows. □

By the lemma we can find $\mathbf{e}_i$, $\mathbf{f}_i$ $i = 1, \dots, m$ by induction such that

$$V = \bigoplus_{i=1}^{m} (\mathbb{R}\mathbf{e}_i \oplus \mathbb{R}\mathbf{f}_i, \Omega_0) \oplus V_0.$$

Here Ω_0 is as above and Ω is zero on V_0.

Now we consider the case $\Omega = \omega_x$. It is obvious that Ω is non-degenerate if and only if V_0 is 0.

On the other hand, we may write

$$\omega_x = \mathbf{e}^1 \wedge \mathbf{f}^1 + \cdots + \mathbf{e}^m \wedge \mathbf{f}^m$$

where $\mathbf{e}^i$, $\mathbf{f}^i$ are dual basis to $\mathbf{e}_i$, $\mathbf{f}_i$.

Therefore $(\omega_x)^n \neq 0$ if and only if $m = n$. Here $2n = \dim V$.

3.2. **Kähler manifold.**

Definition 43. We say (X, ω, J) is a Kähler manifold if:

(1) ω is a symplectic structure.
(2) J is an almost complex structure which is compatible with ω.
(3) J is integrable. (In other words (X, J) is a complex manifold.)

Lemmas 37 and 41 imply:

Lemma 44. *Complex submanifold of a Kähler manifold is Kähler.*

In the next subsection we show that a complex projective space $\mathbb{C}P^n$ has a canonical Kähler structure. Therefore its complex submanifold is also Kähler. It is a classical theorem of Chow, that a complex submanifold of $\mathbb{C}P^n$ is an algebraic variety. A smooth algebraic variety is said to be projective if it is a complex submanifold of $\mathbb{C}P^n$. So a smooth complex algebraic variety is a Kähler.

3.3. **Projective space.**

We consider $\mathbb{C}^{1+n}$ the $n+1$ dimensional complex vector space.

Definition 45. $\mathbb{C}P^n$ is the set of all one dimensional complex linear spaces of $\mathbb{C}^{1+n}$.

We recall that $\mathbb{C}P^n$ is a complex manifold. Let $\pi_i : \mathbb{C}P^n \to \mathbb{C}$ be the projection to i-th factor. ($i = 0, \dots, n$.) We put:

$$U_i = \{L \in \mathbb{C}P^n \mid \pi_i(L) \neq 0\}.$$

If $L \in U_i$ then there exists unique $\vec{z} \in L$ such that $z_i = 1$. Therefore U_i is identified with $\mathbb{C}^n$ by

$$\phi_i : (w_1, \dots, w_n) \mapsto (w_1, \dots, w_{i-1}, 1, w_i, \dots, w_n).$$

Excercise 46. Show $\phi_j \circ \phi_i^{-1}$ is a diffeomorphism between open subsets.

Suppose $L \in U_i$, we define a complex structure of $T_L\mathbb{C}P^n$ such that $D_{(\phi_i)^{-1}(L)}\phi_i : T_{(\phi_i)^{-1}(L)}\mathbb{C}^n \to T_L\mathbb{C}P^n$ is complex linear. Show that this complex structure of $T_L\mathbb{C}P^n$ is independent of i.

We next define a symplectic (Kähler) structure on $\mathbb{C}P^n$. The construction below is a special case of the construction of symplectic quotient, which we will discuss systematically later.

We consider the standard symplectic form ω on $\mathbb{C}^{n+1}$ where

$$\omega = \sum dx^i \wedge dy^i,$$

$(z_i = x_i + \sqrt{-1}y^i.)$

$$S^{2n+1} = \{\vec{z} \in \mathbb{C}^{n+1} \mid \|\vec{z}\| = 1\}.$$

and the map

$$\Pi : S^{2n+1} \to \mathbb{C}P^n$$

where $\Pi(\vec{z}) := \mathbb{C}\vec{z}$.

Lemma 47. *There exists a unique differential form $\overline{\omega}$ on $\mathbb{C}P^n$ such that*

$$\Pi^*\overline{\omega} = \omega|_{S^{2n+1}}.$$

Proof. Let $\vec{z} \in S^{2n+1}$. $T_{\vec{z}}S^{2n+1} \cap JT_{\vec{z}}S^{2n+1}$ is $2n$ dimensional. In fact

$$T_{\vec{z}}S^{2n+1} = (T_{\vec{z}}S^{2n+1} \cap JT_{\vec{z}}S^{2n+1}) \oplus \mathbb{R}\vec{z}.$$

Note $\mathbb{C}^{n+1}$ is Kähler. For $W \in T_{\vec{z}}S^{2n+1} = (T_{\vec{z}}S^{2n+1} \cap JT_{\vec{z}}S^{2n+1})$ $JW \in T_{\vec{z}}S^{2n+1} = (T_{\vec{z}}S^{2n+1} \cap JT_{\vec{z}}S^{2n+1})$ so JW is perpendicular to $\vec{z}$. Therefore

$$\omega(\vec{z}, W) = -g(\vec{z}, JW) = 0. \tag{31}$$

Put $L = \Pi(\vec{z})$. By (31) there exists $\overline{\omega}_L \in \Lambda^2_L\mathbb{C}P^n$ such that

$$\omega|_{T_{\vec{z}}S^{2n+1}} = D\Pi \circ \overline{\omega}_L.$$

We claim that such $\overline{\omega}_L$ is independent of $\vec{z}$ such that $L = \Pi(\vec{z})$. To see this we consider the $S^1 = \{\alpha \in \mathbb{C} \mid |\alpha| = 1\}$ action on S^{2n+1} given by

$$\alpha \cdot (z_0, \dots, z_n) = (\alpha z_0, \dots, \alpha z_n).$$

Then the independence of $\overline{\omega}_L$ of $\vec{z}$ is a consequence of the next two facts.

(1) $\alpha^*\omega = \omega$. Here $\alpha : S^{2n+1} \to S^{2n+1}$ is defined as above.
(2) If $\Pi(\vec{z}) = \Pi(\vec{w})$ then there exists $\alpha \in S^1$ such that $\vec{w} = \alpha \cdot \vec{z}$.

□

Lemma 48. $d\overline{\omega} = 0$.

Proof. $\Pi^*d\overline{\omega} = d\Pi^*\overline{\omega} = d\omega_{S^{2n+1}} = 0$ since $d\omega = 0$. Since $D_{\vec{z}}\Pi : T_{\vec{z}}S^{2n+1} \to T_{\Pi(\vec{z})}\mathbb{C}P^n$ is surjective the lemma follows. □

Lemma 49. *$\overline{\omega}$ is a symplectic form.*

Proof. It suffices to show that $\overline{\omega}$ is non-degenerate on $T_L\mathbb{C}P^n$. This follows from the fact that

$$D_{\vec{z}}\Pi : T_{\vec{z}}S^{2n+1} \cap JT_{\vec{z}}S^{2n+1} \to T_{\Pi(\vec{z})}\mathbb{C}P^n \tag{32}$$

is an isomorphism and that ω is non-degenerate on $T_{\vec{z}}S^{2n+1} \cap JT_{\vec{z}}S^{2n+1}$. □

Lemma 50. *Lemma 39 (1)(2)(3) holds for $\overline{\omega}$.*

Proof. They hold for ω on $\mathbb{C}^{n+1}$. Therefore they hold for ω on $T_{\vec{z}}S^{2n+1} \cap JT_{\vec{z}}S^{2n+1}$. Therefore the lemma follows from the fact that (32) is an isomorphism (of complex vector spaces). $\square$

We thus proved:

Theorem 51. *$\mathbb{C}P^n$ is Kähler.*

The symplectic form $\overline{\omega}$ is called the Fubini-Study form. Let us calculate it explicitly by a coordinate.

We define

$$f(z_1, \dots, z_n) = \sqrt{1 + |z_1|^2 + \cdots + |z_n|^2}.$$

Put

$$\varphi_0(z_1, \dots, z_n) = \frac{1}{f(z_1, \dots, z_n)}(1, z_1, \dots, z_n) \in S^{n+1}.$$

$\Pi \circ \varphi_0$ is an isomorphism $\mathbb{C}^n \to U_0$. Actually it coincides with ϕ_0. By definition

$$(\Pi \circ \varphi_0)^*\omega = \phi_0^*\overline{\omega}.$$

We will calculate the left hand side.

Let $(w_0, \dots, w_n)$ be the standard coordinate of $\mathbb{C}^{n+1}$ and $w_i = X_i + \sqrt{-1}Y_i$. $\omega = \sum dX^i \wedge dY^i$. Note

$$X_i = \mathrm{Re} z_i/f, \qquad Y_i = \mathrm{Im} z_i/f,$$

for $i \neq 0$ and

$$X_0 = 1/f, \qquad Y_0 = 0.$$

We put $z_i = x_i + \sqrt{-1}y_i$. Then

$$\begin{aligned} \overline{\omega} &= \sum_{i=1}^n dX_i \wedge dY_i \\ &= \frac{1}{f^2}\sum_i dx_i \wedge dy_i - \frac{1}{f^4}\sum_i (x_i dx_i \wedge df - y_i dy_i \wedge df) \end{aligned}$$

(We use $df \wedge df = 0$ here.) Since

$$df = \frac{1}{f}\sum (x_i dx_i + y_i dy_i)$$

we have

$$\overline{\omega} = \frac{1}{f^2}\sum_i dx_i \wedge dy_i - \frac{2}{f^4}\sum_{i,j} x_i y_j dx_i \wedge dy_j. \tag{33}$$

We can calculate $(\Pi \circ \varphi_i)^*\omega$ in the same way, by renaming the variables.

Excercise 52. Show (33) is a symplectic form by a direct calculation. Show also that it is a Kähler form with respect to the standard complex structure by a direct calculation. Show (33) together its analogue on U_i define a global 2 form by a direct calculation.

3.4. **Space of compatible almost complex structures.** Let (X, ω) be a symplectic manifold. We consider

$$\mathcal{J}(X, \omega) = \{J \mid J \text{ is an almost complex structure compatible with } \omega\}.$$

Theorem 53. *$\mathcal{J}(X, \omega)$ is weakly contractible.*

We recall:

Definition 54. A space X is said to be weakly contractible, if all the maps $S^n \to X$ extends to $D^{n+1} \to X$. Where $S^m = \partial D^{n+1}$.

Remark 55. In fact we need to specify the topology of $\mathcal{J}(X, \omega)$. Most of the natural topology works. For example we can take the topology of C^∞ convergence.

Theorem 53 is a consequence of a standard fact on linear algebra and a general statement on fiber bundles. We first discuss the former.

We consider $2n$-dimensional vector space $\mathbb{R}^{2n}$ together with its standard symplectic form Ω. Let $\mathcal{J}$ be the set of all linear map $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that $J^2 = -1$ and that J satisfies Lemma 39 (1)(2) with ω replaced by Ω. We prove:

Proposition 56. *$\mathcal{J}$ is contractible.*

For the proof, we need a few things about Lagrangian Grassmannian. We define

$$\mathcal{LAG}(\mathbb{R}^{2n}; \Omega) = \{L \subset \mathbb{R}^{2n} \mid n\text{-dimensional linear subspace, } \Omega|_L = 0\}.$$

An element of $\mathcal{LAG}(\mathbb{R}^{2n}; \Omega)$ is said to be a Lagrangian linear subspace of $(\mathbb{R}^{2n}; \Omega)$.

We take $L_0 \subset \mathbb{R}^{2n}$ such that $\Omega|_{L_0} = 0$. More explicitely x_i, y_i $(i = 1, \dots, n)$ is a basis of $\mathbb{R}^{2n}$ such that $\Omega = \sum dx_i \wedge dy_i$ and $y_1, \dots, y_n$ is a basis of L_0.

Lemma 57. *The set*

$$\mathcal{LAG}(\mathbb{R}^{2n}; \Omega; L_0)\{L \in \mathcal{LAG}(\mathbb{R}^{2n}; \Omega) \mid L \cap L_0 = \{0\}\}.$$

is contractible.

We prove the lemma later. Let $J \in \mathcal{J}$. We put:

$$\pi(J) = J(L_0).$$

We remark that for $v \in L_0$, $v \neq 0$, $\Omega(v, J(v)) \neq 0$. Therefore $J(v) \notin L_0$. (In fact $\Omega = 0$ on L_0.) Therefore $\pi(J) \in \mathcal{LAG}(\mathbb{R}^{2n}; \Omega; L_0)$. (We also use the fact that Ω is J invariant. It implies that $J(L_0) \in \mathcal{LAG}(\mathbb{R}^{2n}; \Omega)$.

Lemma 58. *$\pi : \mathcal{J} \to \mathcal{LAG}(\mathbb{R}^{2n}; \Omega; L_0)$ is a fiber bundle.*

We prove Lemma 58 in Subsection 3.5. (We define the notion of fiber bundle also in Subsection 3.5.)

Lemma 59. *The fiber $\pi^{-1}(L)$ is contractible.*

Lemma 60. *If $\pi : E \to B$ is a fiber bundle such that B and $\pi^{-1}(p)$ are contractible. ($p \in B$.) Then E is contractible.*

We prove Lemma 60 in Subsection 3.5. Proposition 56 follows from Lemmas 57-60. □

Proof of Lemma 57. We observe that $\mathbb{R}^{2n}$ can be identified with the cotangent bundle $T^*\mathbb{R}^n$. In fact if $x_1, \dots, x_n$ are coordinate of $\mathbb{R}^n$ then by defining y_i by $v = \sum y_i dx_i$ (for tangent vector v), $x_1, \dots, x_n, y_1, \dots, y_n$. The symplectic form $\sum dx_i \wedge dy_i$ is minus of the symplectic form we used in the last section.

A digression: We generalize the situation a bit more since we use those cases later. Let T^*M be a cotangent bundle and $L \subset T^*M$ be an $n = \dim M$ dimensional submanifold. We assume for each x two submanifolds T^*_xM and L intersection transversally at one point. Then there exists $u(x) \in T^*_xM$ such that $u(x) \in L$. $x \mapsto u(x)$ becomes a diffeomorphism $I : M \to L$. $x \mapsto u(x)$ may be regarded as a differential 1 form.

Lemma 61. *$I^*\theta = u$.*

Proof. Immediate from the definition. □

In particular $\omega|_L = 0$ is equivalent to $du = 0$.

We go back to the proof of Lemma 57. We are given $L \subset T^*\mathbb{R}^n$. It is a linear subspace and $L \cap T^*_0\mathbb{R}^n = 0$. (Note that $T^*_0\mathbb{R}^n = L_0$.) It implies $L \cap T^*_x\mathbb{R}^n = 0$ for any x easily. Therefore there exists a closed one form u on $\mathbb{R}^n$ such that $L = \{u(x) \mid u \in \mathbb{R}^n\}$. Put $u(x) = (x, y(x))$ where $y(x) \in \mathbb{R}^n$. Since L is a linear subspace, the map y is linear. Since u is closed on $\mathbb{R}^n$ it is exact. There exists $f\mathbb{R}^n \to \mathbb{R}$ such that $u = df$. We may require $f(0) = 0$. Then f is unique. Since u is linear f is quadratic, that is,

$$f(x) = \sum a_{ij} x_i x_j$$

$a_{ij} = a_{ji}$.

When a quadratic function f is given we put $u = df$ and $L = \{u(x) \mid u \in \mathbb{R}^n\}$. Then $L \in \mathcal{LAG}(\mathbb{R}^{2n}; \Omega; L_0)$.

Thus $\mathcal{LAG}(\mathbb{R}^{2n}; \Omega; L_0)$ is diffeomorphic to the space of all quadratic functions, which is contractible. In fact it is diffeomorphic to $\mathbb{R}^{n(n+1)/2}$ □

Proof of Lemma 59. We define $I : L \to L_0^*$ by $I(v)(w) = \Omega(w, v)$. Since Ω is non-degenerate, $L_0|_{\Omega_0} = 0$, $L \cap L_0 = 0$, and $\dim L_0 = n$ the map I is an isomorphism.

Let $J \in \mathcal{J}$, $J \in \pi^{-1}(L)$. We define a inner product h_J on L_0 by

$$h_J(v, w) = \Omega(v, Jw). \tag{34}$$

We can show $h_J(w, v) = h_J(v, w)$ in the same way as the proof of Lemma 39. Moreover h_J is strictly positive definite. (This is the consequence of the compatibility of J with ω.)

We claim that if $h_{J_1} = h_{J_2}$ then $J_1 = J_2$. In fact if $h_{J_1} = h_{J_2}$ then $I(J_1 w) = I(J_2 w)$ for any $w \in L_0$. Therefore, since I is an isomorphism, $J_1 = J_2$.

On the other hand, if h is a strictly positive definite inner product on L there exists unique map $J_0 : L_0 \to L$ such that (34) holds with h_J replaced by h and J replaced by J_0. Since h is non-degenrate J_0 is an isomorphism. Note that $\mathbb{R}^{2n} = L_0 \oplus L$. We define $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ by $(a, b) \mapsto (-J_0^{-1}(b), J_0(a))$. It is easy to see that $J^2 = -1$ and (34) holds with h_J replaced by h. Moreover $J \in \mathcal{J}$, $J \in \pi^{-1}(L)$.

Thus $\pi^{-1}(L)$ is diffeomorphic to the set of all positive definite inner product on L_0. It is easy to see that the latter is contractible. □

Proof of Theorem 53. We consider the set

$$\mathcal{J}(X) = \{(x, J_x) \mid x \in X, J_x \in \mathcal{J}(T_x X, \omega_x)\}. \tag{35}$$

Here $\mathcal{J}(T_x X, \omega_x)$ is the space $\mathcal{J}$ above when we replace $\mathbb{C}^n$, Ω by $T_x X$, ω_x.

Lemma 62. *$\pi : \mathcal{J}(X) \to X$ which sends (x, J_x) to x is a fiber bundle.*

This is actually easy. See Subsection 3.5.

Then Theorem 53 is a consequence of Proposition 56, Lemma 62 and the next proposition.

Proposition 63. *Let $\pi : M \to N$ be a fiber bundle whose fiber F is contractible, then*

$$S = \{s : N \to M \mid \pi \circ s = \mathrm{id}\}$$

is contractible.

We prove Proposition 63 in Subsection 3.5. The proof of Theorem 53 is complete modulo the points we show in Subsection 3.5. □

Proof of Proposition 56. □

3.5. **A quick review of fiber bundle.**

Definition 64. Let $\pi : M \to N$ be a C^∞ map between C^∞ manifolds. We say it is a fiber bundle with fiber F (a smooth manifold) if the following holds.

For each $x \in N$ there exists its neighborhood U_x and a diffeomorphism $\varphi_x : \pi^{-1}(U_x) \to U_x \times F$ such that

$$\pi_{U_x} \circ \varphi_x = \pi.$$

Here $\pi_F : U_x \times F \to U_x$ is the projection.

Proof of Lemma 58. Let $L, L' \in \mathcal{LAG}(\mathbb{R}^{2n}; \Omega; L_0)$. We have isomorphism $I_L : L \to L_0^*$ by $I_L(v)(w) = \Omega(w, v)$. We also have $I_{L'} : L' \to L_0^*$.

We define $I'_{L'}$: as the composition

$$I'_{L'} = I_L \circ I_{L'}^{-1} : L' \to L.$$

By definition

$$\omega(w, I'_{L'}(v)) = \omega(w, v) \tag{36}$$

for $w \in L_0$, $v \in L'$.

We remark that

$$L \oplus L_0 \cong \mathbb{R}^{2n} \cong L' \oplus L_0.$$

We define $\varphi'_L : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ by

$$\varphi'_L(x + y) = (I'_{L'}(v) + y)$$

where $x \in L'$, $y \in L_0$. (36) and $\Omega(x_1, x_2) = \Omega(x'_1, x'_2) = \Omega(y_1, y_2) = 0$ for $x_1, x_2 \in L$, $x'_2, x'_2 \in L'$, $y_1, y_2 \in L_0$ implies

$$(\varphi'_L)^*\Omega = \Omega.$$

Now we define

$$\varphi(J) \to (\varphi_{J(L)}, \varphi'_L \circ J \circ (\varphi'_L)^{-1}).$$

It defines a diffeomorphism

$$\mathcal{J} \to \mathcal{LAG}(\mathbb{R}^{2n}; \Omega; L_0) \times \pi^{-1}(L_0).$$

□

The poof of Lemma 57 is actually over, since we obtained a global isomorphism. Lemma 60 is also true. We mention it a bit.

Let $\pi : M \to N$ be a fiber bundle. $f : N' \to N$ a smooth map. We put

$$f^*M = \{(x, y) \in M \times N' \mid \pi(x) = f(y)\}.$$

Lemma 65. *f^*M is a smooth manifold. $f^*M \to N$, $(x, y) \mapsto y$ defines a fiber bundle.*

Proof. Let $y'_0 \in N'$. Put $y_0 = f(y_0)$. There exists a neighborhood U_{y_0} such that $\varphi : \pi^{-1}U_{y_0} \cong F \times U_{y_0}$ and the diffeomorphism preserves projections. Put $U'_{y'_0} = f^{-1}(y_0)$. We can define diffeomorphism $\pi'^{-1}(U'_{y'_0}) \cong F \times U'_{y'_0}$ by $(x, y') \mapsto (\pi_F(\varphi(x)), y')$. □

We call (f^*M, N', π') the pull back bundle.

Important fact in the theory of fiber bundle is the following.

Theorem 66. *Let $\mathfrak{F} = (\pi : M \to N)$ be a fiber bundle and $f_i : N' \to N$ a smooth map. Suppose f_1 is homotopic to f_2. Then $f_1 * \mathfrak{F} \cong f_2^*\mathfrak{F}$.*

Here two fiber bundles $\mathfrak{F}_i = (M_i, N, \pi_i)$ are isomorphic if there exists diffeomorphism $g : M_1 \to M_2$ such that $\pi_2 \circ g = \pi_1$.

We postpone the proof of Theorem 66.

Proof of Lemma 60. We observe id : $B \to B$ is homotopic to the constant map. The pull back of $M \to B$ by id is original fiber bundle. On the other hand the pull back by constant map is the direct product. □

To prove Theorem 53 it suffices to prove the next proposition.

Proposition 67. *Let $M \to N$ be a fiber bundle such that its fiber is contractible. Then the space of its sections are weakly contractible.*

Proof. It suffices to prove it in the case when N is a simplicial complex. By Lemma 60 $M = N \times F$ with F being contractible. The space of the sections is identified with $\mathrm{Map}(N, F)$ Let $\partial D^n \to \mathrm{Map}(N, F)$ be a map. It induces $f : \partial D^n \times N \to F$. Since F is contractible standard algebraic topology implies that it extends to $D^n \times N \to F$. □

3.6. An example of symplectic manifold which is not Kähler.

4. Daroux's and Moser's Theorems.

4.1. **Moser's Theorem.** Let X be a compact manifold without boundary and ω_t a $t \in (-1.1)$ parametrized family of symplectic structures on X.

Theorem 68. *If the de-Rham cohomlology class $[\omega_t]$ is independent of t, there exists a t parametrized family of diffeomorphisms $\varphi_t : M \to M$ such that $\varphi_t^* \omega_0 = \omega_t$.*

Example 69. We consider t parametrized family of complex submanifolds

$$X_t := \{[x_0, \ldots, x_5] \in \mathbb{C}P^5 \mid \sum_{i=0}^{5} x_i^5 + t x_0 x_1 x_2 x_3 x_4 x_5 = 0\}.$$

X_t is a smooth complex submanifold of $\mathbb{C}P^5$ for $t \neq 0$. Therefore X_t is Kähler. Theorem 68 implies that X_t is symplectomorphic to $X_{t'}$ for $t, t' \neq 0$.

One may say that a 'constant' family of symplectic manifolds X_t suddenly becomes singular at $t = 0$.

Note that the particular form of the equation $\sum_{i=0}^{5} x_i^5 + t x_0 x_1 x_2 x_3 x_4 x_5 = 0$ is not important. Suppose we have t parametrized family of homogeneous polynomials $P_t(x_0, \ldots, x_n)$ such that the zero of P_t is non-singular except finitely many t's. (This is the case of 'generic' family.) Then the symplectic structure of the hypersurface of $\mathbb{C}P^n$ obtained as zero set of P_t is independent of the generic t.

The complex structure does depend on t.

The technically most non-trivial part of the proof of Theorem 68 is the next proposition.

Proposition 70. *Let u_t be a $t \in P$ parametrized family of differential k-forms on X. We assume $du_t = 0$ and $[u_t] = 0$ in de Rham cohomology. Then there exists a family of $k - 1$ forms v_t depending smoothly on t such that $dv_t = u_t$.*

Maybe a shortest proof is using harmonic analysis but it is not so much elementary. We provide an elementary proof (using a proof of de Rham theorem) later.

Proof of Theorem 68. Put

$$\alpha_t = \frac{d\omega_t}{dt}.$$

α_t is a family of closed 2 forms representing 0 in de-Rham cohomology. Therefore there exists a family of 1-forms β_t such that

$$d\beta_t = \alpha_t.$$

We will find $\varphi_t : X \to X$ such that

$$\varphi_t^* \omega_0 = \omega_t. \tag{37}$$

We can find a t parametrized family of vector fields V_t such that

$$\varphi_t = \exp^t_{V_*}.$$

Differentiating (37) we have

$$L_{V_t} \omega_t = d\beta_t.$$

Now we *define* V_t by the formula

$$i_{V_t} \omega_t = \beta_t.$$

By Cartan's formula

$$L_{V_t} \omega_t = d\beta_t.$$

By doing calculation in the opposite direction, it implies $\varphi_t = \exp^t_{V_*}$ satisfies $\varphi_t^* \omega_0 = \omega_t$. □

Example 71. We consider $X_\epsilon = \{[x : y : z] \in \mathbb{C}P^2 \mid zx^2 + y^3 + \epsilon z^3 = 0\}$. The intersection of X_t with $\mathbb{C}^2$ $(= \{[x : y : z] \mid z \neq 0\})$ is the solution set of $x^2 + y^3 + \epsilon = 0$. It is singular when $\epsilon = 0$. For $\epsilon \neq 0$, X_ϵ is a non-singular 2 manifold. It is actually a T^2 (elliplic curve). We consider one parameter family $\epsilon_t = \exp(2\pi\sqrt{-1}t)$. By Theorem 68 there exists a one parameter family of symplectic diffeomorphisms $\varphi_t : X_{\epsilon_0} \to X_{\epsilon_t}$. Note $\epsilon_t = \epsilon_0 = 1$. Therefore $\varphi_1 : X_1 \to X_1$ is a symplectic diffeomorphism. This diffeomorphism cannot be an identity map. We can show in homology it becomes $(a, b) \mapsto (a, b + a)$. We can use this fact to show that such φ_1 cannot be biholomorphic. (On the other hand, as we discussed above, it is realized as a symplectic diffeomorphism.)

4.2. **Family version of de Rham's theorem.** In this subsection we prove Proposition 70. We first review Čeck cohomology and proof of de Rham theorem using it. Let $\mathfrak{F}$ be one of, Λ^k $(k = 0, 1, 2, \dots)$ or $\mathbb{R}$. For U an open set of an manifolds $\mathfrak{F}(U)$ is the set of k forms on U if $\mathfrak{F} = \Lambda^k$, the set of real valued locally constant functions on U if $\mathfrak{F} = \mathbb{R}$. (In the latter case $\mathbb{R}(U) = \mathbb{R}$ when U is connected.)

Let $\mathcal{U} = \{U_i \mid i \in I\}$ be an open covering of X where the index set I is a finite set. For $\vec{i} = (i_0, \dots, i_k) \in I^{k+1}$, we put $U_{\vec{i}} = \bigcap_j U_{i_j}$. and

$$C^k(\mathfrak{F}; \mathcal{U}) = \bigoplus_{\vec{i} \in I^{k+1}} \mathfrak{F}(U_{\vec{i}}).$$

Its element is written as $(x_{i_0,\dots,i_k})$. We define

$$\delta : C^k(\mathfrak{F};\mathcal{U}) \to C^{k+1}(\mathfrak{F};\mathcal{U})$$

by $(x_{i_0,\dots,i_k}) \mapsto (y_{i_0,\dots,i_{k+1}})$, where

$$y_{i_0,\dots,i_{k+1}} = \sum_j (-1)^j i^* x_{i_0,\dots,\check{i}_k,\dots,i_k}$$

Here $i^* : \mathfrak{F}(U_{i_0,\dots,\check{i}_k,\dots,i_k}) \to \mathfrak{F}(U_{i_0,\dots,i_k})$ is the restriction map.

It is easy to check $\delta \circ \delta = 0$.

We define

$$\check{H}^k(X;\mathfrak{F};\mathcal{U}) = \frac{\mathrm{Ker}\delta : C^k(\mathfrak{F};\mathcal{U}) \to C^{k+1}(\mathfrak{F};\mathcal{U})}{\mathrm{Im}\delta : C^{k-1}(\mathfrak{F};\mathcal{U}) \to C^k(\mathfrak{F};\mathcal{U})}$$

the Čeck cohomology group of $\mathfrak{F}$ with respect to the covering $\mathcal{U}$.

Lemma 72. *If $\mathfrak{F} = \Lambda^\ell$ then*

$$\check{H}^k(X;\Lambda^\ell;\mathcal{U}) = 0 \tag{38}$$

for $\ell > 0$ and

$$\check{H}^0(X;\Lambda^\ell;\mathcal{U}) = \Lambda^\ell(X) \tag{39}$$

the space of of differential ℓ forms on X.

Proof. Let $\chi_i : U_i \to [0,1]$ be the partition of unity associated to $\mathcal{U}$. We extend it to all X by putting $\chi_i = 0$ outside U_i. We define

$$\Delta : C^{k+1}(\mathfrak{F};\mathcal{U}) \to C^k(\mathfrak{F};\mathcal{U})$$

by $(y_{i_0,\dots,i_{k+1}}) \mapsto x_{i_0,\dots,i_k}$ such that

$$x_{i_0,\dots,i_k} = \sum_{i\in I} \sum_{j=0}^{k+1} (-1)^j \chi_i y_{i_0,\dots,i_{j-1},i,i_{j+1},\dots,i_{k+1}}$$

By a direct calculation we can easily check

$$\Delta \circ \delta + \delta \circ \Delta = \mathrm{id}.$$

Let $k > 0$ and $\delta u = 0$ with $u \in C^k(\mathfrak{F};\mathcal{U})$. Then $u = \delta\Delta u$. Thus (38) holds.

Note that $\check{H}^0(X;\Lambda^\ell;\mathcal{U}) = \mathrm{Ker}\delta : C^0(\Lambda^\ell;\mathcal{U}) \to C^1(\Lambda^\ell;\mathcal{U})$. Suppose (x_i) is in the kernel. x_i is a differential ℓ form on U_i. $\delta(x_i) = 0$ implies that $x_i = x_j$ on $U_i \cap U_j$. Therefore they determine a differential ℓ form on X. On the contrary a differential ℓ form x on X determine an element (x_i) is in the kernel by $x_i = x|_{U_i}$. Thus (39) holds. □

Definition 73. An open covering $\mathcal{U}$ of a manifold X is called a simple cover if $U_{\vec{i}}$ is either empty or is diffeomorphic to $\mathbb{R}^n$.

Lemma 74. *Simple cover exists for any manifolds.*

We omit the proof. See ???.

We also use the next proposition.

Proposition 75. (parametrized version of Poincaré's lemma.) *Let α_t be a t parametrized family of differential k forms on $\mathbb{R}^n$ such that $d\alpha_t = 0$. Assume $k > 0$. Then there exists a t-parametrized family of differential $k-1$ forms β_t on $\mathbb{R}^n$ such that $d\beta_t = \alpha_t$.*

We will prove it in Subsection 4.5.

We take and fix a simple cover $\mathcal{U}$. We define

$$C^{k,\ell} = C^k(\Lambda^\ell;\mathcal{U}).$$

We have defined the operator:

$$\delta : C^{k,\ell} \to C^{k+1,\ell}.$$

We next define

$$d : C^{k,\ell} \to C^{k,\ell+1}.$$

An element of $C^{k,\ell}$ is $(x_{i_0,\dots,x_{i_k}})$ where $x_{i_0,\dots,x_{i_k}}$ is a differential ℓ form on $U_{i_0,\dots,i_k}$. We define

$$d(x_{i_0,\dots,x_{i_k}}) = (dx_{i_0,\dots,x_{i_k}}).$$

We have

$$d \circ d = \delta \circ \delta = 0, \qquad d \circ \delta = \delta \circ d.$$

Lemma 72 implies:

$$\mathrm{Ker}\delta : C^{k,\ell} \to C^{k+1,\ell} = \mathrm{Im}\delta : C^{k-1,\ell} \to C^{k,\ell}$$

for $k \geqslant 1$. Moreover

$$\mathrm{Ker}\delta : C^{0,\ell} \to C^{1,\ell} = \Lambda^\ell$$

the set of differential ℓ forms.

Proposition 75 implies:

$$\mathrm{Ker}d : C^{k,\ell} \to C^{k,\ell+1} = \mathrm{Im}d : C^{k,\ell-1} \to C^{k,\ell}$$

for $\ell \geqslant 1$. Moreover

$$\mathrm{Ker}\delta : C^{k,0} \to C^{k,1} = C^k(\mathbb{R};\mathcal{U}).$$

Thus we have the next commutative diagram. The vertical and horizontal lines are exact except -1-th ones.

We first review how we prove de Rham's theorem using this diagram. (This proof is due to A. Weil.) We consider the case of degree 2 form which is the case we used. Let α be a differential 2 form with $d\alpha = 0$. The we obtain $\alpha^{20} \in C^{20}$ such that $d\alpha^{20} = 0$. Therefore we have $\alpha^{10} \in C^{10}$ such that $d\alpha^{10} = \alpha^{20}$. We obtain $\alpha^{11} \in C^{11}$ by $\alpha^{11} = \delta\alpha^{10}$. $d\alpha^{11} = \delta d\alpha^{10} = \delta\alpha^{20} = 0$. Therefore we have $\alpha^{01} \in C^{01}$ with $d\alpha^{01} = \alpha^{11}$. We put $\alpha^{02} = \delta\alpha^{01}$. Then $d\alpha^{02} = 0$. Therefore it comes from $\beta \in C^2(\mathbb{R},\mathcal{U})$. We can show $\delta\beta = 0$. Thus $[\alpha] \mapsto [\beta]$ $H^2_{dR}(X) \to H^2(X;\mathbb{R};\mathcal{U})$. We can check that this map is independent of the choices we made. For example we may replace α^{10} by $\alpha^{10} + dx$. Then α^{01} changes to $\alpha^{10} + \delta x$. Therefore $\alpha^{02} = \delta\alpha^{10}$ does not change.

If we change α^{01} to $\alpha^{01} + y$ where $y \in C^2(X;\mathbb{R};\mathcal{U})$ then β changes to $\beta + \delta y$.

$$
\begin{tikzcd}
0 \arrow[r] & C^2(\mathbb{R}) \arrow[r] & C^{02} \arrow[r] & C^{12} & & \\
0 \arrow[r] & C^1(\mathbb{R}) \arrow[r] \arrow[u] & C^{01} \arrow[r] \arrow[u] & C^{11} \arrow[u] & & \\
0 \arrow[r] & C^0(\mathbb{R}) \arrow[r] \arrow[u] & C^{00} \arrow[r] \arrow[u] & C^{10} \arrow[r] \arrow[u] & C^{20} & \\
 & & \Lambda^0(X) \arrow[r] \arrow[u] & \Lambda^1(X) \arrow[r] \arrow[u] & \Lambda^2(X) \arrow[r] \arrow[u] & {}
\end{tikzcd}
$$

$\xrightarrow{d}$ $\quad \uparrow \delta$

FIGURE 1. double complex.

Thus in a similar way we can show $[\alpha] \mapsto [\beta]$, $H^2_{dR}(X) \to H^2(X;\mathbb{R};\mathcal{U})$ is well defined.

We can change the role of k and ℓ and repeat the same argument to show that it is an isomorphism.

Proof of Proposition 70. Let α_t be a t parametrized family of 2 forms with $d\alpha_t = 0$. We obtain α_t^{20} etc. in a similar way as above and then $\beta_t \in C^2(X;\mathbb{R};\mathcal{U})$. (We use Lemma 76 below here.) Note $\delta\beta_t = 0$. Moreover we assume that $[\alpha_t] = 0$ in de Rham cohomology. Then $[\beta_t] \in H^2(X;\mathbb{R};\mathcal{U})$. Using the fact $C^k(X;\mathbb{R};\mathcal{U})$ is finite dimensional, it is easy to find $\gamma_t \in C^1(X;\mathbb{R};\mathcal{U})$ depending smoothly on t such that $\delta\gamma_t = \beta_t$. Then $\delta(\alpha_t^{11} - \gamma) = 0$. Therefore there exists $\gamma_t^{00} \in C^{00}$ such that $\delta(\gamma_t^{00}) = \alpha_t^{11} - \gamma_t$. Then $\delta(\alpha_t^{10} - d\gamma_t^{00}) = 0$. Therefore there exists $\theta_t \in \Lambda^1$ such that $\theta_t = \alpha_t^{10} - d\gamma_t^{00}$ in C^{10}. Now it is easy to see $d\theta_t = \alpha_t$. $\square$

Lemma 76. *Let (C^k, δ) be a finite dimensional chain complex over $\mathbb{R}$. If $x_t \in C^k$ is a t parametrized family such that $x_t \in \mathrm{Im}\delta$ then there exists y_t t parametrized family such that $x_t = \delta y_t$.*

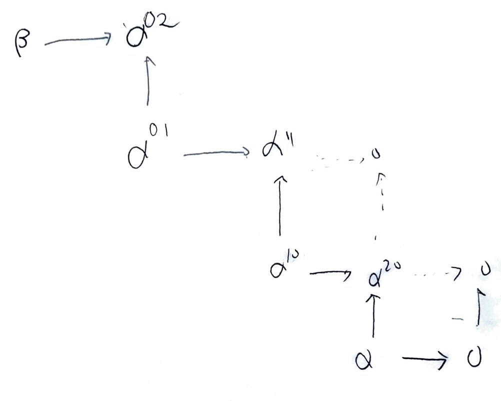

FIGURE 2. diagram chase.

Proof. $\mathrm{Im}(\delta : C^{k-1} \to C^k)$ is a finite dimensional subspace there exists $\Delta : \mathrm{Im}(\delta : C^{k-1} \to C^k) \to C^{k-1}$ such that $\delta \circ \Delta = \mathrm{id}$. $y_t = \Delta(x_t)$ has the required property. $\square$

4.3. **The group of symplectic diffeomorphisms.** We continue discussion on group of symplectic diffeomorphisms. We first explain that the group of Hamiltonian diffeomorphisms coincides with $\mathrm{Aut}(X, \omega)$ modulo finite dimension.

Let v be a differential one form on X. We assume $dv = 0$. By non-degeneracy of ω there exists unique vector field $\mathfrak{H}_v$ such that

$$i_{\mathfrak{X}_v}\omega = v.$$

By the same calculation as the proof of Lemma 15 we can show

$$L_{\mathfrak{X}_v}\omega = dv = 0.$$

Suppose (X, ω) is compact symplectic manifold.

Lemma 77. *If $\varphi : X \to X$ is a diffeomorphism with $\varphi^*\omega = \omega$. Suppose φ is sufficiently close to the identity map, then there exists a t parametrized family of differential one forms v_t with $dv_t = 0$ such that*

$$\varphi = \exp^1_{\{\mathfrak{X}_{v_t}\}}.$$

We postpone its proof to later.

Morally speaking Lemma 77 implies that the Lie algebra of $\mathrm{Aut}(X, \omega)$ is the set of closed 1-forms. We consider the next exact sequence:

$$0 \to \mathbb{R} \to \{df \mid f \in C^\infty(X)\} \to \{u \in \Omega^1(X) \mid du = 0\} \to H^1(X; \mathbb{R}) \to 0.$$

This is linearization of the following exact sequence of groups:

$$1 \to \widetilde{\mathrm{Ham}}(X, \omega) \to \widetilde{\mathrm{Aut}}_0(X, \omega) \to H^1(X; \mathbb{R}) \to 0, \tag{40}$$

where $\widetilde{\mathrm{Aut}}_0(X,\omega)$ is the connected component of the universal covering space of $\mathrm{Aut}(X,\omega)$ and

$$\widetilde{\mathrm{Ham}}(X,\omega) = \{[\varphi_t] \in \widetilde{\mathrm{Aut}}_0(X,\omega) \mid \varphi^1 \in \mathrm{Ham}(X,\omega)\}$$

where $\mathrm{Ham}(X,\omega)$ is the group of Hamiltonian diffeomorphisms. The Lie algebra of $\widetilde{\mathrm{Ham}}(X,\omega)$ is $\{df \mid f \in C^\infty(X)\}$.)

The homomorphism $\widetilde{\mathrm{Aut}}_0(X,\omega) \to H^1(X;\mathbb{R})$ appearing in (40) is call the flux homomorphism and is defined as follows.

Definition 78. Let $\varphi \in \widetilde{\mathrm{Aut}}_0(X,\omega)$ we choose a path φ^t such that φ^0 is the identity map and $\varphi = \varphi^1$. (By an abuse of notation we write an element of $\widetilde{\mathrm{Aut}}_0(X,\omega)$ and its image in $\mathrm{Aut}_0(X,\omega)$ by the same symbol.) There exists a closed one form v_t such that

$$\varphi^t = \exp^1_{\mathfrak{X}_{v_t}}. \tag{41}$$

(Again by Cartan's formula.) We define

$$\mathrm{Flux}(\varphi) = \int_0^1 [v_t] \in H^1(X;\mathbb{R}).$$

Lemma 79. $\mathrm{Flux}(\varphi)$ *depends only on* $\varphi \in \widetilde{\mathrm{Aut}}_0(X,\omega)$.

Proof. Let $\gamma : S^1 \to X$ be a loop. We define $u : [0,1] \times S^1 \to X$ by

$$u(t,s) = \varphi^t(\gamma(s)).$$

We claim:

$$(\int_0^1 [v_t]) \cap [\gamma] = \int_{[0,1]\times S^1} u^*\omega \in \mathbb{R}. \tag{42}$$

We observe that the claim implies (42) implies the lemma. In fact if φ'^t is another path then u changes to u' such that u' is homotopic to u relative to the boundary $\partial[0,1] \times S^1$. Therefore by the right hand side does not change (since ω is closed.)

We will prove (42) in the rest of the proof. We put

$$f(t) = (\int_0^t [v_t]) \cap [\gamma], \qquad g(t) = \int_{[0,t]\times S^1} u^*\omega.$$

It is easy to see

$$\frac{df}{dt} = [v_t] \cap [\gamma].$$

On the other hand,

$$\frac{df}{dt} = \lim_{\epsilon\to 0} \frac{1}{\epsilon} \int_{[t,t+\epsilon]\times S^1} u^*\omega.$$

Note that $Du(\frac{\partial}{\partial t}) = \mathfrak{X}_{v_t}$. Therefore

$$u^*\omega = \omega(\mathfrak{X}_{v_t}, \partial/\partial s) dt \wedge ds = (i_{\mathfrak{X}_{v_t}}\omega)(\partial/\partial s) dt \wedge ds = v_t(\partial/\partial s) dt \wedge ds.$$

Therefore
$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{[t,t+\epsilon]\times S^1} u^*\omega = \int_{S^1} v_t(\partial/\partial s)ds = [v_t]\cap[\varphi^t\gamma] = [v_t]\cap[\gamma],$$
as required. □

Lemma 80. Flux *is a group homomorphism. Namely*
$$\mathrm{Flux}(\psi\circ\varphi) = \mathrm{Flux}(\varphi) + \mathrm{Flux}(\psi).$$

Proof. Let φ^t, ψ^t represent φ and ψ, respectively. Then we consider the family parametrized by $t \in [0,2]$ and defined by
$$\phi^t = \begin{cases}\varphi^t & \text{when } t \leqslant 1\\ \psi^{t-1}\circ\varphi^1 & \text{when } t \geqslant 1,\end{cases}$$
which represents $\psi\circ\varphi$. (This is the definition of the covering group.) The lemma follows from (42) and the fact that φ^1 is homotopic to the identity through the diffeomorphisms preserving symplectic form. □

Proposition 81. *The sequence (40) is exact.*

Proof. We first show the surjectivity of the Flux homomorphism $\widetilde{\mathrm{Aut}}_0(X,\omega)\to H^1(X;\mathbb{R})$. Let v be a closed one form representing an element of $H^1(X;\mathbb{R})$. Then, for $\varphi^1_{\mathfrak{X}_v}$ (Here $\mathfrak{X}_v$ is a t-independent vector field), $\mathrm{Flux}(\varphi^1) = [v]$ is immediate from definition.

The fact the composition $\widetilde{\mathrm{Ham}}(X,\omega)\to\widetilde{\mathrm{Aut}}_0(X,\omega)\to H^1(X;\mathbb{R})$ is also obvious from definition.

The most important step is to show that if φ^t is an element of $\widetilde{\mathrm{Aut}}_0(X,\omega)$ such that its Flux is 0 then it is equivalent to an element of the image of $\widetilde{\mathrm{Ham}}(X,\omega)$. We prove it now. Let $[\varphi^t]\in\widetilde{\mathrm{Aut}}_0(X,\omega)$. We have a t parametrized family of closed 1 forms v_t such that
$$\varphi^t = \exp^t_{\mathfrak{X}_{v*}}.$$
Here $i_{\mathfrak{X}_{v_t}}\omega = v_t$. We assume
$$\int_{t=0}^1 [v_t]dt = 0 \in H^1(X;\mathbb{R}).$$
Therefore there exists a function f such that
$$\int_{t=0}^1 v_t = df.$$
We consider $\psi^t = \exp^t_{-\mathfrak{X}_f}$ Note (ψ^t) represents an element of $\widetilde{\mathrm{Ham}}(X,\omega)$. The composition $(\psi_t)\circ(\varphi_t)$ is $\exp^t_{\mathfrak{X}_{v'_*}}$ with $\int_{t=0}^2 v'_t = 0$. (Here $v'_t = v_t$ for $t\in[0,1]$ and $v'_t = -df$ for $t\in[1,2]$.) Thus it suffices to consider $[\varphi^t]$ with $\varphi^t = \exp^t_{\mathfrak{X}_{v*}}$ and $\int_{t=0}^1 v_t = 0$. We will study this case below.

For $s\in[0,1]$ we take $w^s = \int_s^1 v_t dt$. We consider $\exp_{\mathfrak{x}_{w^s}}(t)$ $(t\in[0,1])$. ($\mathfrak{x}_{w^s}$ is a t independent family of vector fields.). It gives an element $\widetilde{\mathrm{Aut}}_0(X,\omega)$

which we define by ψ_s. On the other hand $\exp_{v*}(t)$ $(t \in [0,s])$ gives an element of $\widetilde{\mathrm{Aut}}_0(X,\omega)$ which we define by φ_s. Note

$$\mathrm{Flux}\varphi_s = \int_0^s [v_t]dt = -\int_s^1 [v_t]dt = -\mathrm{Flux}\psi_s$$

Therefore $\mathrm{Flux}(\psi_s \circ \varphi_s) = 0$. It follows that

$$X_s := \frac{D}{ds}\psi_s \circ \varphi_s$$

satisfies $L_{X_s} = 0$, $[i_{X_s}\omega] = 0 \in H^2(X;\mathbb{R})$.

Moreover since $w_0 = w_1 = 0$ we have

$$\psi_0 \circ \varphi_0 = \mathrm{id}, \qquad \psi_1 \circ \varphi_1 = \varphi_1$$

Therefore there exists H_s such that $X_s = \mathfrak{X}_{H_s}$. Thus $[\psi_s \circ \varphi_s] \in \widetilde{\mathrm{Ham}}(X,\omega)$.

We thus proved exactness at $\widetilde{\mathrm{Aut}}_0(X,\omega)$. Using simply connected-ness of $H^1(X;\mathbb{R})$ we can then show $\widetilde{\mathrm{Ham}}(X,\omega) \to \widetilde{\mathrm{Aut}}_0(X,\omega)$ is injective. □

Proposition 81 shows that $\widetilde{\mathrm{Ham}}(X,\omega)$ is a *closed* subgroup of $\widetilde{\mathrm{Aut}}_0(X,\omega)$.

Theorem 82. (Ono) $\mathrm{Ham}(X,\omega)$ *is a closed subgroup of* $\mathrm{Aut}_0(X,\omega)$ *in* C^1 *topology.*

This is a deep theorem and its proof uses various modern technique in symplectic geometry.

Conjecture 83. (C^0-Flus conjecture) $\mathrm{Ham}(X,\omega)$ *is a closed subgroup of* $\mathrm{Aut}_0(X,\omega)$ *in* C^0 *topology.*

In other words, if φ_i is a sequence of Hamiltonian diffeomorphisms which converges to a symplectic diffeomorphism φ then φ is also a Hamiltonian diffeomorphism.

The next famous result is older and it is discovered around the time when the study of global symplectic geometry started.

Theorem 84. (Eliashberg) $\mathrm{Aut}(X,\omega)$ *is closed in* $\mathrm{Diff}(X)$ *in* C^0 *topology.*

In other words if φ_i is a sequence of symplectic diffeomorphisms which converges to a diffeomorphism φ then φ is also a symplectic diffeomorphism.

4.4. **Darboux's theorem.** We next show the following:

Theorem 85. *Let* (X,ω) *be a symplectic manifold and* $p \in X$. *Then there exists a diffeomorphism* $\varphi : D^{2n} \to X$ *onto an open subset such that* $\varphi(0) = p$ *and*

$$\varphi^*\omega = c\sum_{i=1}^n dx^i \wedge dy^i$$

for some positive constant c. Here $x^1,\ldots,x^n,y^1,\ldots,y^n$ *is a standard coordinate of* D^{2n}.

Here D^{2n} denotes the open ball of radius 1 in $\mathbb{R}^{2n}$. We first explain the rough idea. We take a smooth map $\Phi : D^{2n} \to X$ with $\Phi(0) = p$. By assumption $\Phi^*\omega$ at 0 is a anti-symmetric bi-linear form ω_0. We may change a coordinate of D^{2n} such that $\omega_0 = \sum_{i=1}^n dx^i \wedge dy^i$ *at origin.* (See solution of Excersice 2 in Subsection 3.1.) Put $\Phi^N(x,y) = \Phi(x/N, y/N)$ then

$$\|N^2(\Phi^N)^*\omega - \sum_{i=1}^n dx^i \wedge dy\|_{C^k} < C/N$$

We may choose N large such that

$$\omega_t = tN^2(\Phi^N)^*\omega + (1-t)\sum_{i=1}^n dx^i \wedge dy$$

is non-degenerate on D^{2n} for $t \in [0,1]$.

Since $H^2(D^{2n};\mathbb{R}) = 0$ the de Rham cohomology class of ω_t is zero. So if we could apply Theorem 68, then there exists $\varphi_t : D^{2n} \to D^{2n}$ such that $\varphi_t^*\omega_t = \omega_0$. It will imply Theorem 85.

However we assumed compactness of X in Theorem 68. In fact we need to *integrate* the vector field $\mathfrak{X}_{v_t}$ appearing in the proof.

So we need to adapt the proof of Theorem 68 carefully so that it works in our non-compact situation.

We use the following variant of Poincaré's lemma for this purpose. A domain D in $\mathbb{R}^n$ is said to be star-shaped if there exists $p_0 \in D$ such that the line segument $\overline{p_0p}$ is in D for any $p \in D$.

Proposition 86. (A variant of Poincaré's lemma) *Let D be a star-shapced domain. If u is a differential k-form on D with $du = 0$. Then there exists $k-1$ form v on D such that $dv = u$. Moreover:*

1. *If u_t is t parameter family with $du_t = 0$ then we can take t parameter family v_t such that $dv_t = u_t$.*
2. *There exists $C(k,D)$ depending only of k and D such that*

$$\|v_t\|_{C^k} \leqslant C(k,D)\|u_t\|_{C^k}.$$

We prove Proposition 86 in Subsection 4.5

Proof of Theorem 85. We use notations above. Note

$$\|\frac{d}{dt}\omega_t\|_{C^k} \leqslant C(k)/N.$$

Therefore we have β_t such $d\beta_t = \frac{d}{dt}\omega_t$ and

$$\|\beta_t\|_{C^k} \leqslant C'(k)/N$$

We take a vector field X_t on $D^{2n}(2)$ such that

$$i_{X_t}\omega = \beta_t$$

and

$$\|X_t\|_{C^k} < C''/N. \tag{43}$$

Now we claim

Lemma 87. *If N is large then for each $p \in D^{2n}$ there exists $\gamma_p(t)$ such that $\gamma(0) = p$,*

$$\frac{d}{dt}\gamma_p = V_t(\gamma(t))$$

and the length of γ_p is smaller than $1/2$.

Using (43) we can prove the lemma in the same way as the standard proof of the existence of the solution of ODE. The map $p \mapsto \gamma_p(t)$ becomes a smooth map $D^{2n} \to D^{2n}(2)$ which is a diffeomorphism to an open subspace. We write it φ^t. Now by the same calculation as the proof of Theorem 68 we can show $(\varphi^t)^*\omega_0 = \omega_t$ and can complete the proof of Theorem 85. □

4.5. **Poincaré's lemma with estimate.** We begin with the following:

Lemma 88. *Let $U \subset \mathbb{R}^n$ be an open subset. There exists a map*

$$I : \Lambda^k([0,1] \times U) \to \Lambda^{k-1}([0,1] \times U)$$

such that

$$(d \circ I + I \circ d)u = u - \pi^*(u|_{\{0\}\times U}). \tag{44}$$

Proof. We use t as the coordinate of $[0,1]$. We write

$$u = dt \wedge u_1 + u_2$$

where u_1, u_2 does not contain dt. We put

$$I(u)(s,x) = \int_0^s u_1(t,x)dt.$$

We calculate

$$\begin{aligned} I(du)(s,x) &= I(-dt \wedge d_x u_1 + d_x u_2 + dt \wedge \frac{\partial u_2}{dt})) \\ &= -\int_0^s d_x u_1(t,x)dt + u_2(s,x) - u_2(0,x). \\ d(Iu)(s,x) &= \int_0^s d_x u_1(t,x)dt + dt \wedge u_1(s,x). \end{aligned}$$

The lemma follows. □

We remark that, explicit formula of I implies that, if u_t is t-parametrized family then $I(u_t)$ is also a t-parametrized family. Moreover I is C^k bounded.

Proof of Proposition 86. We put $U = D$. We define $H : [0,1] \to D$ by $H(t,p) = (1-t)p_0 + tp$. Then we apply Lemma for H^*u. Note $H^*u|_{\{0\}\times U} = 0$. Therefore using $du = 0$.

$$dI(H^*u) = H^*u.$$

Since $H(1,p) = p$ we obtain $dI(H^*u)|_{\{1\}\times D} = u$. Thus $v = I(H^*u)|_{\{1\}\times D}$ has the required properties.

Item (1)(2) in Proposition 86 follows from the corresponding properties of I which we remarked above. □

4.6. **Weinstein neighborhood theorem.** We recall that

Definition 89. Let (X,ω) be a $2n$-dimensional symplectic manifold.

An (embedded) Lagrangian submanimfold L is an n dimensional (embedded) submanifold such that $\omega|L = 0$.

An immersed Lagrangian submanimfold is a pair $L = (\tilde{L}, i_L)$ where $\tilde{L}$ is an n-dimensional manifold and $i_L : \tilde{L} \to X$ is an immersion such that $i_L^*\omega = 0$.

We discuss Lagrangian submanifold systematically in Part **??**. In the case when $X = T^*M$ is a cotangent bundle the fibers T_p^*M are Lagrangian submanifold. Moreover for a closed 1 form u its graph $\{(p, u(p)) \in T^*M \mid p \in M\}$ is a Lagrangian submanifold. In particular the zero section $0_M := \{(p, 0) \in T^*M \mid p \in M\}$ is a Lagrangian submanifold.

Theorem 90. *Let L be an embedded Lagrangian submanifold of a symplectic manifold (X, ω_X). Then there exists an open neighborhood U_L of the zero section 0_L in the cotangent bundle T^*L and an open embedding $i : U_L \to X$ such that:*

1. *$i^*\omega_X = \omega$. Here ω_X is the symplectic form of X and ω in the right hand side is the canonical symplectic form in Lemma 22.*
2. *$i|_{0_L}$ is the identity map.*

Proof.

Lemma 91. *Let $i_L : L \to X$ be the identity map and $i_0 : L \to T^*L$ is the identification with zero section. Then there exists an isomorphism of vector bundle $I : i_L^*TX \cong i_L^*TT^*X$ such that:*

1. *I preserves the symplectic forms.*
2. *The restriction of I to i_L^*TL is the identity map $TL \to T0_L$.*

Proof. We can choose a rank n subbundle W of i_L^*TL such that $W_p \cap T_pL = \{0\}$ for $p \in L$ and the symplectic form vanish on W.

Excercise 92. Prove it.

Using symplectic form there is a canonical isomorphism $W \cong TL^*$. We consider the subbundle $V \subset i_L^*TT^*X$ such that $V_p = T_pT_p^*L$. (Here we regard $p \in 0_L$ and T_p^*L a submanifold of T^*L.) Using symplectic structure of T^*L there is a canonical isomorphism $W \cong V$. Note $i_L^*TX = W \oplus TL$ and $i_L^*TT^*X = V \oplus TL$. Therefore the isomorphism $W \cong V$ and the identity map $TL \cong TL$ induces a bundle isomorphism $I : i_L^*TX \cong i_L^*TT^*X$. Using the fact that the fibers of V, W are Lagrangians, T_pL is also a Lagrangian, and the construction of $W \cong V$ it is easy to see $I : i_L^*TX \cong i_L^*TT^*X$ preserves the symplectic form. ∎

Excercise 93. Show that there exists an open subset U'_L of the zero section 0_L and an open embedding $i' : U'_L \to X$ such that:

1. i'_L is the identity map on L.

(2) For $p \in L$ the derivative $D_p i'$ is the isomorphism in Lemma 91.

We consider
$$\omega_t = (1-t)\omega + t(i')^*\omega_X.$$
We may replace U'_p by a smaller neighborhood so that ω_t is a symplectic form for any t. Then the rest of the proof is mostly the same as the proof of Moser's theorem, except again the domain is not compact. We can go around this trouble in the same way as the proof of Darboux's theorem using the next lemma.

Proposition 94. (A variant of De Rham's theorem with estimage) *Let M be a (not necessary smooth). If u_t is a t-parametrized family of differential k-form on M with $du_t = 0$. Assume the de Rham cohomology classes of u_t are 0. Then there exists a t-parametrized family $k-1$ form v_t on M such that $dv_t = u_t$. Moreover there exists $C(k,M)$ depending only of k and M such that*
$$\|v_t\|_{C^k} \leqslant C(k,M)\|u_t\|_{C^k}. \tag{45}$$

Proof. Except (45), this is Proposition 70. On one chart (45) is Proposition 86. We can use a similar argument to the proof of Proposition 70 (diagram chase of double complex $C^{k,\ell}$) and one chart version of (45), to prove (45) for M. □

The proof of Theorem 90 is now complete. □

We can use Theorem 90 to prove Lemma 77 as follows. Let $\varphi : X \to X$ be a symplectic diffeomorphism which is C^1 close to identity map. We consider $-X \times X$, which is a symplectic manifold $(X \times X, -\pi_1^*\omega + \pi_2^*\omega)$, where ω is the symplectic form of X and $\pi_1 : X \times X \to X$ is the projection to the first factor, π_2 is the projection to the second factor. The diagonal $\Delta = \{(x,x) \mid x \in X\}$ is a Lagrangian submanifold and $\mathrm{Grap}\varphi = \{(x,\varphi(x) \mid x \in X\}$ is also a Lagrangian submanifold. We apply Theorem 90 to a Lagrangian submanifold Δ of $-X \times X$. We may assume $\mathrm{Grap}\varphi \subset i_\Delta(U_\Delta)$. Then $i_\Delta^{-1}\mathrm{Grap}\varphi$ is a Lagrangian submanifold of T^*X which is C^1 close to the zero section. Therefore there exists closed one form u on L such that $i_\Delta^{-1}\mathrm{Grap}\varphi = \{(x,u(x)) \in T^*X \mid x \in X\}$. The map
$$x \mapsto i_\Delta((x, tu(x))$$
is a Lagrangian embedding $X \to -X \times X$ which is sufficiently close to the diagonal embedding. Therefore $x \mapsto \pi_1(i_\Delta((x,tu(x)))$ is a diffeomorphism. It implies that there is a t parametrized family of symplectic diffeomorpisms $\varphi_t : X \to X$ such that
$$\{i_\Delta((x,tu(x)) \mid x \in X\} = \{(x,\varphi_t(x)) \mid x \in X\}.$$
We put
$$V_t(\varphi_t(x)) = \frac{d}{dt}\varphi_t(x).$$

Then $v_t = i_{V_t}\omega$ is the required family of closed 1 forms. The proof of Lemma 77 is complete. $\square$

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