(Semi)classical limits of
internally quantum systems.

Given an algebra $[\hat{O}_a, \hat{O}_b] = i f_{ab}^{\;\;c} \hat{O}_c$ (1)
of Herm. operators $\hat{O}_a$ and an integer
number j I can go the new algebra
acting in the space $V_j = S^{\otimes j} V$; here V was
a pepresentation of algebra (1)
In the example studied before V was $\mathbb{C}^2$
and algebra (1) was the algebra of
SU(2) acting in the fundamental representation.

we would introduce new operators, acting on V_j:
$\hat{O}^{cl}_a \rightarrow$ operators, having classical limit
$$\hat{O}^{cl}_a = \frac{1}{j} \hat{O}_a .$$
In particular, the range of eigenvalues of $\hat{O}_a$ acting on V_j is j-times larger than that of $\hat{O}_a$ acting on V_1, while the range of eigenvalues of $\hat{O}^{cl}_a$ is the same for all j.

In the studied example $V = \mathbb{C}^2$,
$$O_3 = T_3 = \left(z_1 \frac{\partial}{\partial z_1} - z_0 \frac{\partial}{\partial z_0}\right) \qquad O_3 \ (z_1^N) = N$$
grows as $N \to +\infty$

however $$O_3^{cl} = \frac{1}{N}\left(z_1 \frac{\partial}{\partial z_1} - z_0 \frac{\partial}{\partial z_0}\right)$$
$$O_3^{cl}(z_1^N) = 1$$

Operators O_a^{cl} satisfy
$$A_j^{cl}: \ [O_a^{cl}, O_b^{cl}] = \frac{1}{j} f_{ab}^{c} O_c^{cl} \quad (\text{acting in } S^{\otimes j} V)$$
and this algebra tends to commutative when $j \to +\infty$

Thus we may consider the spectrum $\operatorname{Spec}(A^{cl}_\infty)$ - of the limiting algebra - it would be some manifold (see Leyenson lectures).

Now, the first term in $\frac{1}{j}$ gives the bilinear operation on O_a^{cl} (now considered as coord. functions on $\operatorname{Spec}(A^{cl}_\infty)$).

what about dynamics?

What should we take for Hamiltonian on S^jV?

There are many Herm. operators, what if we take just $\hat{O}^{cl}_a$?

Then we will get, unfortunately, trivial dynamics in the $j \to \infty$ limit since $\lim_{j\to\infty} [\hat{O}^{cl}_a, \hat{O}^{cl}_b] = 0$.

However, if we enlarge the Hamiltonian and consider as $\hat{H} = \underset{en}{j} \cdot \hat{O}^{cl}_a = \hat{O}_a \leftarrow \underline{\underline{\text{old}}}$

we will get interesting dynamics:

$$\frac{d}{dt} \langle O^{cl}_b \rangle = i \langle [O_a, O^{cl}_b] \rangle = -f_{ab}^{c} \langle O^{cl}_c \rangle,$$

so here may take $j \to \infty$ limit and get nontrivial answers.

In particular, in the model studied last time

$$(O^{cl}_1)^2 + (O^{cl}_2)^2 + (O^{cl}_3)^2 = 1 + \frac{1}{j}$$

$\xrightarrow{j \to \infty}$ sphere

If we take $\hat{H} = \hat{O}_3 = j \, \hat{O}^{cl}_3$, then $\langle \hat{O}^{cl}_a \rangle$ is a point on S^2; (here we consider exp. values of

3 operators), and due to dynamics we have:

It is a well-known formula from the theory of the angular momentum

Moreover, one may consider other herm. operators, like

$$j(\hat{O}_a^{cl}\hat{O}_b^{cl}+\hat{O}_b^{cl}O_a^{cl})$$

Then we will see interesting trajectories of the $\langle O_c^{cl}\rangle$ on the sphere

These trajectories would be algebraic curves of the second order, since in the class. limit $H\to$ quadratic function on a sphere, and evolution preserves H.

Look, evolution of $\langle O_c^{cl}\rangle$ for given $H=\langle O_a\rangle$ seems to be indep. of j? So what is the diff. between such 2 systems?

In our simle example for $SU(2)$, $N=1$ the $\langle O_3\rangle$ on a state $\binom{a}{b}=az_1+bz_0$ $\frac{a^2-b^2}{a^2+b^2}$, however this average is achieved as follows. Experiment show $+1$ with the probability $\frac{a^2}{a^2+b^2}$

and -1 with the probability $\frac{b^2}{a^2+b^2}$ ($N=1$ case)

However, in $N\to\infty$ case I can choose another state $(az_1 + bz_0)^N$

It is possible to check that this state is "almost" eigenstate for operators O_a^{cl}, in particular for

$$O_3^{cl} = \frac{1}{N}\left(z_1\frac{\partial}{\partial z_1} - z_0\frac{\partial}{\partial z_0}\right)$$

I will do this computation next time

Poissonian dynamics appears.

Have alternative look at the Poissonian dynamics from functional integral perspective

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Reminder of how P.D. arizes from f.i.

Consider the space Φ with coordinates φ_i and a 1-form $\alpha = \sum_i \alpha_i(\varphi)\,d\varphi^i$

Consider $H(\varphi)$ - function on Φ called Hamiltonian.

Consider the action (H)

$$S = \int_0^T \Big(\underbrace{\alpha_i(\varphi)\frac{d\varphi^i}{dt}}_{1} - \underbrace{H(\varphi)}_{2}\Big)dt = \int_{[0;T]} \varphi^*(\alpha) - \varphi^*(H)\,dt$$

$$\delta S = \underbrace{\delta\varphi^j\left(\partial_j\alpha_i\frac{d\varphi^i}{dt}\right)}_{1} + \alpha_i\frac{d}{dt}(\delta\varphi^i) - \underbrace{\frac{\partial H}{\partial \varphi^j}\delta\varphi^j}_{3}$$

$$\alpha_j\frac{d}{dt}(\delta\varphi^j)$$

due to int. by parts

$$\underbrace{-\frac{\partial\alpha_j}{\partial\varphi^i}\frac{d\varphi^i}{dt}}_{2}$$

① + ② $\rightarrow$ $$\underline{\underline{(\partial_j \alpha_i - \partial_i \alpha_j)}} \frac{d\varphi^i}{dt} \delta\varphi^j - \frac{\partial H}{\partial \varphi^j} \delta\varphi^j$$

Altogether: $$\omega_{ji} = (d\omega)_{ji}$$

$$\omega_{ji} \frac{d\varphi^i}{dt} = \frac{\partial H}{\partial \varphi^j}$$

$$\frac{d\varphi^i}{dt} = (\omega^{-1})^{ij} \frac{\partial H}{\partial \varphi^j}$$ – Poissonian (P) dynamics

derived from Extr. action principle.

Idea is to understand the Extr. action principle as an ass. of the F.I. for

$$\underline{I} = \int \mathcal{D}\varphi \exp \frac{i}{\hbar} S(\varphi) \qquad (\underline{I})$$

$\hbar \to 0$

the oscillating integral is dominated on its extremal values of $S(\varphi)$.

Observables in $(\underline{I})$ are just functions on the space Φ: $\mathcal{O} \in \mathrm{Fun}(\Phi)$

Expectation value of $\mathcal{O}$ is: (F.I. formula)

$$\langle \mathcal{O}(\varphi) \rangle_t = \int \mathcal{D}\varphi\, \mathcal{O}(\varphi(t)) \exp\left(\frac{i}{\hbar} S(\varphi)\right)$$ is dominated on $\varphi_{cl}(t) \rightarrow$ solutions to E.V. problem.

put boundary conditions on the fields φ

In advance $\rightarrow$ boundary conditions are given by Lagrangian submanifolds $\underbrace{\text{in } \Phi}_{\mathcal{L}_0, \mathcal{L}_T}$

$$\langle \mathcal{O}(\varphi) \rangle_{t, \mathcal{L}_0, \mathcal{L}_T}$$

States in functorial approach are "functions" on $\mathcal{L}$.

$$\langle \mathcal{O}(\varphi)\rangle_{t, \mathcal{L}_0, \mathcal{L}_T} = \langle \text{out}| e^{\frac{i(T-t)H}{\hbar}} \hat{\mathcal{O}} \, \underline{\underline{e^{\frac{itH}{\hbar}}}} |\text{in}\rangle$$

corresponds to dependence on $\mathcal{L}_T$

corresponding to dep. on $\mathcal{L}_0$

$$\frac{\partial}{\partial t}\langle \mathcal{O}(\varphi)\rangle_{t, \mathcal{L}_0, \mathcal{L}_T} = \langle \text{out}| e^{\frac{i(T-t)H}{\hbar}} \left(-\frac{i}{\hbar}H\hat{\mathcal{O}} + \frac{i}{\hbar}\hat{\mathcal{O}}H\right) e^{\frac{itH}{\hbar}}|\text{in}\rangle = -\frac{i}{\hbar}\langle \text{out}| e^{\frac{i(T-t)H}{\hbar}}[H, \hat{\mathcal{O}}] e^{\frac{itH}{\hbar}}|\text{in}\rangle =$$

$$= -\frac{i}{\hbar}\langle [H, \hat{\mathcal{O}}]\rangle_{t, \mathcal{L}_0, \mathcal{L}_T}$$

come from Functorial approach

come from F.I.

|| use least action principle

replace $\frac{\partial \mathcal{O}(\varphi)}{\partial t}$ by $\{H, \mathcal{O}\}$

$$\langle \{H, \mathcal{O}\}\rangle = \frac{i}{\hbar}\langle [H, \hat{\mathcal{O}}]\rangle$$

Comparing we get that $i[\hat{H}, \hat{\mathcal{O}}]$ increased by $\frac{i}{\hbar}$ goes to P.B.

The naive way to get the semiclassical limit ($j \to \infty$) would be to study just the action (H) in the functional integral.

However, there is a problem:

For ω corresponding to P.B. $\{\mathcal{O}^{cl}_a, \mathcal{O}^{cl}_b\} = \epsilon_{abc}\mathcal{O}^{cl}_c$ there is no α! Actually, this form is just closed but not exact.

There are two ways to save the F.I.A.

Way 1.

Consider instead of theory of maps $\overset{\bullet\!\!\longrightarrow\!\!\bullet}{0 \quad T} \to \mathbb{C}P_1$ a 1-dim gauge theory for $\mathbb{C}^2$.

Take an action

$S_0 = \sum_{\alpha=0}^{1} \bar{z}_\alpha \frac{dz_\alpha}{dt}$ and gauge the $U(1)$ symmetry

$$(\epsilon): \ z_\alpha \to e^{i\epsilon} z_\alpha \quad \bar{z}_\alpha \to e^{-i\epsilon}\bar{z}_\alpha$$

(ϵ) is a symmetry of S_0 for constant ϵ, but we would like to go to ϵ dep. on t.

$$S_A = \sum_{\alpha=0}^{1} \bar{z}_a\left(\frac{d}{dt} - iA\right) z_a \qquad (A)$$

where no ϵ in (ϵ) depends on t

A - is a 1-form on an interval:

$A \to A + d\epsilon$.

Then (A) is invariant.

However, as we will see S_A is not interesting

$\boxed{S_{K,A} = S_A + K\int A}$ Reason is like this

I may consider the $S_{K,A}$ theory in following way:

1) It is quadratic in fields $\underline{z, \bar{z}}$ so I understand this theory. For given A states - funct. on boundary conditions are funct. of coordinates z_0, z_1.

$$\bar{z}_\alpha = \frac{\partial}{\partial z_\alpha}$$

2) $$e^{i\int A\left(z^\alpha \frac{\partial}{\partial z_\alpha} - K\right)}$$

Taking integral over A we have an extra condition for the state

$$\left[\sum_{\alpha=0}^{1} \left(z^{\alpha} \frac{\partial}{\partial z^{\alpha}} \right) - K \right] \Psi(z) = 0$$

It says that $\Psi(z)$ is homogenious of degree K (quantum version of the Gauss law)

We understand K as N.

For $K=0$ system is not interesting - its space of states ($\mathbb{C}P^0$) is a point.

Another way → First try to integrate over A and then integrate over Z.