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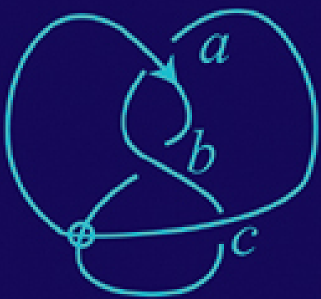
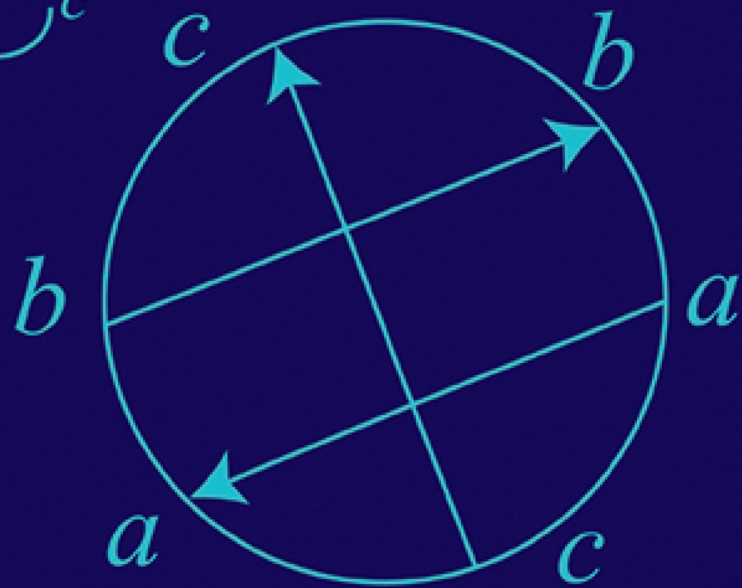
Virtual Knots

The State of the Art



Vassily Olegovich Manturov
and
Denis Petrovich Ilyutko



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Virtual Knots
The State of the Art

Series on Knots and Everything — Vol. 51

Virtual Knots

The State of the Art

Vassily Olegovich Manturov
Peoples' Friendship University of Russia, Russia

Denis Petrovich Ilyutko
Moscow State University, Russia &
Yaroslavl State University, Russia

World Scientific

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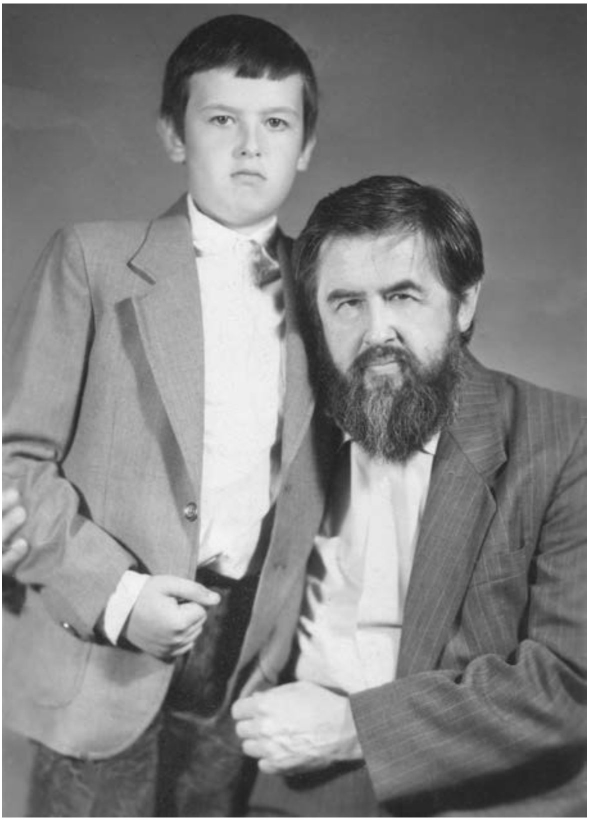
Dedication

This book is dedicated to the memory of my father Oleg Vassilievich Manturov (July 3, 1936 – July 23, 2011).



V. O. Manturov and O. V. Manturov, Moscow, 1988.

Oleg Vassilievich Manturov was a remarkable mathematician with a profound scientific sight. I mention two of his most significant achievements: the classification of homogeneous Riemannian spaces with irreducible rotation group (early 1960s) and the classification of complex vector bundles over compact homogeneous spaces (early 1970s). The first mentioned result is related to the classification of the most natural wide class of homogeneous spaces since E. Cartan classified his symmetric spaces (some years later, O. V. Manturov's result was rediscovered by J. Wolf; afterwards these spaces were called Manturov–Wolf spaces). The second mentioned result was a breakthrough in K-theory. Before my father's work, K-rings were constructed explicitly for a very small class of spaces; for a very huge class of homogeneous spaces O. V. Manturov classified all complex vector bundles explicitly in terms of maps to the Grassmann space; to this end, he handled the Bott isomorphism "by hand".

Since 1983, he was one of co-chairmen of the World's well known "Seminar on Vector and Tensor Analysis" (this seminar works in the Moscow State University since 1920s; in 1930s E. Cartan and J. Schouten gave talks at this seminar). The scope of my father's mathematical interests was very wide; he was working on product integration, calculation of homology groups of homogeneous spaces, decompositions of tensor products of Lie algebra representation; he led about 40 disciples to their Ph.D.'s.

He paid attention to my early interest in mathematics. When I was six and discovered a criterion of divisibility by 43 of positive integers, my father immediately explained to me how to find such divisibility criteria by other integers; some years later, he taught me how to construct the 17th square root of unity by compass and ruler, and how to classify the sections of the four-dimensional cube by hyperplanes. In the same period of time he brought me to "kruzhoks" (Russian-type seminars) held for schoolchildren in the Moscow State University ("small" mechanical and mathematical faculty).

It is important to mention the deep understanding of my father of the general importance of many problems I was working on. An advice concerning the general importance often led me to the true way. Understanding the deep importance of quantum invariants, the interrelation between knot theory and Lie group representation theory, he played a key role in the foundation of the seminar "Knots and the Representation theory" in 2000; in the mean time, this seminar is working very actively. My father warmheartedly supported my research in the virtual knot theory, especially the relations between atoms and virtual knots. He praised highly the parity theory. He himself made an impact to the parity theory, when he constructed in a joint work with myself an invariant of free knots valued in conjugacy classes of the infinite dihedral group (and its generalizations), which later turned out to be a sliceness obstruction for free knots.

Up to the very last days Oleg Vassilievich was very interested in various areas of mathematics, which he discussed with me.

I hope the book will be a modest contribution to the memory of an outstanding mathematician and a remarkable person.

Vassily Olegovich Manturov, Moscow, 2012.

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Preface by Louis H. Kauffman

This book, is the first full-length book on the theory of virtual knots and links. I coined the term virtual knot theory and wrote a first paper[1] on this subject in 1999. From my point of view, the idea was to simultaneously have a diagrammatic theory that could handle knots in thickened surfaces and would generalize knot theory to arbitrary oriented, not necessarily planar, Gauss codes. These non-planar Gauss codes act in knot theory as abstract graphs behave in graph theory. They can be projected into the plane, and will acquire virtual crossings in the process. They can be embedded in surfaces of higher genus, and one can search for the surface of least genus that will support them. The abstract Gauss codes (often depicted as so-called Gauss diagrams) have Reidemeister moves defined for them irrespective of a choice of embedding of the code in a surface. Once embedded, they yield a knot in a thickened version of that surface. Classical knot theory embeds in virtual knot theory.

Many invariants of classical knots generalize directly to virtual knots, the fundamental group and the quandle among them. The first quantum invariant that has an easy generalization is the Jones polynomial. The bracket state sum model of the Jones polynomial generalizes directly to virtual knot theory and has a number of startling properties. One of these properties is that a non-trivial classical knot can be transformed to a virtual knot by changing only the orientation of some of its crossings (we call this process the *virtualization of the crossing*). If a subset of crossings is chosen so that switching those crossings unknots the classical knot, then the virtual knot obtained by reversing their orientations will be a non-trivial virtual knot with Jones polynomial equal to one. Thus there are infinitely many

[1]Kauffman, L. H. (1999). Virtual knot theory, *European J. Combin.* **20**, 7, pp. 663–690.

non-trivial virtual knots with unit Jones polynomial. We would like to know if any of these are actually classical knots. We suspect that this is not the case and the research on this problem has led to the invention/discovery of new invariants by myself and by many other workers in virtual knot theory, including particularly the authors of this book and Vassily Manturov most particularly. This problem about the Jones polynomial is a driving force behind much of the research on virtual knot theory. Of course, in back of that problem is the question whether the Jones polynomial detects the unknot for classical knots.

All quantum invariants of classical knots generalize to a special category of virtual knots and links that I call the *rotational virtuals* and are called in this book the *rigid virtuals*. Using the present terminology, a rigid virtual knot does not have a flat version of the first Reidemeister move available for its diagrams. In the paper[1] I began the study of this version of virtual knot theory and it is still in need of development. The reader will find a short exposition of rigid virtual knot theory in the present book in Chap. 4, in the paper[1] and in the paper[2] where we relate this theory both to a categorical approach to the virtual braid group and to quantum invariants derived from Hopf algebras. In any case, this underlines the meaning of "State of the Art" in the title of the present book. There is much that is in the course of development in virtual knot theory, and this book is a snapshot of that development from the point of view of the authors of the book.

Turning to the book itself, the reader will find a remarkable and encyclopedic treatment of the basics, with the first chapter devoted to the diagrammatic and combinatorial definitions and to a discovery of mine of which I am very fond – the self-linking number of a virtual knot[3]. This self-linking number is, in the Gauss-diagram language the sum of the crossing signs for those chords in the diagram that are *odd*. A chord in a chord diagram is odd if it is intersected by an odd number of other chords in the diagram. The appearance of odd chords is a feature that can only happen in a virtual knot or link. The self-linking number is surprisingly powerful and certainly the simplest combinatorial invariant that one can find for virtual knots. It is also the tip of the iceberg of *parity*, a subject that Manturov has expanded for virtual knot theory and found ways to exploit that are highly significant for virtual theory and, we expect for classical theory as

[2]Kauffman, L. H. and Lambropoulou, S. (2011). A categorical model for the virtual braid group, preprint, arXiv:math.GT/1103.3158.

[3]Kauffman, L. H. (2004). A self-linking invariant of virtual knots, *Fund. Math.* **184**, pp. 135–158.

well in an indirect way. Later in the book, in Chap. 8, this full development of Manturov Parity Theory for Virtual Knots is given in its entirety. Here again, we go from the basics of the first chapter to the frontiers of research. The surface structure of parity that one sees in the Gauss diagrams for virtual knots and links is certainly just an indication of deeper matters of parity that are in the works below that surface. The future of virtual knots and their relationship with classical knots turns on this deep parity.

Continuing to Chap. 2, we find a discussion of Kuperberg's Theorem and the surface genus of virtual knots. Kuperberg proved the remarkable theorem that if a virtual knot or link is represented in its minimal genus surface, then this embedding type is unique. This means that the virtual knots have a definite topological interpretation in their minimal genus surfaces. This result of Kuperberg has been very fruitful for getting deeper invariants of virtual knots. Heather Dye and myself found ways to use the Kuperberg theorem to get stronger version of the Jones polynonmial for virtuals[4] and Manturov did a similar analysis that will be found in this book. This chapter proves that virtual knots are algorithmically recognizable by generalizing the technique of Haken and Hemion for classical knots and it proves that connected sums of non-trivial virtual knots are non-trivial. This is particularly significant in the light of the phenomenon that connected sums of trivial virtual diagrams can yield non-trivial virtual knots!

In Chap. 3 the authors consider quandles and their generalizations for virtual knots. One of the most fruitful generalizations of invariants that has emerged from virtual knot theory is the use of the biquandle[5,6], a generalization of the quandle (which in turn generalizes the fundamental group of a knot or link) and its powerful applications to the theory. Here also the reader will find Manturov's remarkable uses of Lie algebraic techniques for invariants. In this same chapter one will find long virtual knots and clever invariants for them using Manturov's application of the precendence order that is available there and flat virtuals where one is looking at immersions of curves in surfaces. This leads to a notion of mine called the (flat) hierarchy where we have as many virtual crossings as one likes, ordered by some choice of an ordinal, in terms of their ability to move across one another.

[4]Dye, H. A. and Kauffman, L. H. (2005). Minimal surface representations of virtual knots and links, *Algebr. Geom. Topol.* **5**, pp. 509–535.

[5]Fenn, R., Jordan-Santana, M. and Kauffman, L. H. (2004). Biquandles and virtual links, *Topology Appl.* **145**, pp. 157–175.

[6]Kauffman, L. H. and Manturov, V. O. (2005). Virtual biquandles, *Fund. Math.* **188**, pp. 103–146.

Manturov's first steps at some invariants are found here for the hierarchy.

Chapter 4 does the basics of the Jones polynomial via the bracket state sum and introduces the concept of atoms (surfaces, orientable or not, bearing the virtual knot diagram). Now the pace picks up as we go to Chap. 5 and meet Khovanov homology for virtual knots. Manturov gave a first combinatorial solution to the question of constructing integral Khovanov homology for virtual knots and links. His theory is quite interesting and it needs to be compared with the Khovanov–Rozansky Categorified Link Homology and it needs to be computed. There is a rich vein of research material for the active reader who gets to this point in the book.

We are not done! Chapter 6 deals with virtual braids and the work of Kamada and Kauffman and Lambropoulou[7]. It ends with an exposition of Manturov's invariants of virtual braids.

Chapter 7 treats Vassiliev invariants and returns to another source of virtual knot theory – the work of Goussarov, Polyak and Viro who used virtual knots in the guise of general Gauss diagrams to construct a theory of Gauss diagram formulas for virtual knots. Here we find that theory and a exposition of Manturov's proof of a conjecture of Vassiliev about these diagrams. Then comes Chap. 8 on Parity, alluded to before. This chapter includes work on the Goldman bracket, and the Turaev cobracket and on cobordisms of free knots. What is a *free knot*? In this book's terminology it is a flat virtual knot taken up to virtualization consisting in allowing one to exchange adjacent virtual and real crossings. This frees up part of the restriction at a vertex. The theory of free knots has been pursued with much energy by Manturov and his collaborators. At first it was thought (a conjecture of Turaev) that free knots were trivial, but Manturov showed, using parity that this is not the case and that there are non-trivial cobordism classes of free knots. The analogous virtual knot theory is to study virtual knots (with over and undercrossings) up to change of orientation of the crossings as we describe at the beginning of the preface, that is to study virtual knots up to virtualization equivalence. The theory of free knots sheds light on these questions of virtualization. And finally there is Chap. 9 on Graph-Links, a further combinatorial generalization of the theory that has much promise in illuminating its original problems.

We hope that the reader of this preface is motivated to delve into the adventure presented by this remarkable book.

[7]Kauffman, L. H. and Lambropoulou, S. (2006). Virtual braids and the L-move, *J. Knot Theory Ramifications* **15**, 6, pp. 773–811.

Preface by Vassily Olegovich Manturov & Denis Petrovich Ilyutko

This book is the first systematic research completely devoted to a comprehensive study of virtual knots and classical knots as its integral part.

Classical knot theory counts more than two hundred years. As a mathematical theory, it appeared in 19th century with an important impact to knot theory due to Vandermonde, Dehn [67, 68], Gauss [104], Klein, Tait, Reidemeister [271], Alexander [5, 7] and other outstanding scientists. Over the last decades, knot theory was enriched by numerous methods and subtle invariants, which nowadays constitute a powerful machinery in knot theory and low-dimensional topology. A breakthrough in knot theory which led to its modern state-of-art and solutions to many long-standing problems is due to discoveries of Conway [63] and Jones [141], and later, of Vassiliev [304, 305], is dated by the last thirty years of the 20th century (the Conway and Jones polynomials, Vassiliev's finite type invariants). For their impact to knot theory which related knot theory to various branches of mathematics and physics, Jones, Witten, Drinfeld (1990) and Kontsevich (1998) were awarded the highest honor of mathematics, the Fields medals. We recommend the following textbooks and monographs on knot theory: [154, 156, 162, 215, 267].

Classical knot theory (i.e. the theory of knots in three-dimensional Euclidean space or in the three-sphere), is an integral part of a much larger theory, knots in 3-manifolds. For the latter theory, the approach is much less developed.

Virtual knots were discovered by Kauffman in 1996, [158, 159], and independently by Goussarov, Polyak and Viro [114]. When virtual knot theory arose, it became clear that classical knot theory was a small integral part of a larger theory, and studying properties of virtual knots helped one to understand better some aspects of the classical knot theory and stimulated the statement of some problems, see [86]. Virtual knot theory finds its applications in classical knot theory. By means of virtual knot theory, the problem of existence of combinatorial formulae for finite type invariants for classical knots was solved [114]. Virtual knot theory occupies an intermediate position between theory of knots in arbitrary 3-manifolds and classical knot theory. On one hand, it is much wider than classical knot theory, and on the other hand, it is very close to classical knot theory because of many reasons we are going to describe below. Consequently, many methods and invariants of classical knot theory can be extended to virtual knot theory. This extension often uses additional ideas presented in the present book. Among such ideas we can mention new powerful invariant, Khovanov homology (1997). The latter is the homology of an algebraic complex which is constructed with a diagram of a knot (link). As it was shown by Kronheimer and Mrówka [187], Khovanov homology detects the unknot, which seriously enlarges horizons of the Khovanov homology theory.

In 1990s, some branches of low-dimensional geometry and topology experienced a rapid development; these theories are connected to knot theory and have their own interest. The theory of Legendrian knots, lying in the junction of knot theory and contact geometry [55, 83, 84, 100], has revealed, and many important problems in this theory were solved. This theory has numerous analogues and is connected to various branches of geometry, topology, optics and physics. The Khovanov theory [176] associates with each knot diagram a chain complex, whose homology is a knot invariant, and the Euler characteristic of this complex coincides with the Jones polynomial. The chains of the complex constructed with a knot diagram, correspond to formal smoothings of this diagram at all classical crossings. Every smoothing of such sort gives rise to a collection of some number k of circles; with this state we associate a chain space which is the kth tensor power of some two-dimensional space. Differentials of the Khovanov complex are defined combinatorially, and the homology is invariant under Reidemeister moves.

In order to extend Khovanov homology theory to virtual knots, the first-named author (V. O. Manturov) had to revisit completely the definition of the usual Khovanov homology and construct a complex homotopically equivalent to the "usual" Khovanov complex. The problem of such a construction in the case of virtual knots is related to the well definedness in order for the square of the differential to be equal to zero. Such a construction required a number of new ideas: orientation and enumeration of the state curves, twisted coefficients in the Frobenius algebra representing the homology of the unknot, usage of exterior products instead of usual (symmetric) products. A key role in this construction was played by the notion of atom (introduced by Fomenko [98]). Atoms and d-diagrams (special type of chord diagrams with two families of pairwise split chords, for definition see page 137) played the key role in the proof of Vassiliev's conjecture (Chap. 7). The d-diagram theory was developed in [203–205].

Let us describe a common point of view which allows us to treat classical and virtual knots uniformly. A classical knot (link) can be given by a planar diagram. In such a diagram there are classical crossings and curves connecting crossings to each other. If we locate crossings  on the plane in an arbitrary way and indicate how they are connected to each other, then sometimes the curves connecting crossings can be chosen to be non-intersecting (this happens in the case of a classical knot), and in some cases it is impossible to situate these curves without additional intersections. These new intersections are marked as virtual (encircled) and we get a virtual diagram. Virtual crossings appear every time when a 4-valent graph defined by classical crossings and ways of connection, is not planar, which happens quite often. An example of a virtual diagram is given in Fig. 0.1.

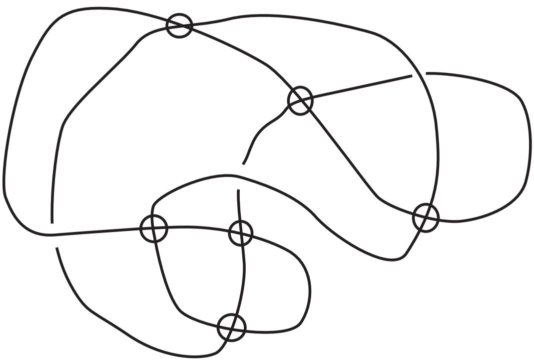

Fig. 0.1 A virtual diagram.

Thus, virtual knots are related to classical knots approximately in the same way as graphs are related to planar graphs.

Herewith, the equivalence (isotopy) of classical knot diagrams is defined by means of formal combinatorial transformations (Reidemeister moves), which are applied to crossings lying close to each other. For virtual knots given by a collection of classical crossings and ways to connect them, the equivalence is given by the same Reidemeister moves applied to classical crossings.

From the topological point of view, virtual knots are knots in thickened surfaces (products of sphere with handles with an interval) considered up to isotopy and stabilizations (see, e.g. [158, 160]). Herewith, when projecting the sphere with handles S_g to the plane, crossings of the diagram on S_g are projected to classical crossings; besides that, new (virtual) crossings appear as intersection points between different branches of the diagram S_g; such virtual crossings are depicted as encircled.

Note that in this generalization new theories were born: Virtual multidimensional knots (abstract knots, Kamada and Kamada [151]), virtual spatial graphs (Flemming and Mellor [93]) and others.

The recognition problem for classical knots was one of the central problems in low-dimensional topology. Its first solution is related to Haken's normal surface theory, and the final steps belong to Matveev. The result of algorithmic recognizability of a certain object in low-dimensional topology is important because in low-dimensional topology, the algorithmic *non-realizability* takes place for many objects. When virtual knot theory appeared, the problem of algorithmic realizability of virtual knots arose. This problem is resolved positively in Chap. 2 of the present book; this result relies not only on Haken's theory but also on Kuperberg's theorem.

Virtual knot theory, its constructions and methods are closely related to various branches of classical knot theory, in particular, to Vassiliev's invariant theory. Vassiliev invariants occupy a special position in classical knot theory; it turned out just when this theory appeared, that all polynomial and quantum invariants were expressible in terms of Vassiliev invariants. In the case of virtual knots, the theory of Vassiliev knot invariants is much more complicated; even the space of order zero invariants is infinite-dimensional. To combinatorial aspects of the Vassiliev invariant theory, we dedicate Chap. 7. In this chapter, by using atoms and d-diagrams, we prove Vassiliev's conjecture about planarity of framed 4-valent graphs (graphs where at each vertex four half-edges are split into two pairs of opposite ones); this conjecture solved positively by the first-named author, plays a key role in Vassiliev's work on the existence of integer-valued combinatorial formulae for invariants of finite order.

In virtual knot theory, there are many unexpected invariants which do not take place in the classical case.

The most striking example of such a theory is the *parity theory* conceived by the first-named author, where all classical crossings are either *even* or *odd*, herewith the property of being even is naturally preserved by Reidemeister moves. Roughly speaking, by parity we mean any such natural way of labeling of all classical crossings which is defined for all knots from this theory. By means of parity, one can construct functorial mappings from knots to knots, filtrations on the space of knots, refine many invariants and prove minimality of many series of knot diagrams. The existence of different parities and different projections (from knots to knots) allows one to establish various filtrations on the space of knots. Besides that, such projections allows one to "lift" invariants from classical knots to virtual knots.

Chapter 8 is devoted to parity in knot theory.

The passage from classical knots to virtual knots can also be motivated by representing Reidemeister moves in the language of Gauss diagrams. Every Gauss diagram is a circle with a collection of pairs of points (all points mutually disjoint); every pair of points is endowed with an arrow from one point to the other and a sign. Each chord diagram of such sort has an *intersection graph*. Vertices of the intersection graph correspond to chords, and two vertices are adjacent whenever the corresponding chords are linked. To such graphs, one can extend Reidemeister moves. Note that not all simple (without multiple edges and loops) graphs originate from chord diagrams. When passing from intersection graphs of chord diagrams to arbitrary graphs and extending Reidemeister moves to such graphs, we end up with the *graph-link theory* due to the authors of the present book. An analogous theory was constructed by Traldi and Zulli [293]. Chapter 9 of the present book is devoted to the graph-link theory. Graph-links can be treated as "diagramless knot theory": Such "links" have crossings, but they do not have arcs connecting these crossings since the corresponding graphs are not intersection graphs of any chord diagrams and thus they are not "drawable" on the plane. It turns out, however, that to graph-links one can extend many methods of the classical and virtual knot theory, in particular, the parity theory. We have constructed various invariants, proved the equivalence of two approaches to graph-knots: the one suggested by the authors and the one suggested by Traldi and Zulli. We have constructed various invariants showing non-realizability of graph-links (the fact that a graph-link has no drawable representative). A remarkable achievement in the graph-link theory is the work by Nikonov [256, 257] who constructed Khovanov homology theory for graph-links with coefficients from $\mathbb{Z}_2$. Unlike the usual Kauffman bracket when one had to count the number of "non-existing" state-circles, for this problem one had to understand how these "non-existing" circles might interfere in order to construct the differential in the Khovanov complex.

The theories mentioned above are related to different problems of combinatorics, three-dimensional topology and four-dimensional topology, representation theory for Lie groups and algebras. Representation theory is the starting point for constructing *quantum invariants* of knots and 3-manifolds, see, e.g. [259, 297].

Acknowledgments

It is impossible to overestimate the influence of the creator of virtual knot theory Louis Hirsch Kauffman on our work. We express especial gratitude to him.

During the work on this book, it was extremely important for us to discuss various aspects of virtual theory with V. A. Vassiliev, O. V. Manturov, A. T. Fomenko, O. Ya. Viro, M. G. Khovanov, V. V. Chernov, P. Cahn, S. Carter, R. A. Fenn, M.-L. Ge, I. M. Nikonov, A. V. Chernavsky, H. A. Dye, V. G. Turaev, V. F. R. Jones, J. S. Carter, J. Przytycki, N. Yu. Netsvetaev, M. W. Chrisman, D. M. Afanasiev, D. A. Fedoseev, V. P. Ilyutko. We apologize for not being able to mention all specialists in virtual knot theory!

For over 10 years, different questions of virtual knot theory were discussed in the seminar "Knots and the representation theory" and "Seminar on Tensor and Vector Analysis" (the latter exists since 1920s) in the Moscow State University.

The first-named author was partially supported by grants of the Russian Government 11.G34.31.0053, RF President NSh – 1410.2012.1, Ministry of Education and Science of the Russian Federation 14.740.11.0794. The second-named author was partially supported by grants of the Russian Government 11.G34.31.0053, RFBR 10-01-00748-a, RF President NSh – 1410.2012.1, Ministry of Education and Science of the Russian Federation 14.740.11.0794.

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Chapter 1

Basic Definitions and Notions

1.1 Classical knots

A (*classical*) *knot* is the image of a smooth embedding of the standard circle S^1 in the three-dimensional sphere S^3 (or in the space $\mathbb{R}^3$); two knots are called *isotopic*, if one of them can be transformed to the other by a diffeomorphism of the ambient space S^3 (or $\mathbb{R}^3$) onto itself, preserving the orientation of the sphere S^3 (or $\mathbb{R}^3$) (such a diffeomorphism is called an *isotopy*) (it is well known that each of such diffeomorphisms is homotopic to the identical one in the class of diffeomorphisms [53]). If we embed a disjoint union of several circles $S^1 \sqcup \cdots \sqcup S^1$ in the sphere S^3 (or $\mathbb{R}^3$), then we get a *classical link*; the image of each circle, a knot, is called a *component of the link*. We call such a smooth embedding of a circle (a disjoint union of circles) also a *knot* (a *link*).

The main question of knot theory is the following: Which two knots are isotopic and which are not? This problem is called the *knot recognition problem*. A classification of knots in S^3 is equivalent to a classification of knots in $\mathbb{R}^3$. Having an isotopy equivalence relation, one can speak about knot isotopy classes. When seen in this context, we shall say "knot" when referring to the knot isotopy class.

If we fix an orientation of the circle S^1, then we have an *oriented knot* (respectively, in the case of an *oriented link* we require orientations of the circles, i.e. the preimages of the component of the link); under an isotopy of oriented links we require that the diffeomorphism of the ambient space preserves both the orientation of S^3 (or $\mathbb{R}^3$) and the orientations of all the components.

If the components of a link are numbered and, moreover, we require that the numbers of the components preserve under isotopies, then we have a *colored link.*

Definition 1.1. An oriented link is called *framed* if at each point of it a unit vector which is orthogonal to a component of the link and depends continuously on a point is specified; two framed links are called (*framed*) *isotopic* if there exists a mapping of the ambient space (the sphere S^3) on itself which preserves the orientation and takes one link to the other one in a way compatible with the vector fields.

Framed links can be treated as embeddings of a set of bands $S^1 \times I$ and Möbius bands $S^1 \tilde{\times} I$ in the three-dimensional Euclidean space (the three-dimensional sphere); the corresponding vector field points out a way in which these bands are twisted.

Usually, knots are encoded by plane diagrams, which are 4-valent graphs with an additional structure. Let us give main definitions concerning 4-valent graphs and plane knot diagrams (knot diagrams).

Let G be a graph with the set of vertices $V(G)$ and the set of edges $E(G)$ (we consider only finite graphs, i.e. graphs with finite sets of vertices and edges). We think of an *edge* as an equivalence class of the two half-edges constituting the edge. We say that a vertex $v \in V(G)$ has the *degree* k if v is incident to k half-edges. A graph whose vertices have the same degree k is called k*-valent* or a k*-graph.* The *free loop*, i.e. the graph without vertices but with one *cyclic edge*, is considered as k-valent graph for any k.

A graph is called *simple*, if it has no multiple edges and loops.

Definition 1.2. A 4-graph is called *framed* if for every vertex the four emanating half-edges are split into two pairs of (formally) opposite edges, i.e. we have the structure of opposite edges. The edges from one pair are called *opposite to each other.*

Definition 1.3. By an *isomorphism* between framed 4-graphs we assume a framing-preserving isomorphism. All framed 4-graphs are considered up to isomorphism. Denote by G_0 the framed 4-graph isomorphic to the circle (the free loop).

When drawing framed graphs on the plane, we always assume the framing to be induced from the plane $\mathbb{R}^2$.

Fix a knot, i.e. the image of an embedding $f : S^1 \to \mathbb{R}^3$. Consider a plane $h \subset \mathbb{R}^3$ (say, $h = Oxy$) and the projection of the knot on it. Without loss of generality we can assume that the projection of the knot is a finite embedded framed 4-valent graph being the image of a smooth immersion of the circle in the plane. Usually, we shall call a part (i.e. the image of an interval in the three-dimensional space, a part of the circle) of a knot a *branch* of it. Sometimes (when it is clear from the context) we also use the term *branch* for the image of a branch in the plane. Each vertex of this graph (also called a *crossing of the diagram of the link*) is endowed with the following additional structure. Let a and b be two branches of a knot, whose projections intersect each other in the point v. Since a and b do not intersect in $\mathbb{R}^3$, the two preimages of v have different z-coordinates. So, we can say, which branch (a or b) goes over (forms an *overcrossing*); the other one goes under (forms an *undercrossing*), see Fig. 1.1. Half-edges of overcrossings are depicted by continuous lines, and half-edges of undercrossings are depicted by lines having a break at the crossing. This image of a knot on the plane is called a *plane knot diagram* or a *knot diagram*.

All crossings of a diagram of an oriented link are divided into *positive* ones  and *negative* ones . It is easy to check that in the case of a knot the sign of a crossing does not depend on an orientation of a knot.

A link diagram is called *alternating*, if when moving along each component one passes overcrossings and undercrossings alternately.

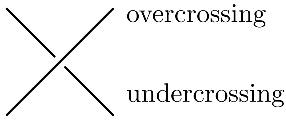

Fig. 1.1 The local structure of a crossing.

The simplest examples of diagrams of classical knots are depicted in Fig. 1.2.

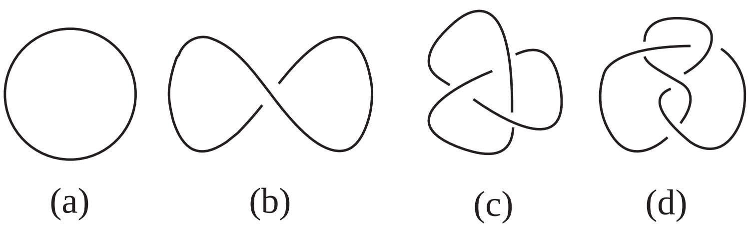

Fig. 1.2 The simplest knots.

The first two diagrams represent the *unknot* or the *trivial knot*, the third diagram represents the *trefoil*, and the knot shown in the fourth diagram is called the *figure eight knot*. All these diagrams are alternating.

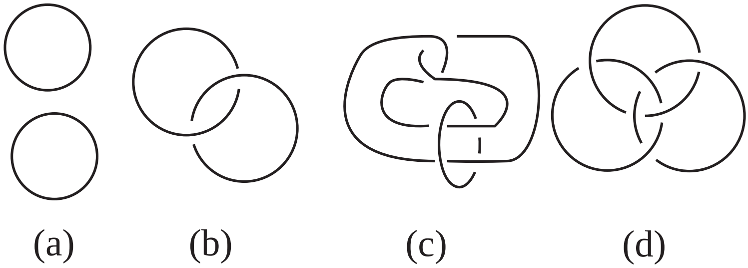

Fig. 1.3 The simplest links.

In Fig. 1.3(a) the *trivial link with two components* is depicted. In Figs. 1.3(b), 1.3(c) and 1.3(d) the *Hopf link*, the *Whitehead link* and the *Borromean rings* are shown, respectively. Each of these links is not trivial.

By an *arc* of a planar link diagram we mean a connected component of the diagram (thus, each arc always goes "over"; it starts and stops at undercrossings).

The framed 4-valent graph without an over/undercrossing structure is called the *shadow* of the knot or the *underlying graph*. The *complexity* of a knot is the minimal number of crossings for knot diagrams of a given isotopy type. We shall call a diagram of a link *connected* if its shadow (its underlying graph) is connected as a framed 4-graph. In particular, any diagram of a knot is connected.

Let us define knots by using the language of framed 4-valent graphs. In [271] Reidemeister presented a list of local topological moves (known as *Reidemeister moves*). The three basic Reidemeister moves are shown in Fig. 1.4, the others are obtained from these moves by using their combinations. Reidemeister proved that any two planar diagrams generated isotopic knots (links) if and only if there existed a finite chain of Reidemeister moves and planar isotopies (diffeomorphisms of the plane on itself preserving the orientation of the plane) from one of them to the other. The Reidemeister theorem allows one to consider isotopy classes of links as combinatorial objects: They represent equivalence classes of planar diagrams modulo the Reidemeister moves. In [119] Hass and Lagarias proved the existence of an exponential upper bound (depending on the crossing number) on the number of Reidemeister moves required to convert a diagram representing the unknot to a trivial knot diagram.

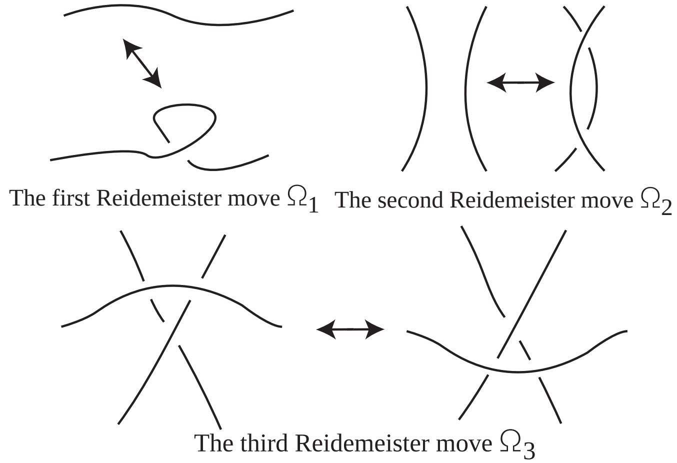

Fig. 1.4 Reidemeister moves $\Omega_1, \Omega_2, \Omega_3$.

The Reidemeister moves are a starting point for a combinatorial definition of virtual knots (see below). To prove the invariance for some functions on knots (given on diagrams of knots), one should check its invariance under the Reidemeister moves.

One of the earliest invariants of knots is a topological invariant, the *knot group*. The *knot group* of a knot (link) $\mathcal{K} \subset \mathbb{R}^3$ (denoted by $\pi(\mathcal{K})$) is the fundamental group of the complement to the knot (link) $\mathcal{K}$, more precisely, to a small tubular neighborhood $N(\mathcal{K})$ of it, namely, we set $\pi(\mathcal{K}) = \pi_1(\mathbb{R}^3 \setminus N(\mathcal{K}))$.

It is known that the knot group is a very strong invariant. For instance, the famous Dehn–Papakyriakopoulos' theorem [265] asserts that a unique knot having the knot group isomorphic to the group $\mathbb{Z}$ is the unknot.

From a diagram of a knot (link) we can construct a presentation of the knot group (which is called a *Wirtinger presentation*). There is also a *Dehn presentation*, which assigns generators to domains of the plane of the projection, and relations to crossings.

The Wirtinger presentation is constructed as follows. Let us consider a knot (link) $\mathcal{K}$ given by some planar diagram K. Consider a point x "hanging" over this plane. Without loss of generality we may assume that

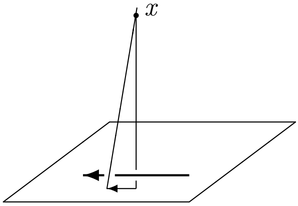

Fig. 1.5 Loops corresponding to edges.

$K \subset \mathbb{R}^2 \times \{0\}$ and $x \in \mathbb{R}^2 \times \{1\}$. Let us classify isotopy classes of loops in $\mathbb{R}^3 \backslash N(\mathcal{K})$ outgoing from this point. It is easy to see (the proof is left to the reader) that one can choose generators in the following way. All generators are classes of loops outgoing from x and hooking the arcs of the planar diagram K. Let us decree that the loop corresponding to an oriented edge is a loop turning according to the right-hand rule (each such loop bounds a disc intersecting exactly one arc of the knot at exactly one point), see Fig. 1.5.

Now, let us find the system of relations for this group.

It is easy to see the geometrical connection between loops hooking adjacent edges (i.e. edges separated by an overcrossing edge). Actually, we have $b = cac^{-1}$, where c separates a and b, see Fig. 1.6.

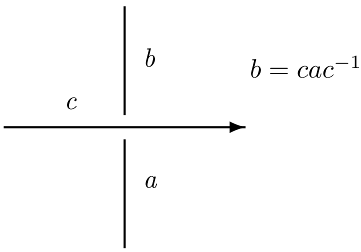

Fig. 1.6 Relation for a crossing.

From the geometrical consideration, it easily follows that the given set of relations generates all relations in the knot group.

Thus, the Wirtinger presentation of the fundamental group of the link complement is constructed as follows. The arcs correspond to the generators and the generating relations come from crossings. We take $cac^{-1} = b^{-1}$ for adjacent edges a and b, separated by c, when the edge b lies on the left from c with respect to the orientation of c.

The main disadvantage of the knot (link) group is that groups given by different presentations are very hard to compare. In the general case, the problem of recognition of a group by its presentation is an unsolved problem. However, one can construct more convenient (but more powerful) invariants of knots and links and prove structural theorems by using a knot group.

The fundamental group is an almost complete knot invariant, in particular, it recognizes prime knots. A knot is called *prime*, if it cannot be represented as the connected sum of two non-trivial knots; see Chap. 2 for the definition of the operation of connected sum. More precisely, two prime oriented knots have isomorphic fundamental groups if they are either isotopic to each other or obtained from each other by changing the orientation of a knot and/or the ambient space. It can be a non-trivial problem to determine whether a given knot is prime or not. There is a prime decomposition for knots [277].

A natural formal generalization of the notion of the fundamental group is a notion of a *distributive groupoid* or a *quandle*, introduced independently by Matveev [242] and (a little later) Joyce [144], also see [71].

The idea is as follows. We change the group relation $b = cac^{-1}$ at a crossing by a formal relation $a \circ c = b$.

Further, we describe a formal object, generators of which are arcs of the diagram and we put formal relations on generators. We want this object to be invariant under the Reidemeister moves. As a result, we immediately get the following necessary axioms:

(1) *idempotency*: $a \circ a = a$;
(2) *existence of left-inverse element*: for fixed elements b, c there exists a unique element x such that $x \circ b = c$. In this case we shall write $x = c/b$;
(3) *right self-distributivity*: $(a \circ b) \circ c = (a \circ c) \circ (b \circ c)$.

These axioms correspond to the first, second and third Reidemeister moves respectively.

This object (the pure axiomatic definition will be given in Chap. 3) is stronger than the fundamental group. A straightforward check of the axioms of a distributive groupoid for the operation $a \circ b = bab^{-1}$ in the group shows that this operation satisfies all axioms of a distributive groupoid and defines a morphism from the category of groups to the category of distributive groupoids, and the preimage of the distributive groupoid of a knot is the fundamental group of the complement to the knot (considered up to isomorphism).

Chapter 3 of the book is devoted to distributive groupoids in theory of virtual and classical knots.

1.2 Virtual knots

The main object of Louis Kauffman's theory (1996) and the present book is a *virtual knot* (or, in the case of many components, a *virtual link*). A virtual knot represents a natural combinatorial generalization of a classical knot: We introduce a new type of a crossing and extend new moves to the list of the Reidemeister moves. The new crossing (which is called *virtual* and depicted by a small circle) should be treated neither as a passage of one branch over the other one nor as a passage of one branch under the other. It should be treated as a diagrammatic picture of two parts of a knot (a link) on the plane which are far from each other, and the intersection of these parts is artifact of such a drawing, see Fig. 1.7. In that sense the following list of *generalized Reidemeister moves* is natural: All classical Reidemeister moves related to classical crossings and a *detour move*. The latter represents the following: A branch of a knot diagram containing several consecutive virtual crossings but not containing classical crossings can be transformed into any other branch with the same endpoints; new intersections and self-intersections are marked as virtual crossings, see Fig. 1.7.

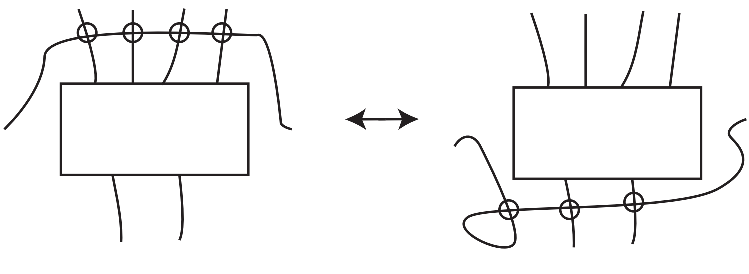

Fig. 1.7 Detour move.

Definition 1.4. A *virtual diagram* (or a *diagram of a virtual link*) is the image of an immersion of a framed 4-valent graph in $\mathbb{R}^2$ with a finite number of intersections of edges. Moreover, each intersection is a transverse double point which we call a *virtual crossing* and mark by a small circle, and each vertex of the graph is endowed with the classical crossing structure (with a choice for underpass and overpass specified). The vertices of the graph are called *classical crossings* or just *crossings*.

A virtual diagram is called *connected* if the corresponding framed 4-valent graph is connected.

A *virtual link* is an equivalence class of virtual diagrams modulo the generalized Reidemeister moves. The latter consist of the usual Reidemeister moves referring to classical crossings and the detour move.

Thereby, in order to define a virtual knot, we need only to know the position of classical crossings and their connections with each other. Moreover, positions of paths connecting classical crossings, their intersections and self-intersections, are not important for us.

Like a classical link a virtual link has the number of *components*. A *virtual knot* is a virtual link with one (*unicursal*) *component*.

The components of a virtual link can be described combinatorially by using virtual diagrams.

Definition 1.5. By a *unicursal component* of a virtual diagram K we mean the following. Consider K as a one-dimensional cell-complex on the plane. Some of the connected components of this complex are circles. We call each of these components a *unicursal component* of K. The remaining part of the cell-complex represents a framed 4-valent graph with vertices which are classical or virtual crossings. *Unicursal components* of K are (besides circles) equivalence classes on the set of edges of the graph: Two edges e, e' are *equivalent* if there exists a collection of edges $e = e_1, \dots, e_k = e'$ and a collection of vertices $v_1, \dots, v_{k-1}$ (some of them may coincide) of the graph such that edges e_i, e_{i+1} are opposite to each other at the vertex v_i.

It is easy to see that the number of components of a virtual diagram is invariant under the generalized Reidemeister moves. Therefore, we can define unicursal components of a link as unicursal components of a diagram of the link. In the classical case, this definition coincides with the definition given above.

The *writhe*, $w(K)$, of a virtual diagram K is the number equal to the number of positive crossings  minus the number of negative crossings .

Remark 1.1. Note that such approach, the standard moves inside a local Euclidean domain and the detour move, was used by Kamada and Kamada [151] for constructing formal theories of multidimensional "virtual knots" and their invariants.

This brings us to the following interpretation of virtual knots described in terms of Gauss diagrams (see, e.g. [57, 123, 124]).

Definition 1.6. By the *Gauss diagram* corresponding to a planar diagram of a (virtual) knot we mean a diagram consisting of an oriented circle (with a fixed point, not a preimage of a crossing) on which the preimages of the overcrossing and the undercrossing (for each classical crossing) are connected by an arrow directed from the preimage of the undercrossing to the preimage of the overcrossing. Each arrow is endowed with a sign coinciding with the sign of a crossing, i.e. it equals 1 if we have  and -1 for .

Remark 1.2. A Gauss diagram is a particular case of a *chord diagram*, see Definition 4.2. More precisely, a Gauss diagram is a chord diagram with an additional structure.

The Gauss diagram of the right-handed trefoil is shown in Fig. 1.8.

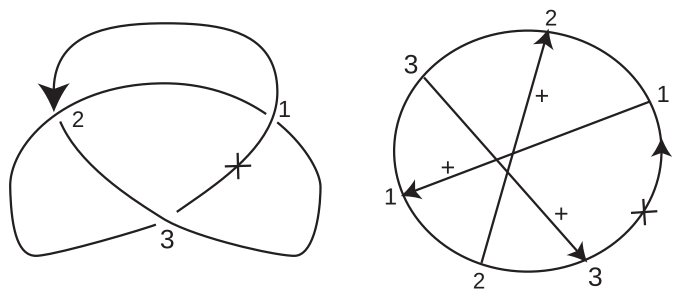

Fig. 1.8 The Gauss diagram of the right-handed trefoil.

Gauss diagrams (with some information lost; we forget about the orientation of chords and signs) can be encoded by double occurrence words, i.e. each letter in a word occurs precisely twice, and, moreover, it does not matter which letters are used, but it is important whether two letters are the same or not. Let us attribute a letter to each classical crossing (different crossings have different letters). To each classical crossing, an arrow of the Gauss diagram is assigned. Let us place each letter corresponding to a crossing near the head and the end of the arrow corresponding to the crossing. By traveling along the circle of the Gauss diagram from the fixed point and writing down consequently the letters met by us, we get the double occurrence word. Let us call this word the *Gauss code* of a knot diagram. For instance, let us attribute the letters a, b, c to the crossings of the classical trefoil. Then the Gauss code is $abcabc$.

Arbitrary Gauss diagrams, generally speaking, can be non-realizable by an embedding of a framed 4-valent graph in the plane, but they can be realized by means of a *generic immersion* in the plane by making points having more than one preimage (in a generic case they have exactly two preimages) virtual crossings, see Fig. 1.9.

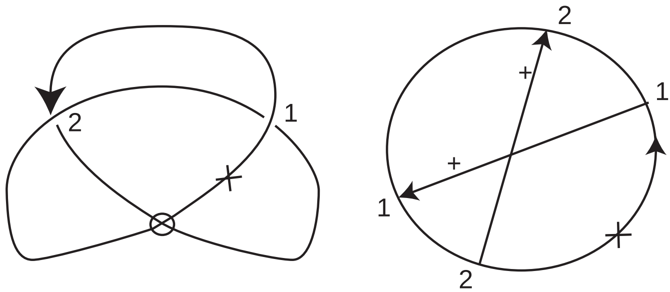

Fig. 1.9 The Gauss diagram of a virtual knot.

This naturally leads us to the next definition of a virtual knot (not links): One has to consider all formal Gauss diagrams and describe formal Reidemeister moves on them (as it was done in the case of classical knots). They will represent combinatorial scheme of transformations of Gauss diagrams. In that case, equivalence classes of Gauss diagrams over formal Reidemeister moves are virtual knots. Note that we do not need the detour move as a Gauss diagram "knows" nothing about position of virtual crossings on the plane, and it "knows" only position of classical crossings and their connections with each other. It means that Gauss diagrams "feel" only classical Reidemeister moves and do not "feel" the detour move. The exact list of Reidemeister moves for Gauss diagrams are given by us in Chaps. 8 and 9. Further, it allows us to construct a new theory, the *theory of graph-links*, see Chap. 9.

Virtual knots (here by a knot we mean also a link) can be defined as knots in thickened oriented surfaces $S \times I$, where S is a two-dimensional oriented closed surface and $I = [0, 1]$ is the segment with a fixed orientation; moreover, thickened surfaces should be considered up to stabilizations, i.e.

up to additions and removals of handles to S in such a way that additional thickened handles do not touch our knot (see more detailed description in Chap. 2 and in [158, 160]). Branches of a virtual knot having a virtual intersection for a virtual diagram and related to two parts of the virtual knot located far away from each other, can move freely on the surface independently from each other.

From now on, we suppose that the structure of direct product is fixed on a thickened surface $S \times I$ and it is pointed toward which side is up, say the side $S \times \{1\}$, and which one is down, say the side $S \times \{0\}$.

In the case of links, one can consider a disjoint union of manifolds, $S = S_1 \sqcup \cdots \sqcup S_k$ (sometimes it is required that in each manifold $S_j \times I$ there is at least one component of a virtual link, [189]).

Links in $S \times I$ can be described by diagrams on S with the under/overcrossing structure specified. In that sense, virtual diagrams are obtained by means of regular generic projections of diagrams from S to the plane: Crossings pass to classical crossings and new intersections (artifacts) are marked by virtual crossings; moreover, it is required that under regular generic projections neighborhoods of classical crossings are mapped into the oriented plane with the orientation preserved. The Reidemeister moves for diagrams on S (the same as in the case of classical diagrams) correspond to the classical Reidemeister moves for virtual diagrams; there are also transformations which do not change the combinatorial structure of diagrams on S, but do change the combinatorial structure of the projection to the plane. The detour move corresponds to these transformations.

The theorem about an equivalence of different definitions of virtual knots was announced in [158] and proved by different authors, including Kauffman. The full detailed proof can be found, e.g. in [221].

In addition to virtual knots there exists a theory of "twisted virtual knots" introduced by Bourgoin [42] which represents virtual knots in oriented thickenings of non-orientable surfaces.

A realization of the detour move by moves on thickened surfaces and their projections is shown in Fig. 1.10.

This leads us to local versions of the detour move which consist of:

(1) *virtual Reidemeister moves* $\Omega_1', \Omega_2', \Omega_3'$, which are obtained from the classical Reidemeister moves by swapping all classical crossings participating in moves for virtual crossings, see Fig. 1.11;
(2) *semivirtual version* Ω_3'' of the third Reidemeister move. Under this move the branch containing two virtual crossings can be carried through a

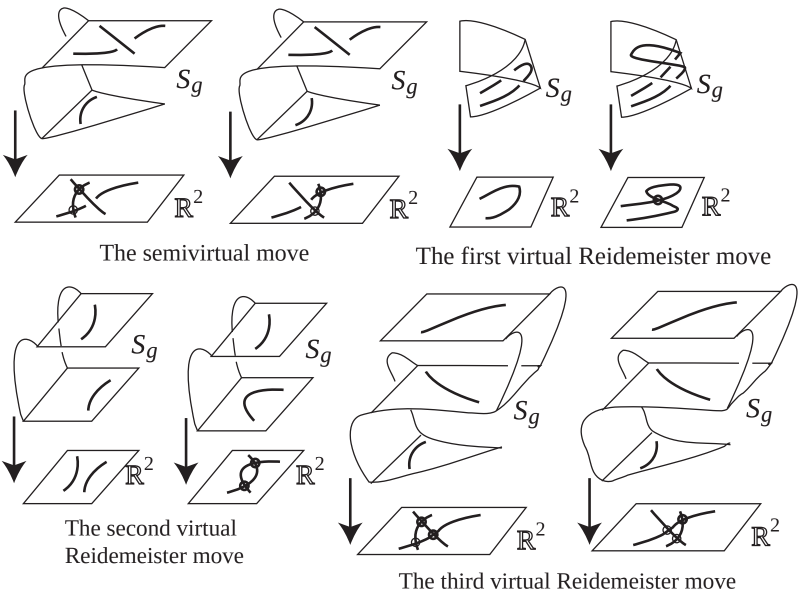

Fig. 1.10 Generalized Reidemeister moves and thickened surfaces.

classical crossing, see Fig. 1.12.

Definition 1.7. We call a Reidemeister move *increasing* (respectively, *decreasing*) if this move increases (respectively, decreases) the number of crossings (the number of classical crossings in the classical case and the number of virtual ones in the virtual case).

For example, the moves Ω_1, Ω_2, Ω'_1, Ω'_2 are increasing "in one direction", and decreasing "in the opposite direction".

The following statement is obvious.

Statement 1.1. *Two virtual diagrams K and K' are obtained from each other by a finite sequence of detour moves if and only if they are obtained from each other by a finite sequence of the moves Ω'_1, Ω'_2, Ω'_3, Ω''_3 and their inverses.*

The reconstruction of a knot diagram in a thickened surface by a virtual diagram on the plane is as follows.

Let K be a virtual diagram on the sphere S^2 (we compactify the plane $\mathbb{R}^2$ by adding one point). Each virtual crossing of this diagram corresponds

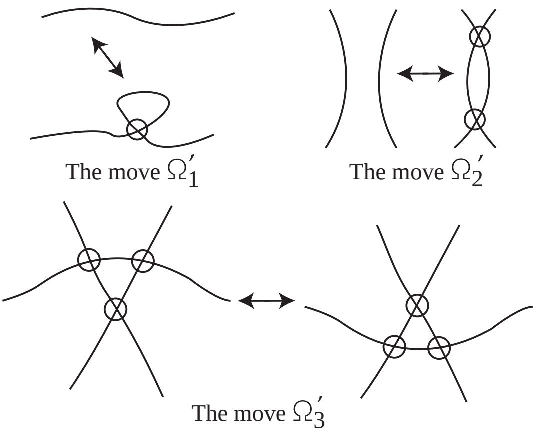

Fig. 1.11 Moves $\Omega_1', \Omega_2', \Omega_3'$.

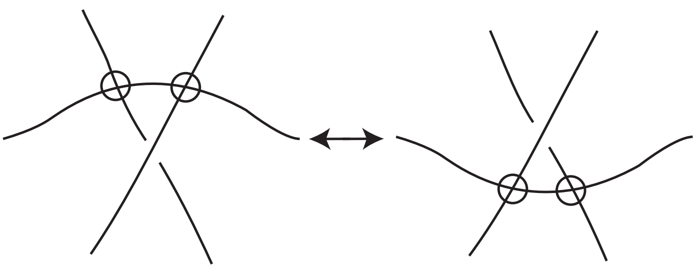

Fig. 1.12 The semivirtual move Ω_3''.

to an intersection of two arcs. Let us choose one of them and construct a handle for its "lifting", see Fig. 1.13. As a result, we get a diagram (with over/undercrossings and virtual crossings) on the torus, the number of virtual crossings of which is less by one than the number of virtual crossings in the initial diagram.

Note that the choice for a position of a handle, up or down, is immaterial since thickened surfaces are considered on their own account without any embedding into $\mathbb{R}^3$.

It is also easy to check that it does not matter which arc we choose for lifting it to a new handle. Two diagrams K_1 and K_2 corresponding to two such lifts to surfaces M_1 and M_2 with handles, i.e. $K_1 \subset M_1$ and $K_2 \subset$

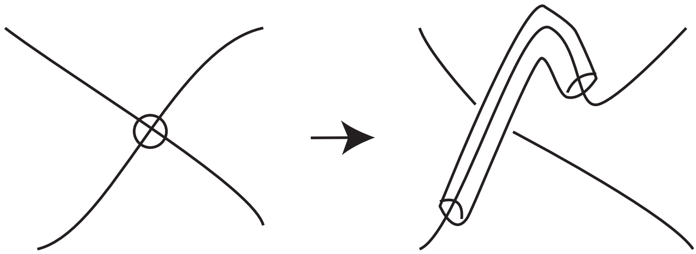

Fig. 1.13 Lifting of a virtual crossing to a handle.

M_2, will be combinatorially equivalent (i.e. there exists a homeomorphism $f\colon M_1 \to M_2$ of one lifting to the other one and f transforms one virtual diagram to the other one, $f(K_1) = K_2$).

Continuing this process we can get rid of all the virtual crossings and obtain a diagram on S_g $(=S_g \times \{\frac{1}{2}\} \subset S_g \times [0,1])$, where g is the number of handles. Here it is convenient to use detour moves. Each of these moves is a merging of subsequently situated handles to one handle and a partition of this handle into new handles situated in other places, see Fig. 1.14.

In Fig. 1.14 (lower part) merging (respectively, partition) consists of elementary moves which are the destabilization (respectively, the stabilization). Meanwhile classical Reidemeister moves are performed locally on some part of the surface S_g obtained from the sphere by adding handles.

It is natural that the surface S_g is automatically oriented. The orientation for S_g arises from the orientation of the sphere S^2 to which we attach handles.

Note that on the surface S_g there is no fixed system of longitudes and meridians. Actually, under the first virtual Reidemeister move this surface goes through *Dehn twist*, see Fig. 1.15.

Note that two moves resembling Reidemeister moves and shown in Fig. 1.16, generally speaking, are not equivalence relations on virtual knots. They are called the *forbidden moves*, see Fig. 1.16.

It turns out (it was noted independently by Goussarov, Polyak and Viro [114] and Kauffman [157, 158]) that if we admit two forbidden moves, then any two virtual knots (but not links!) will become equivalent (see also [255]). If we add only one of these moves and the other one is left forbidden, we obtain a theory of *welded knots*. This theory was proposed by Satoh [273], see also [78, 172, 173].

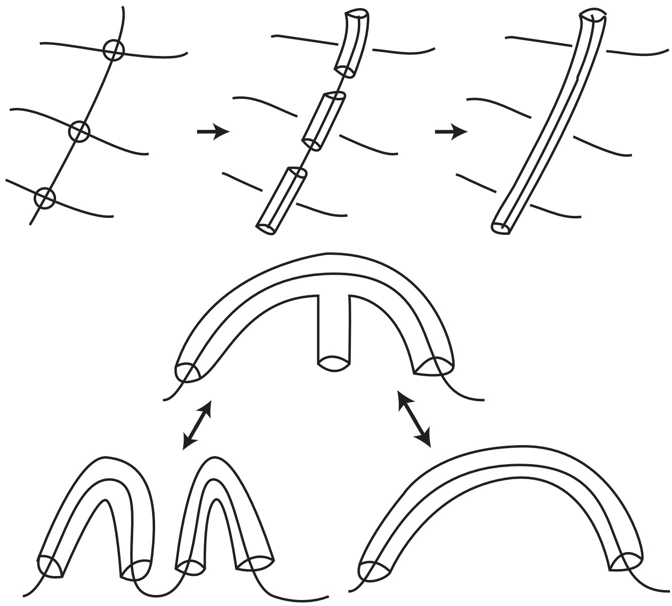

Fig. 1.14 Detour move and stabilization.

It is well known that each classical knot can be transformed into the unknot by a consecutive change of the crossing structure $\times \Longleftrightarrow \times$. This is a starting point for constructing knot invariants (skein-modules, the Conway algebra, the Kauffman polynomial, Vassiliev's invariants and so on). This is not true for virtual knots. Namely, if we factorize the theory of virtual knots over the relation $\times \Longleftrightarrow \times$ (this relation allows one to change the structure of classical crossings), we get a non-trivial theory, the theory of *flat virtual knots*, see, e.g. [151, 175]. This theory can be formalized in the following way. We only use one sort of crossings instead of classical crossings, which is called *flat* or *flat classical*; it is depicted as an ordinary intersection of two lines on the plane; moreover, we admit virtual crossings. The Reidemeister moves for flat virtual knots are depicted in Fig. 1.17.

We can simplify this theory further and obtain a new non-trivial theory. Namely, let us factorize the theory of flat virtual knots over a new move, the *virtualization*, see Fig. 1.18. We get a new theory, the theory of *free knots*. It turns out that this theory is highly non-trivial, see more details in Chap. 8.

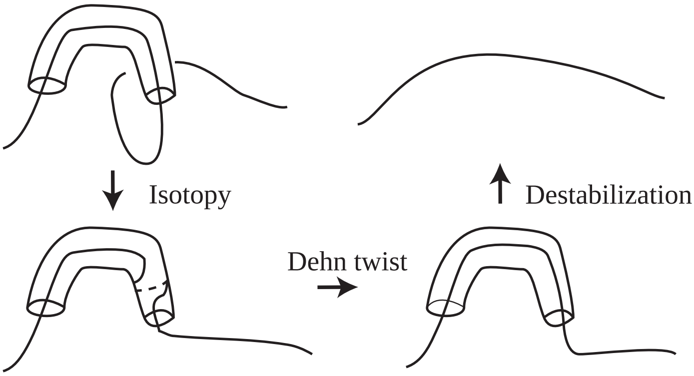

Fig. 1.15 Dehn twist and the move Ω_1'.

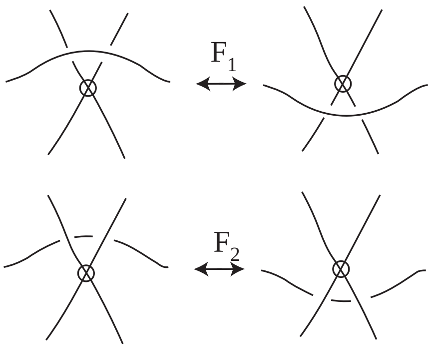

Fig. 1.16 Forbidden moves.

Another simplification of the theory of virtual knots when we keep in mind only the local writhe information (at each classical crossing) is called the *theory of pseudoknots*. But if we keep in mind the information about which branch lies upper and which one is lower, we obtain the theory called the *theory of quasiknots*, see [300].

The simplest example of a flat virtual knot not equivalent to the unknot is depicted in Fig. 1.19.

Virtual knots are lifted to thickened surfaces. If we disregard which branch of a virtual knot at a classical crossing passes over and which one passes under (i.e. we forget the under/overcrossing structure at classical crossings), we shall obtain a natural lifting of flat virtual knots to 2-surfaces

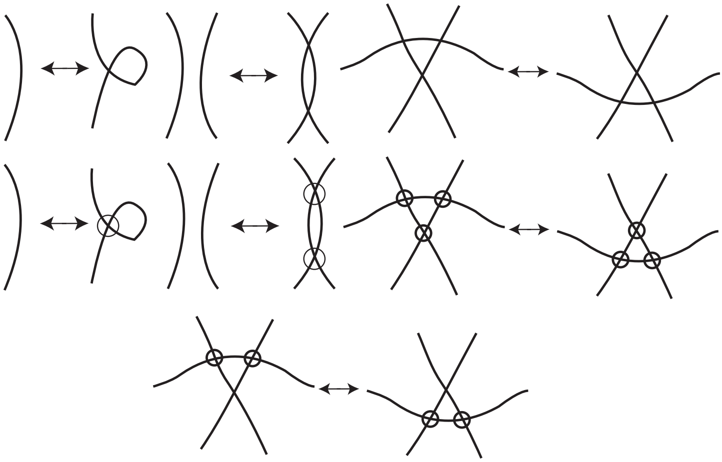

Fig. 1.17 Reidemeister moves for flat knots.

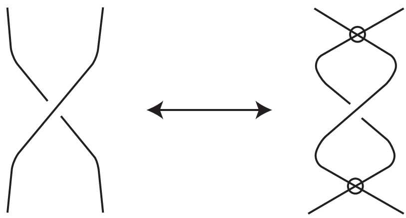

Fig. 1.18 Virtualization.

("thickening" is not required); here we describe another way of lifting distinct from the one described above.

The lift is constructed in two steps. Let a virtual diagram K be given. We construct a surface with boundary as follows. In each classical crossing of K we place a cross (the upper picture of Fig. 1.20), at each virtual crossing we set two non-intersecting bands (the lower picture), cf. [151]. Connecting these crosses and bands by bands (non-overtwisted) along the arcs of K we obtain an oriented two-dimensional manifold with boundary. Denote this manifold by M'.

One can project the diagram of K to M' in such a way that the arcs of the diagram are projected to the middle lines of the bands; herewith classical (flat) crossings correspond to crossings in the crosses. Thus, we

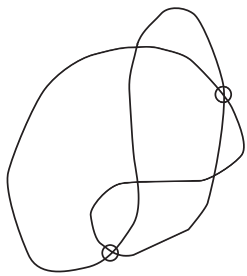

Fig. 1.19 The simplest non-trivial flat virtual knot.

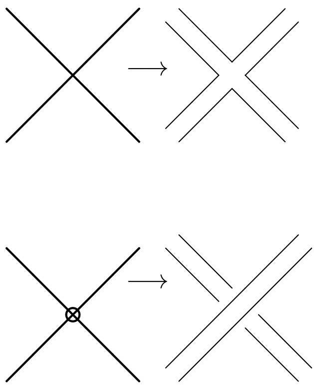

Fig. 1.20 The local structure of M'.

obtain a set of curves $\delta \subset M'$. Attaching discs to the boundary components of M', one obtains an orientable manifold $M = M(L)$ without boundary with the set δ of curves immersed in it.

This leads us to the theorem, see, e.g. [151].

Theorem 1.1. *Flat virtual links are equivalence classes of finite collections of curves in 2-surfaces up to free homotopy, stabilization and destabilization.*

Remark 1.3. Other generalizations of knots on surfaces were studied in [50, 52].

The problem of whether two such representations of a flat virtual link are equivalent in the category of flat virtual links is algorithmically recognized.

Let S be an oriented closed 2-surface of genus g. In [120], pointed to us by Cahn, Hass and Scott showed that if we had two homotopic curves on the surface S such that the numbers of self-intersection points for these curves were minimal, then these two curves could be related to each other by only a finite sequence of third Reidemeister moves.

Our next goal is to prove the following theorem (see also [272] for curves on compact surfaces).

Theorem 1.2. *A flat knot has a unique minimal (with respect to genus) representative (S, K) (closed oriented 2-surface, a curve on it) up to a diffeomorphism $(f\colon (S, K) \to (S', K'),\ K' = f(K))$ and a homotopy of knots on the surface.*

As a corollary from this theorem (by using the result of Hass and Scott) we have the following.

Corollary 1.1. *A flat knot has a unique minimal representative (with respect to genus and the number of crossings) up to a diffeomorphism and third Reidemeister move.*

The proof of Theorem 1.2 is analogous to the proof of Kuperberg theorem [189], see also Theorem 2.1 and its proof.

***Proof of Theorem* 1.2.** Assume the contrary. Let (S, K) be a minimal (with respect to the genus) representative of a flat knot, which can be transformed to distinct minimal representatives with destabilizations and Reidemeister moves. Therefore, on S there are two diagrams K_1, K_2 homotopic to K and two non-trivial closed curves γ_1, γ_2 such that $K_i \cap \gamma_i = \emptyset$, $i = 1, 2$, and the destabilization along γ_i (cutting along a cycle and pasting a disc to boundary) gives us the set of distinct minimums. Note that we choose the representative (S, K) in such a manner that the result (S_i, K_i') of the destabilization for each $i = 1, 2$ has a unique minimum. Let us show that (S_i, K_i'), $i = 1, 2$, can be destabilized to the same flat diagram $(\widetilde{S}, \widetilde{K})$. Then (S_1, K_1') and (S_2, K_2') will have the same minimum coinciding with a minimum for $(\widetilde{S}, \widetilde{K})$, and as a result, we obtain a contradiction. □

Let K_0 be a minimal (with respect to the number of crossings) representative of K on S. Let us use the next lemma.

Lemma 1.1. *Let K and K' be two diagrams on S, and let K' be obtained from K by applying a finite sequence of decreasing first Reidemeister moves, decreasing second Reidemeister moves not causing destabilizations, and also* *third Reidemeister moves and isotopy. Let γ be a non-contractible closed curve such that $\gamma \cap K = \emptyset$. Then γ can be isotoped simultaneously with the diagram K in such a way that the resulting curve and the resulting diagram will not intersect each other in the surface. In particular, there exists a curve γ' isotopic to γ such that $\gamma' \cap K' = \emptyset$.*

Proof. The curve γ lies in a non-trivial (non-homeomorphic to a disc) component of $S \setminus K$. Let us observe what happens with components under moves. Under first, second and third Reidemeister moves a trivial component of the complement disappears (under the third move, one more such component appears), and the other components are isotoped. Under an isotopy all components are isotoped. Therefore, our curve can be isotoped together with all components in such a way that the resulting curve does not intersect the resulting diagram in the surface. □

Now, let us prove Theorem 1.2. Since a minimal (with respect to the crossing number) representative is unique up to isotopy, then using Lemma 1.1 one may assume that $K_1 = K_2 = K_0$ and $K_0 \cap \gamma_i = \emptyset$, $i = 1, 2$. Let us consider the *complete destabilization* of the representative (S, K_0). To this end, we replace all non-trivial components of $S \setminus K_0$ with unions of disjoint discs and glue discs along circles, along which these components were joined to the knot K_0. This is equivalent to performing consecutive destabilization along some set of non-intersecting non-trivial curves lying in $S \setminus K_0$. It is clear that one can choose this set in such a way that this set contains curves γ_i. Therefore, the result $(\widetilde{S}, \widetilde{K})$ of the complete destabilization is obtained by further destabilizations of (S_i, K_i') (along the rest of curves from this set). Theorem 1.2 is proved.

In Chap. 8, we shall consider the theory of free knots which is a dramatic simplification of the theory of virtual knots. Speaking roughly, a free knot is an equivalence class of curves without any connection with a manifold, i.e. we do not consider pairs (M, δ), but do consider only curves and manifolds are arbitrary. In spite of the fact that flat virtual knots are algorithmically recognized, we do not know whether free knots are algorithmically recognizable.

Let us define an equivalence relation up to free homotopy and stabilization more precisely.

Let S be a set of all pairs (M, δ), where M is a smooth oriented closed 2-surface (probably disconnected, but with a finite number of connected components), and δ be an unordered finite set of closed curves immersed in M. Later on, we shall always consider only generic immersions.

Let us define equivalence classes on S by means of the following elementary equivalence relations:

(1) two pairs (M, δ) and (M', δ') are equivalent if there exists a homeomorphism $M \to M'$ identifying the sets δ and δ';
(2) for a fixed manifold M if a set δ of curves is free homotopic to a set δ' of curves in M, then the pairs (M, δ) and (M, δ') are equivalent;
(3) two pairs (M, δ) and (N, δ) are equivalent if N is the manifold obtained from M by deleting two discs not intersecting curves from δ and pasting an oriented handle instead of them (*stabilization*);
(4) for each closed oriented 2-manifolds N, pairs (M, δ) and $(M \sqcup N, \delta)$ are equivalent.

It is natural to consider the inverse operation to each elementary operation; the inverse operation to the stabilization is called *destabilization*.

In relation (4) the sign $\sqcup$ stands for the disconnected sum of the manifold M (with all curves from the set δ on this manifold) and the manifold N (with empty set of curves).

Further, we shall need these combinatorial objects for constructing much stronger invariants of virtual knots (the Ξ-polynomial, see Chap. 4, Sec. 4.3.2).

The space of Vassiliev's invariants of order zero is the space of functions on virtual knots, which are invariant under the relation  $\Longleftrightarrow$ , i.e. functions on flat virtual knots. Thus the linear space spanned by flat virtual knots is dual to the space of Vassiliev invariants of order zero, see Chap. 7 for details.

Besides knot theory, there exists the theory of braids (the notion was first introduced by Artin [12]). By the *group of braids with n strands* we mean the fundamental group of the configuration space of unordered sets of n distinct points on the complex line. Thus, for defining a braid with n strands we have to define a closed path in this configuration space, i.e. to describe a trajectory of moving of n points in the time in such a way that they will run into the place of initial points at the final moment of time (in some order). An order of the points is not fixed. If we fix an order of the points, we shall get the notion of a *pure braid*. Braids can be considered in either smooth category or piecewise-linear category, herewith in the smooth category the operation of smoothing under multiplication of braids is naturally defined.

An initial point in the configuration space is usually the set of real points $\{1, \ldots, n\}$ on the complex line $\mathbb{C}$.

Trajectories of moving of distinct points represent *strands* of a braid. A homotopy in the space of braids represents a continuous deformation of all strands without intersections.

There exists a natural operation for closing a braid: the *closure* of a braid. If we connect each upper end of the braid with the corresponding lower end, we shall get the well-defined class of the link, see Fig. 1.21.

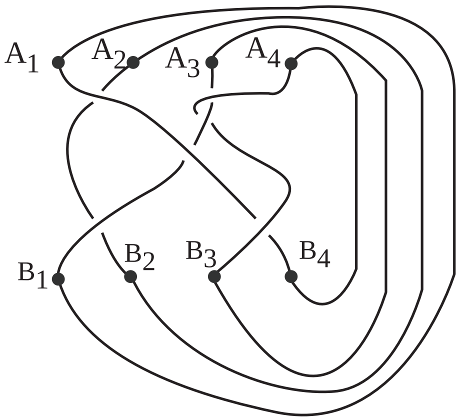

Fig. 1.21 The closure of a braid.

Equivalence classes of braids have a natural group structure.

Simultaneously with virtual knot theory, there exists the theory of virtual braids first considered in papers by Vershinin [307], Kamada [153], see also [166, 209]. The group of virtual braids with n strands represents the group with two families of generators: $n-1$ classical generators (like for classical braids) and $n-1$ virtual generators (like for symmetric group). The relations correspond to all the generalized Reidemeister moves except for the first classical and virtual moves Ω_1, Ω_1' and a remote commutativity (see below). They are divided into three classes. The classical relations (which are related to the classical generators), the relation of the symmetric group (which are related to the virtual generators) and the mixed relations (which correspond the semivirtual move Ω_3'').

Just as the closure of a classical braid gives us a classical link, the closures of virtual braids lead to virtual links. Moreover, such links are automatically oriented. The orientation is inherited from a braid, if all braids are oriented from top to bottom. An example of a virtual braid is

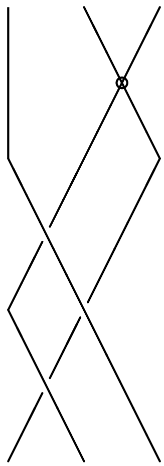

Fig. 1.22 A virtual braid.

depicted in Fig. 1.22.

There are a line of natural aspects of the theory of classical and virtual braids. Let us note some of them.

(1) Any oriented classical link can be obtained by closing a braid. This fact was proved in [6]. Any oriented virtual link can be obtained by closing a virtual braid, see [153, 307].
(2) The closures of isotopic (in the classical case) or equivalent (in the virtual case) braids lead to isotopic (equivalent) closures. This assertion follows from the definition.
(3) Two classical braids having isotopic closures are obtained from each other by a finite sequence of, so-called, *Markov's moves* (see [26, 252]). Two virtual braids having equivalent closures are obtained from each other by a finite sequence of *virtual Markov's moves* (Kamada); another set of sufficient moves was suggested by Kauffman and Lambropoulou [166]. In the series of papers [29–34] it was shown that the unknot can be detected by means of braid theory.
(4) The group of classical braids is naturally embedded in the group of virtual braids. This fact was first proved by Fenn, Rimanyi and Rourke [87]. In Chap. 6 of the book, we give a proof of this fact based on the invariant $\mathcal{F}$ [209].

Note that the assertion that the natural mapping from the set of classical knots to the set of virtual knots is a one-to-one correspondence (with its image), and is not equivalent to the analogous question about braids. Moreover, none of these two assertions is not direct consequence of the other one.

(5) There are a lot of algorithms for recognizing classical braids, see, e.g. [12, 69]. Moreover, for classical braids there are complete invariants clearly described and faithful finite-dimensional representations [24, 186]. For classical braids with any number of strands the problem of conjugacy was solved positively [102]. The Dehornoy's algorithm leads to ordering of classical braids, see [69, 70, 200].
For the case of virtual braids with three strands, an algorithm of recognition was constructed by Bardakov by means of bringing braids to a *normal form*, see [16]. In [109] Godelle and Paris claim that they have a refinement of Bardakov's paper [16] where they prove the word problem for virtual braid groups with any number of strands.
The first-named author of this book constructed an invariant of virtual braids which was a generalization of a complete invariant of classical braids. The question whether this invariant is complete is still open.
Note that the problem of conjugacy for virtual braids is not solved, and the question whether the group of virtual braids is linear is still open.

(6) The problem of constructing invariants of virtual knots by using braids is also interesting. In the classical case, the construction of such invariants was initiated by Jones [142] who constructed an invariant of links by using of Markov's theorem. In what follows, it led to the theory of *quantum invariants* of classical knots. The theory of quantum invariants of virtual knots is not completed, the main obstruction to a complete construction is the virtual first Reidemeister move Ω_1'. If we disregard this move, we shall get the theory of *rigid virtual knots.* All quantum invariants of classical knots can be extended to the theory of rigid virtual knots (Kauffman's theorem from [158]). The theory of rigid knots is more explicit in approaching the theory of classical knots rather than the theory of virtual knots.

1.3 Self-linking number

Let us bring one simple invariant of virtual knots belonging to Kauffman [161] and based on the notion of parity. This invariant is the first one in which parity is used. Further (see Chap. 8), we shall show that the genuine notion of parity gives deep, important and interesting results.

Definition 1.8. Let K be a diagram of an oriented virtual knot. Call a classical crossing v of K *odd*, if in the Gauss code of K the number of letters between two occurrences of the crossing v is odd.

Set:

$$J(K) = w(K)|_{\mathrm{Odd}(K)},$$

where $\mathrm{Odd}(K)$ denotes the collection of odd crossings of K, and the restriction of the writhe $w(K)$ to $\mathrm{Odd}(K)$, denoted by $w(K)|_{\mathrm{Odd}(K)}$, means the summation the signs of the odd crossings in K. Then it is not hard to see that $J(K)$ is an invariant of the virtual knot (link) K. We call $J(K)$ the *self-linking number* of the virtual diagram K. This invariant is simple, but remarkably powerful.

If K is classical, then $J(K) = 0$, since there are no odd crossings at a classical diagram.

Theorem 1.3. *Let K be a virtual knot diagram and let K^* denote the mirror image of K (obtained by switching all the crossings of the diagram K). Then*

$$J(K^*) = -J(K).$$

Hence, if $J(K) \neq 0$, then K is inequivalent to its mirror image. If K is a virtual knot and $J(K)$ is non-zero, then K is not equivalent to any classical knot.

We leave the proof of this theorem and the proof of the invariance of $J(K)$ to the reader. See [161] for more about this invariant.

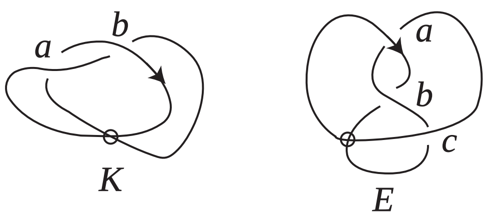

Fig. 1.23 Virtual trefoil K and virtual figure eight E.

In Fig. 1.23, the two virtual knots illustrate an application of Theorem 1.3. In the case of the virtual trefoil K two crossings are odd and, hence, we have $J(K) = 2$. This proves that K is non-trivial, non-classical and inequivalent to its mirror image. Similarly, for the virtual knot E the crossings a and b are odd. We have $J(E) = 2$ and, hence, the knot E is also non-trivial, non-classical and inequivalent to its mirror image. Note that for the knot E the invariant is independent of the type of the crossing c.

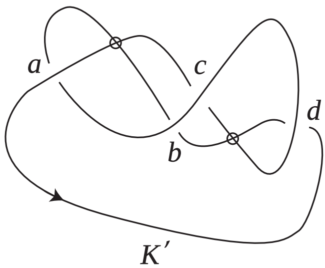

Fig. 1.24 The knot K'.

In Fig. 1.24, the virtual knot K' has $J(K') = 2$. Note that K' would be unknotted if we allowed the second (Fig. 1.16, lower) forbidden move. This example underlines why we forbid such moves in virtual knot theory.

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Chapter 2

Virtual Knots and Three-Dimensional Topology

2.1 Introduction

Three-dimensional topology plays a key role in classical knot theory. The first invariants of classical knots and links were invariants of topological nature related to the complement to a tubular neighborhood of the link. So are the fundamental group (of the complement to the link) and the Alexander polynomial [5, 7]. As it was clarified later, the topology of the knot complement completely defines its isotopy class (the Gordon–Luecke theorem [111]; an analogous statement about links is false). Besides, due to the topological theorem of Waldhausen [313] stating that for some class of 3-manifolds (which are called sufficiently large) every isomorphism of the fundamental group is generated by some homeomorphism of manifolds, a formally invariant of classical links was constructed, see papers of Matveev [243], Joyce [144], Eisermann [81] and Deviatov [71]. Here the key observation is that the complement to a tubular neighborhood of a non-trivial knot is a sufficiently large 3-manifold with boundary.

This chapter is devoted to the proof of the two basic results: We prove the algorithmic recognizability of virtual links and we prove that any non-trivial connected sum of two non-trivial virtual knots is non-trivial.

The first algorithm for recognizing classical knots and links is also related to the three-dimensional topology. Namely, it is based on the *theory of normal surfaces*, suggested by Haken [118] and developed by Hemion [121] and Matveev [243]. Normal surfaces were first introduced by Kneser [183]. The main idea is that in "good" 3-manifolds (possibly, with boundary), there is a class of *normal surfaces*, for example, those surfaces having "right" position with respect to the decomposition of the manifold (for example, to its triangulation). The key statement of the Haken theory states that normal surfaces are obtained from a finite algorithmically decidable collection of basic normal surfaces. Moreover, for many important properties P the following statement holds: If there exists a surface possessing the property P, then there exists a basic normal surface possessing the same property. On the set of normal surfaces the so-called *geometric summation* operation is defined: Every surface is defined by a finite number of non-negative characteristics which are additive with respect to the summation. The question of finding basic normal surfaces is reduced to the question of finding basic non-negative integer-valued solutions of a finite integral system of equations. This problem is algorithmically decidable, but its solution is too huge in concrete cases related to problems of three-dimensional topology.

For a review of the normal surface theory and its application to three-dimensional topology, see [243].

Therefore, many problems in three-dimensional topology can be algorithmically solved if they are reformulated in the language of normal surfaces and their properties (for example, the problem of recognizing a classical knot). The first impeccable proof of all statements of Haken's theory can be found in [243].

From the very beginning, virtual knots are connected to three-dimensional topology. When constructing this theory, Kauffman was motivated by the theory of knots in "thickened surfaces" developed by Jaeger, Kauffman, and Saleur [139]. An important step for understanding this connection is Kuperberg's theorem (Theorem 2.1), stating that every virtual knot has a canonical minimal realization (which is minimal with respect to the surface genus).

We shall use this fact together with key statements of the Haken–Matveev theory for the solution of two problems in virtual knot theory: The proof of the fact that any connected sum of non-trivial virtual knots is non-trivial, and the algorithmical recognition of virtual links.

Note that the algorithmic recognition of links (both virtual and classical) in terms of the normal surface theory is principally a logical question: Though a formal description of all steps of the algorithm (for classical or virtual case) can be described algorithmically, this algorithm is too complicated, and it can hardly be realized by computer (without additional simplification ideas) for concrete practical purposes.

In this chapter, we use the following definition of virtual knots and links.

Definition 2.1. A *virtual link* is an equivalence class of links in thickened surfaces $M \times I$ up to stabilizations/destabilizations (see further).

A virtual link K is *oriented* if the corresponding link in $M \times I$ is oriented, i.e. so is every circle $S^1, \ldots, S^1$ (preimage of any link component).

Analogously, one can define framed virtual links (see Definition 1.1) that can be presented by a collection of bands $S^1 \times I$ and Möbius bands embedded in thickened surfaces.

When defining an equivalence relation of oriented links, we require that the orientation is preserved for all components.

Definition 2.2. By a *thickened surface* we mean the Cartesian product $M \times I$, where I is the interval $[0, 1]$, and M is the closed oriented 2-manifold, not necessarily connected. Herewith, if we have a link, we require that for every connected component M_i of M, the thickened surface $M_i \times I$ contains at least one connected component of the link.

In particular, when we deal with knots, the underlying manifold always has to be connected, i.e. the surface M is homeomorphic to the sphere with some number of handles.

By *destabilization* we mean the following: Let S be a non-contractible circle on the surface M for which there exists a (*vertical*) annulus C homotopic (in the class of *proper* embeddings, see Fig. 2.1) to the annulus $S \times I$, and not intersecting the link (during the homotopy process it is assumed that the intersection of the annulus with $\partial(M \times I)$ coincides with the boundary of the annulus). Then our destabilization is cutting of the manifold $M \times I$ along the annulus and pasting of the newborn components by *plates* (thickened discs) $D^2 \times I$ (Fig. 2.1). If this cutting leads to a new empty component, this component has to be removed in order not to violate the initial agreement (Definition 2.2): In each component $M_i \times I$, there should be at least one link component.

We shall also use the expression *destabilization along an annulus* C.

By *stabilization* we mean the operation opposite to destabilization.

In other words, the stabilization and destabilization are addition and removal of handles to (from) the surface M; herewith these (thickened) handles we are adding/removing should not contain points of the link.

It is evident that every diagram can be stabilized.

An important question is: How long can one destabilize a diagram? Let a virtual link be given by a link in a thickened surface $M \times I$. We say that this representative of the virtual link is *minimal* if no destabilization is possible.

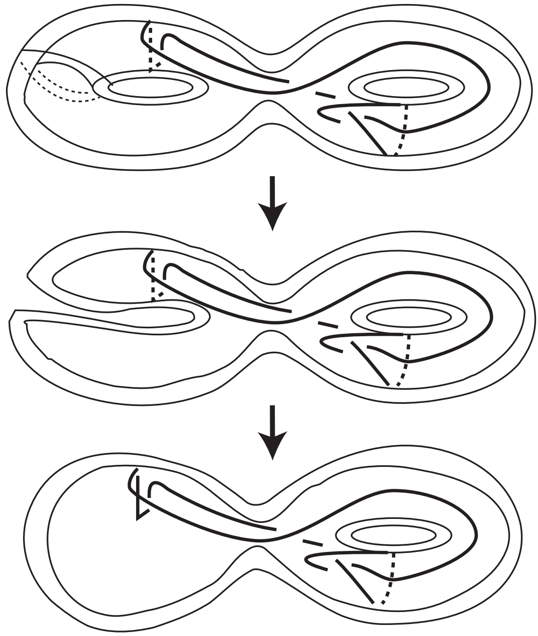

Fig. 2.1 Destabilization of a pair $(S_2 \times I, K)$.

In this chapter, by "manifold" we mean a 3-manifold N, possibly, with a boundary, and by "surface" we mean a *proper* 2-surface $F \subset N$ unless specified otherwise. This means that $F \cap \partial N = \partial F$, where ∂N and ∂F is the boundary of the manifold N and the boundary of the surface F, respectively.

Moreover, in this chapter all 3-manifolds are assumed *oriented*, and the orientation is fixed; by homeomorphism we always mean an orientation-preserving homeomorphism.

2.2 The Kuperberg theorem

Using methods of three-dimensional topology, Kuperberg proved the following important and fundamental theorem, see [189].

Theorem 2.1. *For every virtual link K its minimal representative is unique up to a diffeomorphism of the pair $(M \times I, K \subset M \times I)$ to itself taking the upper boundary component $M \times \{1\}$ to itself.*

Remark 2.1. This theorem is very useful in virtual knot theory. For example, it allows one to understand whether two links are equivalent or not by just considering their minimal representatives, to determine whether a virtual knot diagram is non-classical (and hence non-trivial), see [75, 146]. Also note that some invariants of knots give minimality, i.e. they show that a given representative is minimal, see Chap. 8.

Remark 2.2. In Chap. 1, we proved the theorem for curves which was analogous to Theorem 2.1, see Theorem 1.2.

Let us prove Theorem 2.1. Here we follow the Kuperberg's proof.

***Proof of Theorem* 2.1.** Let us generalize the definition of the destabilization of a link $K \subset M \times I$ to include two other operations:

(1) If a sublink $K' \subset K$ is separated from the rest of K by a sphere or disc $A \subset M \times I$, then we can remove K' from $M \times I$ and place it in a separate thickened sphere $S^2 \times I$.
(2) If an annulus A divides $M \times I$ such that K lies entirely on one side and some genus of M is on the other side, then we can cut $M \times I$ along A, discard the naked component (i.e. the component not containing K), and cap the remaining component. Both operations can easily be reproduced by the destabilization.

Definition 2.3. We say that a surface is *admissible* if it is a vertical annulus, a sphere, or a proper disc; and that an admissible surface is *essential* if it does not bound a three-dimensional ball in $(M \times I) \setminus K$.

Thus, admissible, essential surfaces are those along which we can destabilize $K \subset M \times I$.

Suppose, to the contrary of the conclusion, that some link $K \subset M \times I$ has more than one minimal representative. If M has c components with total genus g, and L has n components, assume that $g + n - c$ is minimal among counterexamples. (Note that $n \geqslant c$.) Then every destabilization of $K \subset M \times I$ has a unique minimal representative, since destabilization always reduces $g + n - c$.

We say that two such destabilizations, $K \subset M_1 \times I$ and $K \subset M_2 \times I$, are *descent equivalent* if their minimal representatives are isomorphic.

The aim is to show that all destabilizations of $K \subset M \times I$ are descent equivalent. Let us prove this fact by the induction on the complexity of intersection between two destabilization annuli by compressing one along an innermost disc of the other.

For example, if A_1 and A_2 are disjoint admissible, essential surfaces, then the resulting destabilizations $K \subset M_1 \times I$ and $K \subset M_2 \times I$ are descent equivalent. This is immediate if A_1 and A_2 are parallel. If they are not, then we can destabilize each $K \subset M_i \times I$ along A_{3-i} to produce a common descendant.

Definition 2.4. A circle in an annulus C is *trivial* if it represents the trivial element in the fundamental group of the annulus C, otherwise the circle is called *horizontal.* An arc in an annulus C is called *trivial* if it connects points from the same boundary component of the annulus; otherwise it is called *vertical.*

Suppose that A_1 and A_2 are descent inequivalent surfaces in general position, and that they intersect in the fewest curves among descent inequivalent pairs in $(M \times I) \setminus K$. If a curve $C \subset A_1 \cap A_2$ is a circle, then it is either horizontal in A_i (if A_i is an annulus and C is parallel to ∂A_i), or it is trivial (if it bounds a disc in A_i). If C is an arc, then it is either vertical (if A_i is an annulus and C connects the two components of ∂A_i), or it is trivial (it bounds a disc together with a part of the boundary of A_i).

If a circle of $A_1 \cap A_2$ is trivial in, say, A_1, then some such circle C is *innermost*, i.e. it bounds a disc D in A_1 which is disjoint from $(A_1 \cap A_2) \setminus C$ (a *naked disk*), as in Fig. 2.2(a). In this case let A_2' and A_2'' be the connected components of the compression of A_2 along the disc D, as in Fig. 2.2(b). Both A_2' and A_2'' are admissible, and at least one is essential, for otherwise A_2 would not be. If A_2' (say) is essential, then it intersects A_1 less than A_2 does, and it does not intersect A_2 at all. But since A_1 and A_2 intersect least among descent inequivalent pairs of essential surfaces, it would follow that A_1 and A_2 are both descent equivalent to A_2', a contradiction.

The same argument applies if $A_1 \cap A_2$ has a trivial arc in A_1, and of course it also applies to trivial circles and trivial arcs in A_2. Thus $A_1 \cap A_2$ consists entirely of vertical segments or horizontal circles in both A_1 and A_2. In particular, both A_1 and A_2 are vertical annuli and not discs or spheres.

Suppose that $A_1 \cap A_2$ consists of horizontal circles. We can assume that none of the four circles of ∂A_1 and ∂A_2 bounds a disc in $M \times I$. If, say, $C \subset A_1$ bounds a naked disc $D \subset M \times \partial I$, i.e. a disc being disjoint from A_2, then we replace A_1 by the disc $D \cup A_1$ and reduce to a previous case without worsening $A_1 \cap A_2$. Otherwise let $C \subset A_1 \cap A_2$ be an *outermost* circle, i.e. it and one component of ∂A_1 bound a naked annulus $A \subset A_1$. The circle C divides A_2 into two annuli A_2' and A_2'', one of which, say A_2',

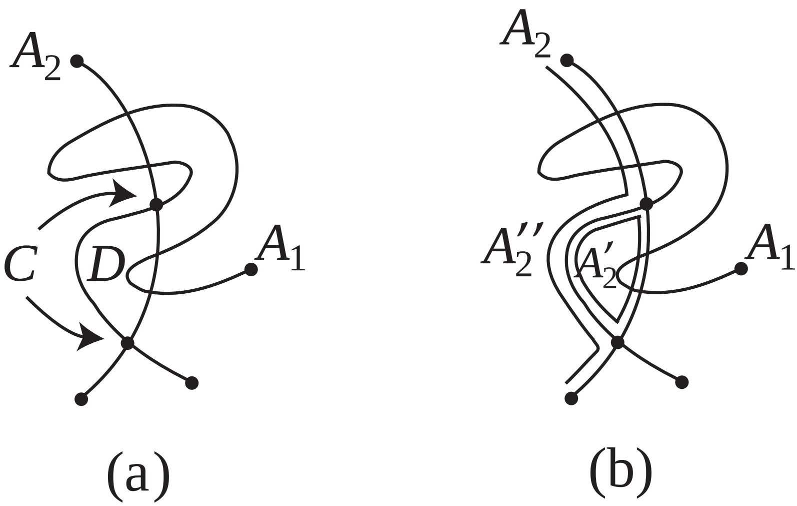

Fig. 2.2 Compressing A_2 along the disc D to simplify $A_1 \cap A_2$ (side view).

makes a vertical annulus together with A. The annulus $A \cup A_2'$ is necessarily essential since its boundary circles do not bound discs in $M \times \partial I$. But after displacement, $A \cup A_2'$ intersects A_1 and A_2 less than they do each other. It is therefore descent equivalent to both.

Finally suppose that $A_1 \cap A_2$ consists of vertical arcs. The boundary of a regular neighborhood of $A_1 \cup A_2$ consists of vertical annuli $B_1, B_2, \ldots, B_n$. Each B_i is disjoint from both A_1 and A_2, so if any of them is essential, it is descent equivalent to both A_1 and A_2, a contradiction. But if they are all inessential, then one of them, say B_1, separates A_1 and A_2 from the link K and bounds a ball that contains A_1 and A_2. This contradicts the hypothesis that A_1 and A_2 are essential. □

In the case of knots or, in general, non-splitting links the corresponding surface has to be connected, i.e. this surface has to be the sphere with g handles, S_g.

Definition 2.5. The *underlying genus* of a non-splitting virtual link K is the genus g of the minimal surface S_g such that the link K can be realized in $S_g \times I$.

Remark 2.3. Classical links and only them have the underlying genus zero.

Remark 2.4. In what follows, we shall define some more characteristics of classical and virtual links, called a word "genus": *the underlying genus of a link, the genus of an atom, Seifert genus.* Sometimes we omit the characteristic "underlying" and write just "genus", if it is clear from the context what we mean. By the genus of a thickened surface $M \times I$ we always mean the genus of the surface M.

From Kuperberg's theorem it follows if two virtual knots K_1 and K_2 have the same genus g, then they are equivalent to each other if and only if one of them can be transformed to the other one in the class of virtual knots of the genus g, i.e. by isotopies in $S_g \times I$ (and homeomorphisms of the surface S_g on itself). In particular, the case $g = 0$ means that two classical diagrams of knots (links) are virtual equivalent if and only if they represent the isotopic links.

Therefore, we have the following theorem.

Theorem 2.2. *Classical links form a subset in the set of virtual links. In other words, if two classical diagrams are equivalent (in the class of virtual diagrams), then they represent isotopic classical links.*

This theorem was first proved in [114] by using the notion of the distributive groupoid. We shall concern this question in Chap. 3.

2.3 Genus of a virtual knot

Definition 2.6. A *disconnected sum of virtual diagrams (of knots or links)* K_1 and K_2 is a planar diagram K such that there exists a line l in the plane and the intersection of K with one of the two half-planes defined by l consists of the diagram K_1, and the intersection with the other half-plane consists of K_2. Denote the obtained disconnected sum by $K_1 \sqcup K_2$.

It is obvious that any disconnected sum of virtual links is well defined: The class of the virtual link generated by the diagram $K_1 \sqcup K_2$ does not depend on a choice of diagram-representatives of the links generated by K_1 and K_2, respectively.

Let us call a virtual diagram *connected* if it cannot be obtained from the disconnected sum of two non-empty virtual diagrams by applying detour move.

Let K_1 and K_2 be two non-intersecting diagrams of oriented virtual knots on an oriented plane P having the property that some 2-disc E intersects $K_1 \sqcup K_2$ in two arcs $AB \in K_1$ and $CD \in K_2$ having opposite orientations, i.e. the first arc is oriented from A to B, and the second one is oriented from C to D, herewith traveling along the circle ∂E in clockwise manner we meet four points in the order A, B, C, D. Assume that there exists a line l intersecting the disc E and not intersecting the diagrams K_1, K_2 such that the diagrams K_1 and K_2 lie on the opposite sides of the line l.

Definition 2.7. A *connected sum* of the diagrams K_1 and K_2 (notation $K_1 \# K_2$) is the diagram obtained from the diagram of disconnected sum $K_1 \sqcup K_2$ by deleting arcs AB and CD and adding arcs DA and CB with extending the orientations of the diagrams K_1 and K_2 on the obtained diagram $K_1 \# K_2$, see Fig. 2.3.

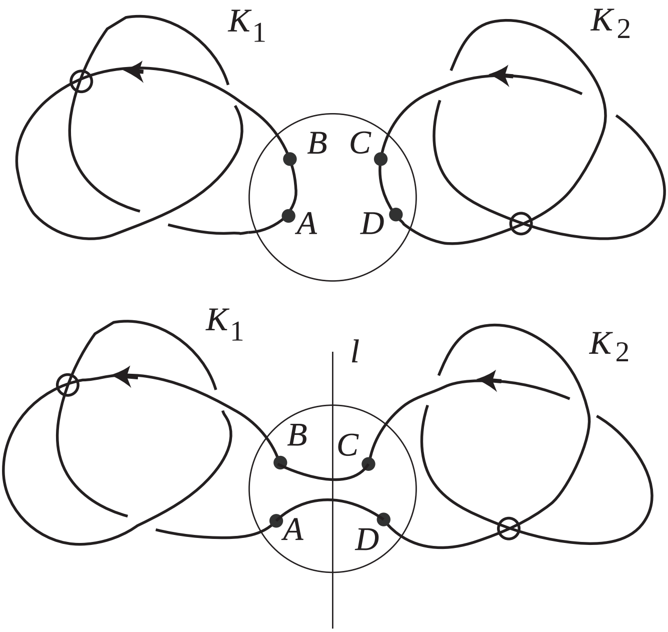

Fig. 2.3 An example of connected sum of virtual knots.

Note that the notion of connected sum is not well defined. In Figs. 2.4 and 2.5 we have two non-equivalent connected sums of the virtual trefoil with itself.

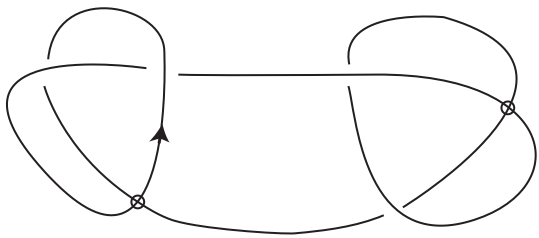

Fig. 2.4 A connected sum $K\#K$.

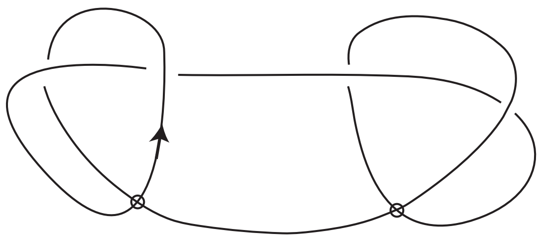

Fig. 2.5 Another connected sum $K\#K$.

The equivalence class of a connected sum of virtual knots is also not well defined. Therefore, we shall use the term a *connected sum* of virtual knots for *any* of connected sums obtained by considering a connected sum of some two diagrams.

It is also interesting to consider a *connected sum of long virtual knots*: it is well defined. Section 3.3 is devoted to long virtual knots.

The goal of this section is to prove the following theorem.

Theorem 2.3. *If at least one of two virtual knots K_1, K_2 is non-trivial, then any of their connected sums $K_1\#K_2$ is a non-trivial virtual knot.*

This theorem will follow from Theorem 2.5 about the (underlying) genus and Theorem 2.1.

In the case of classical links the theorem analogous to Theorem 2.3 was proved by Schubert [277]. In the proof of this theorem he used considerations analogous to the idea of the proof of Theorem 2.5. Namely, the additivity of the *Seifert genus* (a non-negative characteristic of classical knots which is equal to zero only for the unknot) was proved.

Recall that for any classical link $K \subset \mathbb{R}^3$ there exists an oriented connected surface $F \subset \mathbb{R}^3$ such that $\partial F = K$. The minimal genus of such surface is called the *Seifert genus* of the link K.

In the case of classical knots which diagrams lie on the opposite sides of some line l we choose classical diagrams K_1 and K_2 of them in such a way that for some disc E intersecting l in two points on the plane the intersection $E \cap (K_1 \cup K_2)$ consists of two non-intersecting arcs AB and CD lying on the opposite sides of the line l and having opposite directions inside the disc. After that the connected sum of the knots is defined by means of the diagram $K_1 \# K_2 = K_1 \sqcup K_2 \backslash (AB \cup CD) \cup (DA \cup CB)$ with the obvious orientation.

This definition for classical knots gives the *classical connected sum*. The obtained classical knot is well defined: Its isotopic class does not depend on the choice of representatives K_1 and K_2. The proof of this fact is well-known, see, e.g. [46, 221].

We shall call this connected sum the *classical connected sum* of classical knots. By this we emphasize the choice of a connected sum in construction of which we use classical diagrams and do not use virtual crossings. In contrast to the general case of a *virtual* connected sum (which can be used to diagrams of classical knots), the classical connected sum is well defined.

Schubert's theorem is as follows.

Theorem 2.4 ([277]). *If the classical connected sum of two classical knots K_1, K_2 is the unknot, then each of the knots K_i, $i = 1, 2$, is the unknot.*

Definition 2.8. We say that the minimal representative of a virtual knot K as a knot in a thickened surface $M \times I$ is *singular* if there exists a homotopic non-trivial circle $S \subset M$ not separating the surface M into two parts and an annulus $C \subset M \times I$ such that $\partial C = C \cap (M \times \{0, 1\})$, herewith $C \cap (M \times \{1\})$ is homotopic to $S \times \{1\}$, and the intersection $C \cap K$ is transversal and consists of exactly one point.

Remark 2.5. The condition that the intersection consists of one point leads to the fact that the annulus C does not separate $M \times I$ into two connected components.

An example of singular realization is shown in Fig. 2.6.

The key assertion in the proof of Theorem 2.3 is the following.

Theorem 2.5. *For any two virtual links K_1, K_2 and any of their connected sums $K_1 \# K_2$ the inequality $g(K_1 \# K_2) \geqslant g(K_1) + g(K_2) - 1$ holds, and*

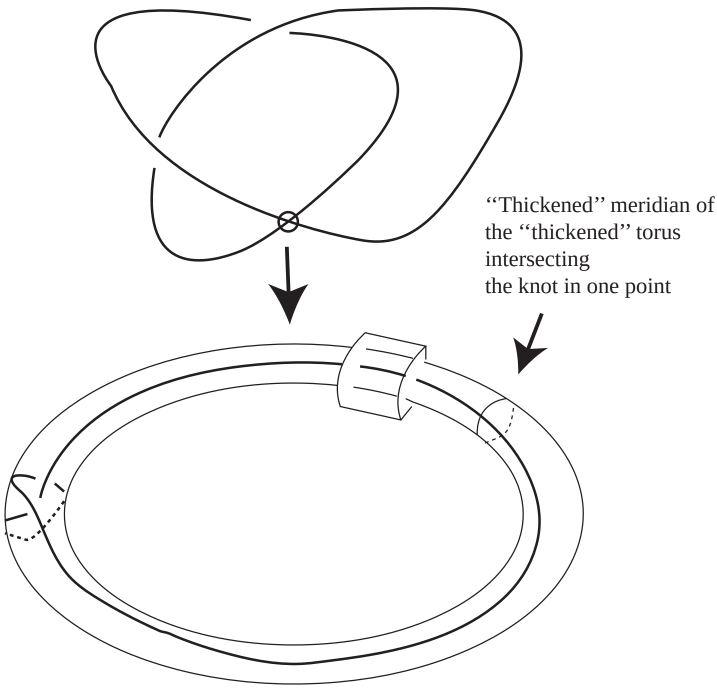

Fig. 2.6 A singular minimal realization.

$g(K_1\#K_2) \geqslant g(K_1) + g(K_2)$ if the minimal representative of any of the virtual knots K_1, K_2 is not singular.

Remark 2.6. Note that these inequalities can be changed by the equalities. In Chap. 4 we shall describe the Kishino knot having the underlying genus two and being a connected sum of two trivial knots (genus zero).

In what follows, we shall need the following.

Lemma 2.1. *If a connected sum of two virtual knots K_1, K_2 has genus zero* (*i.e. it represents a classical knot*), *then both knots K_1 and K_2 have genus zero*, *and this connected sum is equivalent to their classical connected sum.*

A proof of Lemma 2.1 will follow from the proof of Theorem 2.5.

From Theorem 2.5 and Lemma 2.1 Theorem 2.3 follows. Indeed, let K_1 and K_2 be two virtual knots. If at least one of them (say, K_1) has the (minimal) underlying genus greater than zero, then $K_1\#K_2$ also has the genus greater than zero and, therefore, it cannot represent the unknot. Namely, according to Theorem 2.5, the unique possibility for which we have $g(K_1\#K_2) = 0$ is the case when $g(K_1) = 0$, $g(K_2) = 1$ (or $g(K_1) = 1$, $g(K_2) = 0$), herewith both minimal representatives have to be singular, but this is impossible in the case of genus zero by definition (since there are no non-trivial cycles on the sphere).

In the case when both knots are classical we can, according to Lemma 2.1, consider their classical connected sum. The latter represents a non-trivial knot according to Schubert's theorem (with the assumption that at least one of the knots is not trivial).

Let us pass to the proof of Theorem 2.5.

Remark 2.7. In what follows, we shall denote an abstract virtual knot and a knot representing them in a thickened surface by the same letter. For instance, a virtual knot K is represented by a knot K in a thickened surface $S_g \times I$.

2.3.1 *Two types of connected sums*

Given two virtual knots K_1 and K_2 represented by knots in thickened surfaces and let us fix their connected sum. In this case there are two natural possibilities to represent their connected sum as a knot in a thickened surface, see Figs. 2.7(a) and (b).

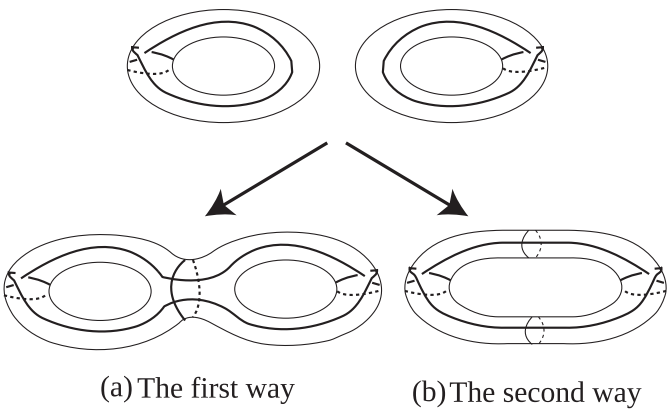

Fig. 2.7 Two types of connected sums.

The first method for making a connected sum goes as follows. We take thickened surfaces $(M_1 \times I) \supset K_1$ and $(M_2 \times I) \supset K_2$ and cut two vertical full cylinders $D_i \times I$, where $D_i \subset M_i$ is a disc, such that the intersection $(D_i \times I) \cap K_i$ is homeomorphic to an interval. Then we paste the obtained boundaries together (by identifying $\partial D_1 \times I$ and $\partial D_2 \times I$ with respect to the orientation of manifolds, the direction of the interval I and the orientation of the knot at the two gluing points). As a result, we obtain $(M \times I) = ((M_1 \# M_2) \times I)$ with the knot $K_1 \# K_2$ inside.

Clearly $g(M) = g(M_1) + g(M_2)$.

Another way to construct a connected sum works only in the cases when realizations of both knots K_1 and K_2 are singular. The second method goes as follows. Suppose K_1 and K_2 lie in $M_1 \times I$ and $M_2 \times I$, where both $g(M_1)$ and $g(M_2)$ are greater than zero, and there exist two non-trivial (not zero-homotopic) curves $\gamma_1 \in M_1$ and $\gamma_2 \in M_2$ such that $(\gamma_i \times I) \cap K_i$ consists of precisely one point (note that in this case such a curve cannot divide the 2-manifold into two parts). Then we cut the thickened surfaces $(M_i \times I)$ along the annulus $\gamma_i \times I$ and paste them together (in this case we paste together two pairs of cylinders). Here we also suppose that pasting agrees with the position of intersection points of the knots and the orientations of the knots.

As a result, we get the manifold $M \times I$, where $g(M) = g(M_1) + g(M_2) - 1$ for some connected sum $K_1 \# K_2$ lying in M.

In what follows, we shall consider only these two types of connected sums when we perform a destabilization and define the genus of a virtual knot.

Kuperberg's theorem is used by us for the notion of the genus of a virtual knot to be well defined.

2.3.2 *The proof plan of Theorem* 2.5

Consider two virtual knots K_1 and K_2 and their connected sum $K_1 \# K_2$. Let us realize this connected sum by curves in thickened surfaces by using the first method. Denote the corresponding surfaces by M_1, M_2, $M_1 \# M_2$, and denote the corresponding knots by K_1, K_2, $K_1 \# K_2$ (we use the same letters for abstract virtual knots and for their representatives in thickened surfaces). Now, we are going to transform $(M_1 \# M_2) \times I$ and knots inside it.

To simplify the notation, let us use the same letters for closed surfaces and surfaces with boundary. We shall write $M = M_1 \# M_2$, and $M =$

$M_1 \cup M_2$. We prefer the second notation to emphasize that both M_1 and M_2 are parts of M, while the first notation will be used when M_1 and M_2 are treated as separate manifolds.

We shall perform the destabilization process on the knot obtained by the connected sum operation described in the previous section.

The process of our destabilization will consist of the following steps.

We shall show that after each step of destabilization the following assertions hold (all notations stay the same during the transformation); further M is a closed surface:

(1) The ambient manifold $M \times I$ can be represented by a connected sum of two parts $M_1 \# M_2$ or by pasting M_1 and M_2 along two cut cylinders (as it was in the second method of the connected sum) in such a way that $M_i \times I$ contains the knot K_i, $i = 1, 2$ (i.e. if we close this manifold, we obtain a surface realization of K_i).
Here by closing we mean the pasting of obtained cylindrical holes $S_i^1 \times I$ by three-dimensional plates $D_i^2 \times I$ in order to get two thickened surfaces.

(2) The intersection $M = M_1 \cap M_2$ consists of one or two components; so $(M_1 \times I) \cap (M_2 \times I)$ consists of one or two annuli.

(3) The knot $K_1 \# K_2$ intersects the manifold $(M_1 \cap M_2) \times I$ precisely at two points; in the case when $M_1 \cap M_2$ is not connected, these intersection points lie in different connected components.

(4) The process of destabilization described in the previous three items stops for finite number of steps. In this moment $g(M_1 \# M_2)$ gives the minimal genus for the knot $K_1 \# K_2$.

Here we use Kuperberg's theorem stating that the minimal representative is unique, and it gives the minimal genus of the knot $K_1 \# K_2$. The process continues unless destabilization is possible, i.e. the genus is not minimal.

If we organize the process as described above, we shall prove Theorem 2.5. Indeed, at each moment of the process we have K_1 and K_2 represented by knots in thickened surfaces of genera g_1 and g_2, respectively. The knot $K_1 \# K_2$ lies in the surface of genus $g_1 + g_2$ if we deal with the connected sum of the first type and in the surface of genus $g_1 + g_2 - 1$ if we deal with the connected sum of the second type. So, the same holds when the process stops, thus we have $g(K_1 \# K_2) = g_1 + g_2$ or $g(K_1 \# K_2) = g_1 + g_2 - 1$, where the last case is possible only if we have the connected sum of the second type (hence, both g_1 and g_2 are greater than zero). Taking into account that g_i is the genus of a surface (not necessarily minimal) representing K_i, we obtain the statement of the theorem.

2.3.3 *The process of destabilization*

In this subsection, we describe how this process works.

Suppose we have a connected sum of type j ($j = 1$ or $j = 2$) of the knots K_1 and K_2. The main statement is the following.

Statement 2.1. *If there is a possibility to decrease the genus of $M_1 \# M_2$ (with the knot $K_1 \# K_2$ inside), then one of the following holds:*

(1) *We can perform a destabilization in M_i (for $i = 1$ or $i = 2$) without changing M_{3-i} and the connected sum type. Thus, we decrease the genus of one of the connected summands M_i by one, as well as that of $M_1 \# M_2$.*
(2) *If we have the first type connected sum, then there is a possibility to transform it to the connected sum of the second type, decreasing the genus of $M_1 \# M_2$ by one without changing the genera of M_1 and M_2.*
(3) *If we have the second type connected sum, then there is a possibility to transform it to the connected sum of the first type, decreasing each of $g(M_1)$, $g(M_2)$, $g(M_1 \# M_2)$ precisely by one.*

Together with all points described above, this statement completes the proof of Theorem 2.5.

***Proof of Statement* 2.1.** First, consider the case of the first type connected sum. We have $M = M_1 \# M_2$. Let A be an annulus separating the manifold M into two parts M_1 and M_2. As was said before for convenience we write $M = M_1 \# M_2$ and sometimes $M = M_1 \cup M_1$.

Suppose we are able to destabilize the pair $((M_1 \# M_2) \times I, K_1 \# K_2)$. Then there is a vertical annulus C in $(M_1 \# M_2) \times I$ (i.e. an annulus the boundary components of which lie in different components of the boundary $(M_1 \# M_2) \times \{0\}$ and $(M_1 \# M_2) \times \{1\}$) which does not intersect the knot $K_1 \# K_2$. If there is such an annulus which does not intersect A, then we can destabilize one of the summands, either (M_1, K_1) or (M_2, K_2), along C; this is the first case of Statement 2.1.

Suppose there is no such an annulus C, i.e. any (non-trivial) annulus C we consider will intersect A. Without loss of generality we assume that the intersection between each C and A is transverse. Let n be the minimal number of connected components of the intersection $C \cap A$ (we suppose that n is not equal to zero).

Since C and A are manifolds with boundary (vertical annuli), their (generic) intersection may consist of:

- trivial circles;
- trivial arcs;
- horizontal circles;
- vertical arcs.

If there is a trivial circle in the intersection $C \cap A$, then we can consider the innermost (by conclusion) circle $\gamma = \partial\Delta$ (herewith $\Delta \subset C$). This circle contains no intersection points with A inside. Because this disc Δ together with a disc from A bounds a 3-ball (see Fig. 2.8), we can slightly change the annulus C in such a way that the total intersection between C and A decreases, and C remains an annulus with non-contractible core (along which we can perform the destabilization). The same situation happens when we have a trivial arc, see Fig. 2.9.

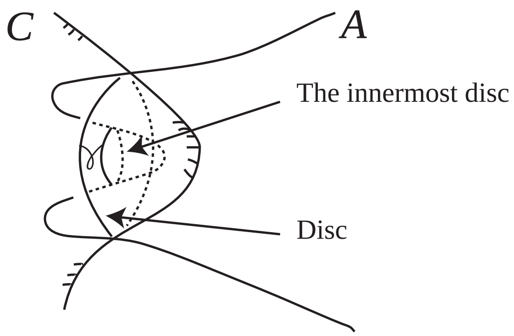

Fig. 2.8 Trivial circles.

Now, let us state two auxiliary lemmas.

Lemma 2.2. *Suppose S_g is an oriented surface of genus g and let Δ be an embedded disc in S_g. Then if a closed curve $\gamma \subset S_g \backslash \Delta$ without self-intersection points is trivial in S_g and not trivial in $S_g \backslash \Delta$, then it is parallel to $\partial\Delta$ (i.e. $\gamma \cup \partial\Delta$ bounds a cylinder in S_g).*

Indeed, if the curve γ not having common points with Δ bounds a disc in S_g, then γ is contractible in $S_g \backslash \Delta$ to the boundary.

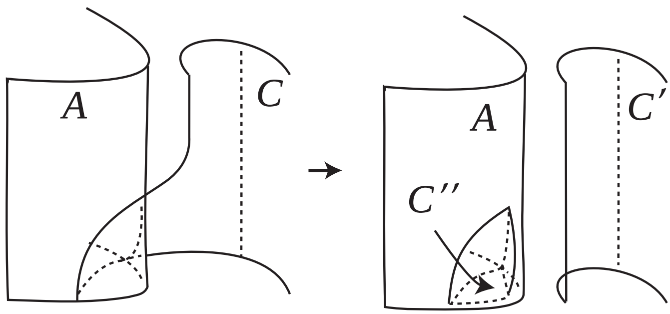

Fig. 2.9 Removing the intersection along a trivial arc.

The following lemma is evident.

Lemma 2.3. *If a proper annulus C' is free homotopic to the annulus A (in the class of proper annuli), then the annulus C' intersects the knot $K_1 \# K_2$.*

Now, we may assume that our intersection $C \cap A$ consists only of vertical arcs, or only of horizontal circles (the existence of a vertical arc contradicts the existence of a horizontal circle).

Suppose we have only horizontal circles. Then C is homotopic to A and, by Lemma 2.3, the annulus C intersects the knot $K_1 \# K_2$. Thus we obtain a contradiction.

Now, suppose the intersection $C \cap A$ consists only of vertical arcs. Then the annulus C is divided into $2k$ parts $C_1, \ldots, C_{2k}$, whereas C_{2l-1} lies in $M_1 \times I$, and C_{2l} lies in $M_2 \times I$, where $l \in \{1, \ldots, k\}$, when meaningful. The annulus C is thus divided into $2k$ sectors by radii (more precisely, radial segments); some of these sectors contain intersection(s) with the knot (they are depicted by thick points), see Fig. 2.10. Denote all these radii by $r_1, \ldots, r_{2k}$.

The annulus A divides $M \times I$ into two parts. Let us call one of them positive, and the other negative. Further, each part C_i of the annulus C is incident to two radii r_j and r_k. These two radii divide the annulus A into two parts; denote these parts (in an arbitrary order) by A^+_{jk} and A^-_{jk}. There are four options with respect to the following questions:

(1) Is it true that any of the two parts A^+_{jk} and A^-_{jk} intersects the knot precisely at one point?
(2) Is it true that an annulus obtained by attaching C_i to one of A^+_{jk} or A^-_{jk}, cuts off a ball (so that if we attach C_i to the other fragment, we

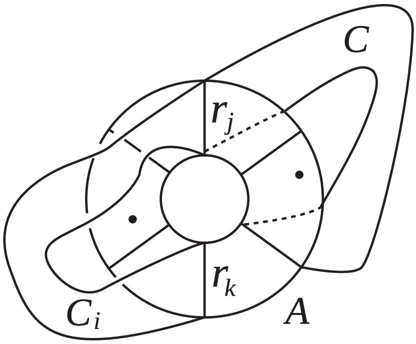

Fig. 2.10 The annulus C divided into sectors.

get an annulus homotopic to A)?

Remark 2.8. Here we mean that a proper surface (say, annulus) $F \subset N$ cuts a ball if $N \backslash F$ has two connected components, one of which is a topological ball. In other words, F bounds a ball together with a part P of the boundary ∂N of the manifold N, so that $P \cup F$ is a 2-sphere.

First, consider the case when the answer to the first question is negative, i.e. one of the parts, say, A^+_{jk} does not intersect the knot (whence the other part A^-_{jk} meets the knot twice).

If the annulus obtained by gluing C_i with A^+_{jk} cuts a ball, then we may "pull" A^+_{jk} through C_i; this would decrease the number of intersection components between C and A. This leads to a contradiction.

If the annulus obtained by pasting C_i and A^-_{jk} cuts off a ball, then the annulus $C_i \cup A^+_{jk}$ is homotopic to A; thus it should intersect the knot. This leads to a contradiction again.

If both answers are affirmative, we get a contradiction: Our knot cannot meet the boundary of a ball precisely at one point.

If both answers are negative, then one of the gluing $C_i \cup A^+_{jk}$ or $C_i \cup A^-_{jk}$ gives a non-trivial annulus not intersecting the knot. By small perturbation we can change this annulus in such a way that a new annulus does not intersect with A. Thus, we get a contradiction with the fact that C has a minimal intersection with A among all annulus along which we can destabilize.

Finally, if, say, A^+_{jk} contains precisely one intersection point with the knot $K_1 \# K_2$, then the number of intersection components of $C \cap A$ should be equal to two.

Indeed, if it is greater than two, then it can be decreased as shown in Fig. 2.11. More precisely, among all parts C_i of the annulus C we take only two parts and compose a non-trivial annulus C', along which $(M_1 \# M_2) \times I$ with the knot $K_1 \# K_2$ inside can be destabilized.

In other words, having more components C_i than two, one can find two of them which can be repasted and thus obtain a new non-trivial annulus C' intersecting A at a smaller number of curves and not intersecting the knot $K_1 \# K_2$.

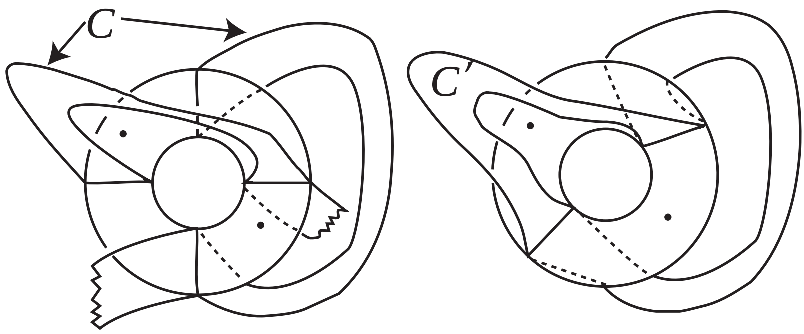

Fig. 2.11 Simplifying the curve C.

Thus we see that if the intersection $C \cap A$ was minimal (over the number of connected components), there would be precisely two connected components.

Let us show that the destabilization along such C just transforms the type of connected sum, i.e. we obtain the connected sum of the second type.

Indeed, the annulus A is just cut into two parts by this destabilization, and two intersection points after the destabilization lie in different connected components.

The proof for the case when we have the connected sum of the second type (and the destabilization takes it to the connected sum of the first type) goes in the same vein. Either it is possible to destabilize only one of M_1 or M_2 with the corresponding knot inside or all annuli along which the stabilization can be performed intersect $M_1 \cap M_2$ (which consists of two components in this case). Considering such an annulus representing minimal intersection with $M_1 \cap M_2$, we get the only possibility when the destabilization transforms the connected sum of the second type to the connected sum of the first type.

Therefore, we have proved not only Theorem 2.5, but also Lemma 2.1. Indeed, if we have a connected sum of genus zero in the end of destabilization, then this connected sum is a connected sum in the thickened sphere $S^2 \times I$, this is obviously equivalent to a classical connected sum of classical knots. □

2.4 Haken's theory and algorithmic recognition of virtual links

The aim of this section is to prove the following.

Theorem 2.6. *There is an algorithm to decide whether two virtual links are equivalent or not.*

The main argument of the proof is an application of an algorithm recognizing some class of 3-manifolds (so-called Haken manifolds) and studying "normal surfaces" in these manifolds. This method was widely used for solving different algorithmic problems in three-dimensional topology. In particular, in the frame of this theory analogous (but simpler) arguments led to a solution in the classical case.

We shall use the result by Moise [249] that each 3-manifold admits a triangulation. In the sequel, each 3-manifold is thought to be triangulated; herewith all considered subsets of 3-manifolds, for example, proper subsets in it, are assumed to be subpolyhedrons of a given triangulation.

In each triangulated 3-manifold N (perhaps, with empty boundary) one can consider *proper* 2*-surfaces* F. Recall that a surface F is called *proper* if $F \cap \partial N = \partial F$. In the class of proper surfaces one can pick out the class of, so-called, *normal surfaces*.

A *normal surface* is a proper surface which, roughly speaking, respects the triangulation. In this sense the definition depends on a triangulation, though later properties of normal surfaces which will be described by us will be universal.

In the class of normal surfaces one can consider the class of *fundamental* or *basis* surfaces. All normal surfaces are obtained from them by applications of *geometric summation* (for details see [243]).

The notion of a normal and fundamental surface can be generalized on manifolds with a *pattern* on the boundary, which is a graph without isolated vertices.

The main statements of Haken's theory of normal surfaces are the following:

(1) the set of fundamental normal surfaces is finite and can be constructed algorithmically;
(2) for many natural properties the following statement holds: If there exists a normal surface F in a manifold N possessing a property P, then there also exists a fundamental surface possessing the property P.

From these statements it follows that the problem of finding a normal surface possessing a "natural" property is algorithmically solvable.

Among the properties mentioned in (2) of the main statements there is the following: the property for a surface F to be a sphere and represent a non-trivial element of the second homotopic group of the manifold N. There is also a possibility of stabilization/destabilization for a given representative of a virtual link (see below).

We are not going to give an accurate definition of a normal surface and a fundamental surface. Our further proof of Theorem 2.6 will be based on Kuperberg's theorem and the sequence of lemmas from the theory of normal surfaces (proofs of the most part of the lemmas can be found in [243]).

We shall need some additional definitions.

A manifold N is called *irreducible* if each embedded 2-sphere in N bounds a 3-ball in N.

We shall use the definition of virtual knots as knots in thickened surfaces $M \times I$ up to stabilizations/destabilizations. Here M is a compact 2-surface, not necessarily connected. Herewith we require that for each connected component M_i of the surface M, the 3-manifold $M_i \times I$ contains at least one component of the link.

Recall that a representative for a virtual link is *minimal* if it cannot be destabilized.

Further, we shall need Theorem 2.1 by Kuperberg [189]. Thus, in order to compare virtual links, it is sufficient for us to be able to find their minimal representatives and compare them. The algorithm to be given below uses a recognition techniques for 3-manifolds with boundary pattern (see definition below) connected to the virtual links in question.

We shall use the following facts from Haken–Matveev theory, see [243].

A *compressing disc* for a surface F in a 3-manifold N is an embedded (non-proper) disc $D \subset N$ which meets F along the boundary of the disc, i.e. $D \cap F = \partial D$.

A surface (possibly disconnected) $F \subset N$ is called *compressible* in one of the two cases:

- it admits a compressing disc D such that ∂D does not bound a disc in F;
- there is a ball B in N such that $B \cap F = \partial B$.

A surface is *incompressible* if it is not compressible.

A surface $F \subset N$ is called *boundary compressible* if there exists a disc $D \subset N$ such that $D \cap (\partial N \cup F) = \partial D$ and $D \cap F$ is a non-trivial arc in F (an arc that does not cut a disc from F). Otherwise, a surface is called *boundary incompressible*.

Also, a 3-manifold N is *boundary irreducible* if for any proper disc $D \subset N$ the boundary ∂D bounds a disc on ∂N.

Given a 3-manifold with boundary. By a *boundary pattern* (first proposed by Johannson [140]) we mean a fixed 1-polyhedron (graph) without isolated vertices on the boundary of the 3-manifold (we assume this graph is a subpolyhedron of the selected triangulation).

The existence of a boundary pattern does not change the definition of incompressible surface and irreducible manifold.

We have straightforward generalizations of a boundary incompressible surface and a boundary irreducible manifold for the case of manifolds with a boundary pattern as described below. A disc $D \subset N$ is called *clean* if it does not intersect the pattern.

For boundary irreducibility we require that for every clean proper disc $D \subset N$, $\partial D \subset \partial N$ the boundary ∂D bounds a disc in $F \subset N$.

Further, a *boundary incompressible disc* (non-proper) for a surface F is a clean disc $D \subset N$ intersecting the surface F along an arc $l \subset \partial D$ and the arc $F \cap \partial N$. In this case a disc D is *inessential* if l cuts out a disc D' from F such that $\partial D'$ consists of the arc l and a clean arc in ∂N. A surface $F \subset (N, \Gamma)$ is called *boundary incompressible* if it has no essential boundary compressible discs.

Recall that an orientable 3-manifold N is *sufficiently large* if it contains a proper incompressible boundary incompressible surface distinct from a sphere S^2 and a disc D.

It is natural to consider the notion of sufficiently large manifold together with properties of irreducibility and boundary irreducibility. This leads to the following definition.

Definition 2.9. A connected 3-manifold without boundary pattern is *Haken* if it is irreducible and sufficiently large.

An irreducible boundary irreducible connected 3-manifold (N, Γ) with a boundary pattern Γ is *Haken* either if it is sufficiently large or if its pattern Γ is non-empty (hence, so is ∂N), and N is a handlebody but not a ball.

Definition 2.10. Let N be an irreducible boundary irreducible 3-manifold (without a boundary pattern). A proper annulus $A \subset N$ is called *inessential* if either it is parallel relatively ∂ to an annulus in ∂N, or the core circle of A is contractible in N. Otherwise A is called *essential.*

Any essential (with respect to the manifold $S_g \times I$ from which the link is deleted, see later) annulus in $S_g \times I$ with two boundary components lying in $S_g \times \{0\}$ and $S_g \times \{1\}$ is precisely an annulus along which we may destabilize.

A manifold with more than one connected component is called *Haken* if any connected component of it is Haken.

We shall use the following proposition (see [243, 287]).

Proposition 2.1 (Jaco–Rubinstein–Thompson). *Any connected irreducible 3-manifold with non-empty boundary is either sufficiently large or a handlebody.*

Later on, we deal with manifolds with non-empty boundary pattern. For this manifold to be Haken, it is sufficient to check that the manifold (more accurate, each of its connected components) in question is irreducible and boundary irreducible but not a ball.

Let us formulate lemmas (see [243, 287]) which we shall use for proving Theorem 2.6. We do not give their proofs.

Lemma 2.4 (Jaco–Rubinstein–Thompson). *There exists an algorithm to decide whether a manifold N is reducible; if it is so, the algorithm constructs a 2-sphere $S^2 \subset N$ not bounding a ball in N.*

Lemma 2.5. *Classical links are algorithmically recognizable.*

This lemma follows from Haken's theory of normal surfaces; the proof is based on the following ideas: For each non-trivial non-split link, the complement in S^3 to a tubular neighborhood of this link is a Haken manifold. Endowing the boundary with a pattern, we shall be able to restore the initial link. After that, the problem is reduced to the recognition problem for Haken manifolds, for details see [243].

Lemma 2.6. *There is an algorithm to decide whether a Haken manifold N has a proper clean essential annulus. If such an annulus exists, it can be constructed algorithmically. Moreover, for two boundary components U, V of the manifold N the problem of finding an essential annulus such that at least one component of its boundary lies in U and none of its boundary components lies in V is algorithmically solved.*

Remark 2.9. We shall use this lemma to define whether a given representative of a virtual link can be destabilized. For this the boundary components which are the boundaries of tubular neighborhoods of the link components play the role of V. Therefore, essential annuli the boundary components of which lie on the boundary components of the thickened surface stay only. For stabilization we shall require that the boundary components of our annulus lie on the different boundary components of the thickened surface.

It is easy to see that the question of existence and finding of such an annulus is also algorithmically recognizable.

Lemma 2.7. *There is an algorithm to decide whether two Haken manifolds (N, Γ) and (N', Γ') with boundary patterns are homeomorphic by means of a homeomorphism that maps Γ to Γ'.*

Consider a virtual link K and an arbitrary representative of K, i.e. a couple (N, K), where $N = M \times I$ for some closed 2-surface M, and K is a link in N (we use the same letter K for denoting both the initial link and a representative of the link in N). Let O be a small open tubular neighborhood of the link K. Cut O from N. We obtain a manifold with boundary. Denote it by N_K. Its boundary consists of the boundary components of N (two, if N is connected) and several tori; the number of tori equals the number of components of the link K. Let us endow each torus with a pattern Γ_K on the boundary, representing the meridian of the corresponding component with three points on it: We also add a vertex to make a graph from the meridional circle. Thus we obtain the manifold (N_K, Γ_K) with boundary pattern.

It is obvious that the virtual link K (and the pair (N, K)) can be restored from (N_K, Γ_K), since we know how to restore the manifold N by attaching full tori to the boundary components of N_K knowing meridians of these full tori.

Lemma 2.8. *Suppose a link K is not a split sum of a (non-empty) classical link and (maybe empty) a virtual link. Then the manifold (N_K, Γ_K) with the boundary pattern Γ_K is Haken.*

Proof. Without loss of generality we may assume that the manifold N_K is connected. Indeed, let us consider the connected components of the manifold N and, if the link K has no classical non-linked components, then each of the components of the manifold (N_K, Γ_K) is a Haken manifold; then, by definition, so is the manifold (N_K, Γ_K).

We consider the connected case. By virtue of Proposition 2.1, it remains to show that (any connected component) this manifold with boundary pattern is irreducible and boundary irreducible (by definition, it cannot be a handlebody).

In the case when $g = 0$ we deal with classical links. Suppose $g > 0$. Then, for any connected orientable 2-surface S_g the manifold $S_g \times I$ is irreducible. Thus, if the link K is not classical (thus $g \neq 0$), then for its neighborhood $O(K)$ the set $(S_g \times I)\backslash O(K)$ might be reducible if and only if it contains a sphere S^2, bounding a ball in $S_g \times \{0, 1\}$ such that this ball contains some components of the link K. This means that these components form a classical sublink of K separated from all other components. The contradiction proves irreducibility.

Furthermore, since K is not a split sum of the unknot with some virtual link, the manifold N_K is boundary irreducible.

Indeed, each curve in $S_g \times \{0\}$ or in $S_g \times \{1\}$ which may bound a disc in $S_g \times I$ is contractible in the boundary. Thus, boundary reducibility can occur only if we have a proper disc with boundary lying on some torus, the boundary of a cut full torus. This should mean that the cut full torus corresponds to the split unknot of the link.

Thus, the corresponding manifold is irreducible and boundary irreducible and thus (by Proposition 2.1), Haken. □

Now, let us prove the main theorem. Let K and K' be virtual links. The recognition algorithm consists of the successful application of the following steps:

(1) Consider some representatives (N, K), (N', K') of the virtual links K, K'. Let us construct the corresponding manifolds with boundary patterns. Denote them by (N_K, Γ), $(N'_{K'}, \Gamma')$.
(2) Define whether one of N_K or $N'_{K'}$ is reducible. If one of them is so, then by Lemma 2.4, we may find a sphere not bounding a ball, and thus separate some classical components of the corresponding link. Now, rename the manifolds with boundary patterns accordingly, i.e. we shall use the previous notations N, N' for what are left from manifolds by

cutting off classical components. We shall keep in mind the deleted classical sublinks.

(3) Define (by Lemma 2.6) whether it is possible to destabilize one of (N, K) and (N', K'). If it is possible, then we shall perform the destabilization. Return to step (2). Let us perform steps (2) and (3) while possible. Obviously, this process stops in a finite period of time since at each step we either delete some number of the link components or decrease the genus of one of the surfaces. Classical links are algorithmically recognizable. Thus, we may compare the split classical sublinks of K and K'. If they are not isotopic, we stop and get that the virtual links in question are not equivalent. Otherwise, we go on. After performing the first three steps, we reduce our problem to the case when there are no split components and their representatives are minimal. From now on, the manifolds in question are Haken manifolds (with a boundary pattern) by Lemma 2.8.

(4) Each connected component of the manifolds (N_K, Γ) and $(N'_{K'}, \Gamma')$ is a Haken manifold with boundary pattern. Thus, we can algorithmically solve the problem whether there exists a homeomorphism $f\colon N_K \to N'_{K'}$ that maps Γ to Γ' (by Lemma 2.7). If such a homeomorphism exists, then virtual links K, K' are equivalent. Otherwise K and K' are not equivalent.

Performing the steps described above, we solve the recognition problem. Theorem 2.6 is proved.

Remark 2.10. The proof given above works also for oriented virtual links and framed virtual links.

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Chapter 3

Quandles (Distributive Groupoids) in Virtual Knot Theory

3.1 Introduction

At the initial stage of knot theory development the key role was played by a topological invariant, *the fundamental group of the knot complement* or *the knot group*. Based on this group, a simple and more convenient invariant, the Alexander polynomial, was constructed, see [5, 7].

The fundamental group of the knot complement can be defined combinatorially by using the Wirtinger presentation which was described in Chap. 1. From Wirtinger's presentation of the fundamental group one can derive the Alexander polynomial in numerous ways; one of them is called the Fox free calculus [66]. The Alexander polynomial and topological invariants of knots are closely related with *torsions*, the review of the combinatorial torsion theory can be found in [299].

A simplest way of proving non-triviality of the trefoil knot is the usage of the so-called *coloring invariant*. One considers a classical diagram K and its arcs (connected components; as usual we think of the undercrossing branch to be broken as it is usually drawn on the plane). We shall color all arcs of the diagram with three colors; we call this coloring *proper* if at every classical crossing either all three arcs have the same color or they all have three different colors. Let us count the number of proper colorings of the knot diagram K. Every classical knot diagram has three *monochrome* colorings. It can be easily proved (see [66]) that *the number* of proper colorings is a link invariant (it can be checked straightforwardly by looking at what happens to colorings under the Reidemeister moves).

Remark 3.1. As it will follow from the general theory, the coloring invariant can be extended to virtual knots and links.

The simplest diagram of the unknot has one (cyclic) arc. Thus, the number of proper colorings of the unknot is equal to three since the trivial diagram has exactly three colorings. Thus, any diagram having a coloring with more than one color, corresponds to a non-trivial knot. Analogously, for the m-component trivial link the number of proper colorings is equal to 3^m. One can easily count that the number of proper colorings of the trefoil knot is equal to nine, since, except the three monochrome colorings, it has six colorings using all three colors: we have three arcs, each of which is incident to every crossing.

This example gives one of the simplest invariants of a deep algebraic theory which leads to numerous invariants of classical and virtual knots considered in this chapter.

Algebraically, the coloring invariant is described as follows: Every color is an element from $\mathbb{Z}_3$; the coloring is *proper* if at every crossing the relation $c \equiv 2b - a \pmod 3$ holds, where a is the overcrossing arc, and b and c are undercrossing arcs (say, approaching a from the right and from the left, respectively).

It turns out that the number of colorings of a knot by elements of an arbitrary ring (with a properly chosen operation at crossings) is a knot invariant. This fact (even in a more general form) will be proved in the sequel.

Both the fundamental group and the quandle can be generalized to an *almost complete*[1] invariant of knots, the *knot quandle* (*distributive groupoid*) which was suggested in late 1970s and early 1980s by Matveev and Joyce. Here one has to mention the paper by Deviatov [71], where the groupoid structure was enlarged to recognize the left and right trefoils.

The completeness of the quandle follows from the invariance of the completeness of the fundamental group with its *peripheral structure*; the latter follows from the celebrated theorem due to Waldhausen stating that for a large class of 3-manifolds the fundamental group is a complete invariant, see [313].

The quandle, like the fundamental group, can be defined geometrically (as the homotopy classes of paths with fixed initial point and final point sliding over the boundary component; for homotopy one requires the initial point to be fixed, and the final point to lie on the boundary). The quandle can also be defined algebraically by means of a formal "presentation" analogous to the Wirtinger presentation.

[1]This invariant detects knots up to a simultaneous change of orientations of the knot and of the ambient space.

Besides its completeness, the quandle is also useful because its particular cases and “preimages” in different categories lead to various invariants of knots, which are easy to work with. Such invariants are the Alexander polynomial as well as some new invariants which were discovered immediately after the quandle itself.

In mid-1980s, Viro [308] and Fenn [88, 89] independently suggested further generalizations of the notion of groupoid based on a more complicated relation structure at classical crossings. Any stronger invariants for classical knots were not constructed, since the quandle itself was almost a complete invariant, however, they turned out to be useful in the case of virtual knots.

The interest to quandles and their generalizations revealed when virtual knots were conceived by Kauffman. A generalization of a complete invariant of classical knots to a new domain automatically shows that the set of classical knots embed in the set of virtual knots, i.e. two virtually equivalent classical knots are classically isotopic [114] (this was mentioned in previous chapter, however, historically the algebraic proof by using quandles was the first one).

However, the quandle itself turned out to be a rather weak invariant for recognition of virtual knots, which led the first-named author [206, 208] to the idea of a deeper approach to algebraic structures and introducing new operations at virtual crossings; the object defined in such a way is called *the virtual quandle* (other authors considered some other generalizations of the quandle, see [85, 174, 276, 280]). Here we mention the work by Afanasiev [1], where the author constructed a refinement of the quandle by using parity, see Chap. 8.

The ideas from [208] were later developed by Fleming and Mellor [93] for the construction of virtual spatial graph theory.

In the way above the first-named author has achieved a number of new invariants of virtual knots; some invariants vanish on classical knots; this allows one to prove non-classicality of many virtual knots. On the other hand, our approach to distributive groupoids (together with some other ideas) led to *new invariants of long knots* based on the notion of the quandle; we shall touch on these results in Sec. 3.3.

The construction of long knot invariants by using quandles is the central result of this chapter. The construction of Sec. 3.3 allowed the first-named author to recognize the non-triviality of long virtual knots having trivial closure by very simple arithmetic means, and also to construct the first example of non-commuting long virtual knots.

Later on, in this chapter we construct a realization of the distributive groupoid which leads to a construction of invariants of classical and virtual knots valued in (possibly, infinite-dimensional) Lie algebras.

Here, we construct and investigate invariants of virtual links originating from planar diagrams. Our goal is to associate with a virtual diagram an object to be invariant under generalized Reidemeister moves. The general method goes as follows. We consider a virtual diagram as a framed 4-valent graph on the plane. With edges of the graph we associate some formal elements which will play the role of generators for our algebraic object to be invariant, and crossings indicate the relations between these elements. The invariance is proved by a direct consideration of the generalized Reidemeister moves.

Toward the end of the chapter, we shall also give some applications of the quandle theory to the theory of flat virtual knots.

In addition, we shall describe hierarchy flat knots, combinatorial generalizations of flat virtual knots for which one can construct invariants by methods similar to those described above.

3.2 Quandles and their generalizations

Let K be a virtual diagram. Let us delete all virtual crossings from it. We get a (disconnected) set $\widetilde{K}$ on the plane consisting of connected components to be called *arcs*. Every arc is either a circle (a link component which has no virtual crossings and forms an overpass at every classical crossing) or an interval. In the second case, the initial and final points correspond either to deleted virtual crossings or to a classical underpass. The subsequent arcs separated by virtual crossings, form *long arcs* which pass from an underpass to the next underpass. In the case of classical links the notion of arc coincides with the notion of long arc.

We call a diagram *proper* if it has no cyclic long arcs. By applying a first increasing classical Reidemeister move to a diagram, we can transform a cyclic long arc to a long arc which is not cyclic.

Every virtual link has a proper diagram. Indeed every diagram of a virtual link having a cyclic arc is transformed to a proper diagram by means of several first Reidemeister moves as shown in Fig. 3.1.

The following statement allows us to restrict ourselves for the case of proper diagrams.

Statement 3.1. *Two proper diagrams generate equivalent virtual links if and only if one of them can be transformed to the other by Reidemeister*

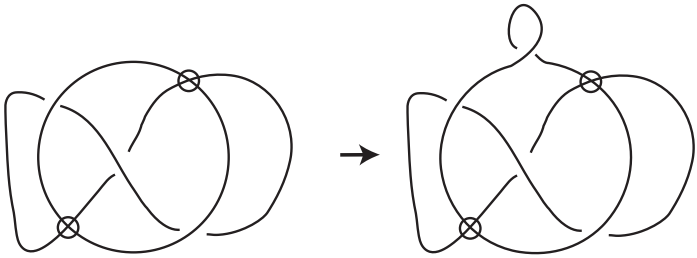


Fig. 3.1 Transformation of a diagram with a cyclic long arc to a proper diagram.

moves in the class of proper diagrams.

Indeed, if when transforming a diagram K to an equivalent diagram K' we meet diagrams with cyclic arcs, we may add auxiliary loops (by means of Ω_1) and then remove them. Thus, the chain of generalized Reidemeister moves $K = K_0 \to K_1 \to \cdots \to K_l = K'$ is transformed into the chain $K = K_0 = K_{0,1} \to \cdots \to K_{0,i_0} \to K_1 \to \cdots \to K_{1,i_1} \to \cdots \to K_{l,1} \to \cdots \to K_{l,i_l} = K_l = K'$, where all diagrams $K_{i,j}$ are proper, and each diagram $K_{i,j}$ is obtained from K_i by an addition/removal of loops with the first Reidemeister move. This concludes the proof. Later in the chapter all virtual diagrams are assumed proper, unless otherwise specified.

Long arcs are split into *arcs*. We may proceed with this subdivision: We divide every arc at a classical crossing where it forms an overpass. In this case we talk about *short arcs*. An example of a diagram with long arcs, arcs and short arcs is shown in Fig. 3.2. Thus, we shall generate our algebraic objects by arcs (long arcs or short arcs).

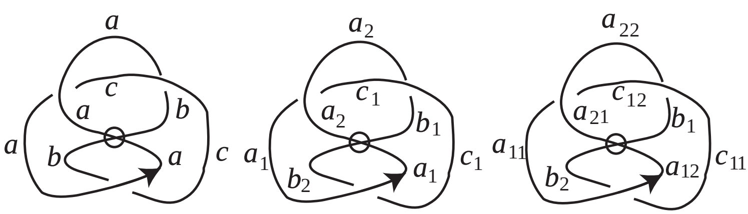


Fig. 3.2 Long arcs, arcs, short arcs.

Let us first consider long arcs of the diagram K. At every classical crossing, locally three long arcs meet each other where one of them forms an overcrossing, and the other two form an undercrossing. Globally, two or three of these arcs can coincide.

Let the number of classical crossings of the diagram K be equal to n. Then the number of long arcs of this diagram is also equal to n (since the diagram K is proper).

Let us mention one important circumstance. When constructing invariants by using long arcs, we ignore the mutual disposition of virtual crossings. Thus, the object we are constructing will be naturally invariant under the detour move (see Fig. 1.7).

Let us associate our long arcs with formal letters (generators) $a_1, \ldots, a_n$. Our goal is to construct an algebraic object invariant under all generalized Reidemeister moves and satisfying some axioms. The relations imposed on the generators $a_1, \ldots, a_n$ of this object will be taken from classical crossings of the diagram.

Following Matveev [242], we define a *quandle* (a *distributive groupoid*) as a set M, endowed with a binary operation $\circ$, satisfying the following properties:

(1) *idempotency*: for every $a \in M$ we have $a \circ a = a$;
(2) *the existence of a left inverse*: for every b, $c \in M$ there exists a unique element $x \in M$ such that $x \circ b = c$ (in this case we write $x = c/b$);
(3) *right self-distributivity*: for every a, b, $c \in M$ the relation $(a \circ b) \circ c = (a \circ c) \circ (b \circ c)$ takes place.

Analogously to groups, quandles (or distributive groupoids) can be generated by generators and relations. This is done as follows. Let $a_1, \ldots, a_k$ be a finite collection of letters (generators). Let us define inductively the set Adm of *admissible words* as the set obtained from the generators by subsequent application of the operations $\circ$ and $/$. Namely, we define the set of admissible words inductively according to the following rules:

(1) all letters a_i are admissible words;
(2) if x, y are admissible words, then the words $(x) \circ (y)$ and $(x)/(y)$ are admissible as well;
(3) there are no other admissible words except for those obtained as in (1), (2).

Remark 3.2. Sometimes in order to simplify the notation we shall omit brackets, when the sense is clear from the context. In particular, instead of $(a_1) \circ (a_2)$ we write $a_1 \circ a_2$.

Now, assume that we are given a set of l relations R_i (called *defining relations*) of the type $r_{i1} = r_{i2}$ for $i = 1, \ldots, l$, where r_{ij} are some admissible words.

On the set Adm of all admissible words there are well-defined operations $\circ$ and $/$.

Now, let us define the quandle generated by $a_1, \ldots, a_k$ subject to the relations $R_1, \ldots, R_l$ and denoted by

$$\Gamma\langle a_1, \ldots, a_k | R_1, \ldots R_l \rangle$$

to be the quotient set of the set Adm modulo the equivalence relation; the latter is defined by the following elementary equivalence relations:

(1) $A \circ A \sim A$, $A/A \sim A$ for any $A \in \text{Adm}$;
(2) $(A \circ B)/B \sim (A/B) \circ B \sim A$ for any A, $B \in \text{Adm}$;
(3) $(A \circ B) \circ C \sim (A \circ C) \circ (B \circ C)$ for any A, B, $C \in \text{Adm}$;
(4) $r_{i1} \sim r_{i2}$ for any $i = 1, \ldots, l$.

For the resulting set, the operations $\circ$ and $/$ are induced from the set Γ; the quandle axioms are obviously satisfied.

Now, we shall consider two definitions: geometric and algebraic definitions of the knot quandle.

3.2.1 *Geometric description of the quandle*

Let $\mathcal{K}$ be an oriented knot in $\mathbb{R}^3$, and let $N(\mathcal{K})$ be its small tubular neighborhood. Let $E(\mathcal{K}) = \overline{(\mathbb{R}^3 \backslash N(\mathcal{K}))}$ be the complement to this neighborhood. Fix a base point $x_{\mathcal{K}}$ on $E(\mathcal{K})$. Denote by $\Gamma_{\mathcal{K}}$ the set of homotopy classes of paths in the space $E(\mathcal{K})$ with fixed initial point at $x_{\mathcal{K}}$ and endpoint on $\partial N(\mathcal{K})$ (these conditions must be preserved during the homotopy). Note that the orientations of $\mathbb{R}^3$ and $\mathcal{K}$ define the orientation of the tubular neighborhood of the knot (right screw rule). Let m_b be the oriented meridian hooking an arc b. Define $a \circ b = [\bar{b}\overline{m_b}\overline{b^{-1}}\bar{a}]$, where for $x \in \Gamma_{\mathcal{K}}$ the letter $\bar{x}$ means a representative path, and the square brackets denote the class that contains the path $\bar{x}$, see Fig. 3.3.

The quandle axioms can also be checked straightforwardly. Also, one can easily check that groupoids corresponding to different points $x_{\mathcal{K}}$ are isomorphic. This statement is left for the reader as an exercise.

There is a natural map from the knot quandle $\Gamma_{\mathcal{K}}$ to the group $\pi_1(\mathbb{R}^3 \backslash E(\mathcal{K}))$. Let us fix a point x outside the tubular neighborhood. Now,

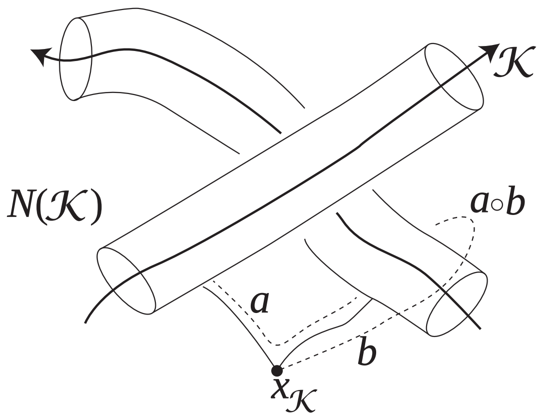


Fig. 3.3 Intuitive description of the quandle operation.

with each element γ of the quandle (path from x to $\partial E(\mathcal{K})$) we associate the loop $\gamma m \gamma^{-1}$, where m is the meridian at the point x.

This interpretation shows that *the fundamental group can be constructed by the quandle*: All meridians can play the role of generators for the fundamental groups, and all relations of type $a \circ b = c$ have to be replaced with $bab^{-1} = c$.

Besides, the fundamental group has the obvious action on the quandle. The path $\delta\gamma$ is again an element of the quandle for each loop δ and element γ of the quandle.

3.2.2 *Algebraic description of the quandle*

Let K be a diagram of the knot $\mathcal{K}$ with n classical crossings. At each classical crossing (with a number i) of this diagram, one of the two arcs forms an overpass. There is a corresponding generator a_{i_1}; we have the (long) arc on the right with the corresponding generator a_{i_2}, and the arc on the left with the corresponding generator a_{i_3}, see Fig. 3.4.

With each classical crossing we associate a relation

$$a_{i_2} \circ a_{i_1} = a_{i_3}. \tag{3.1}$$

Denote this relation by R_i.

Remark 3.3. Note that in Fig. 3.4 the orientation of a_{i_2} and a_{i_3} is not indicated; in fact, it is immaterial. Whatever it is, we shall take the rela-

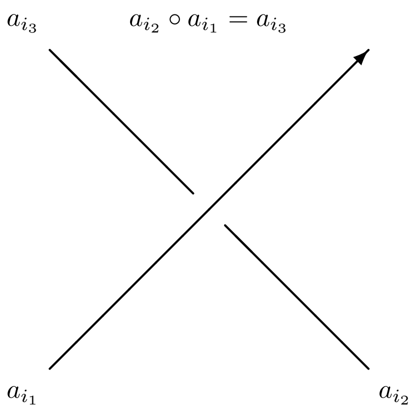


Fig. 3.4 The quandle relation at a crossing.

tion (3.1).

Now, let us associate with the diagram K the quandle

$$\Gamma(K) = \Gamma\langle a_1, \dots, a_n | R_1, \dots, R_n \rangle.$$

Theorem 3.1. *The quandles* $\Gamma_{\mathcal{K}}$ *and* $\Gamma(K)$ *are isomorphic.*

Before proving the theorem, let us first understand its possible interpretations. On the one hand, the theorem shows how to describe generators and relations for the geometrical quandle $\Gamma_{\mathcal{K}}$. On the other hand, it demonstrates the independence $\Gamma(K)$ of the choice of concrete knot diagram. Now, the statement asserting that the number of proper colorings by elements of any quandle Γ is a link invariant evidently follows from this theorem as a corollary because any proper coloring of a knot diagram by elements of Γ is a presentation of $\Gamma(K)$ to Γ.

***Proof of Theorem* 3.1.** With each arc a of the projection K, we associate the path s_a in $E(\mathcal{K})$ in such a way that

(1) the path s_a connects the base point with a point of the part of the torus $\partial N(\mathcal{K})$ corresponding to the arc a;
(2) at all points where the projection of s_a intersects that of K, the path s_a goes over the knot, see Fig. 3.5.

Obviously, these conditions are sufficient for the definition of the homotopy class of s_a.

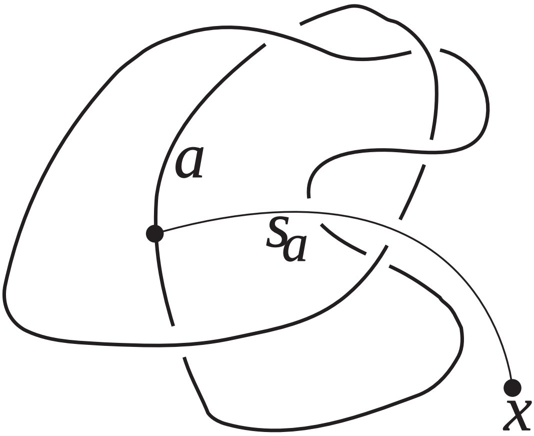


Fig. 3.5 Defining the path s_a.

Consequently, to each generator of $\Gamma(K)$, there corresponds an element of the quandle $\Gamma_{\mathcal{K}}$. Thus we have defined the homomorphism $\phi\colon \Gamma(K) \to \Gamma_{\mathcal{K}}$. In order to define the inverse homomorphism $\psi\colon \Gamma_{\mathcal{K}} \to \Gamma(K)$, let us fix $s \in \Gamma_{\mathcal{K}}$. Then, the path representing s is constructed in such a way that the projection of the path intersects K transversely and contains no diagram crossings.

Denote by $a_n, a_{n-1}, \ldots, a_1$ those arcs of K going over the path s. Denote by a_0 the arc corresponding to the end of s. Now, for each $s \in \Gamma_{\mathcal{K}}$, let us assign the element $((\ldots(a_0\varepsilon_1 a_1)\varepsilon_2 \ldots a_{n-1})\varepsilon_n a_n)$ of the quandle $\Gamma(K)$, where ε_i means the operation / if s goes under a_i from the left to the right, or the operation $\circ$, otherwise, see Fig. 3.6.

It is easy to check that this map is well defined (i.e. it does not depend on the choice of a representative s for the element of $\Gamma_{\mathcal{K}}$) and the maps ϕ and ψ are inverse to each other. This completes the proof. □

The following theorem holds.

Theorem 3.2. *If diagrams K, K' represent equivalent virtual knots, then the quandles $\Gamma(K)$ and $\Gamma(K')$ are isomorphic.*

The proof of this theorem is based on the fact that the quandle axiomatics corresponds to its invariance under the Reidemeister moves. Here, one only needs to check the invariance under the classical Reidemeister moves,

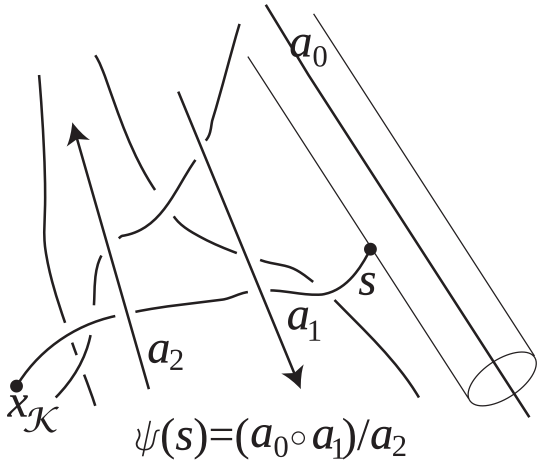


Fig. 3.6 Constructing the map $\psi\colon \Gamma_{\mathcal{K}} \to \Gamma(K)$.

since the disposition of virtual crossings on long arcs is immaterial. Later in this chapter we shall prove a more general theorem (Theorem 3.4).

In the classical case this theorem was first proved by Matveev [242] and independently by Joyce [144]. A generalization of the quandle for the case of virtual knots belongs to Kauffman [157].

Actually, Matveev and Joyce proved much more, namely the following theorem.

Theorem 3.3 (Matveev–Joyce). *If for classical knot diagrams K and K' the quandles $\Gamma(K)$ and $\Gamma(K')$ are isomorphic, then either the diagram K is equivalent to the diagram K', or K is equivalent to the diagram obtained from K' by a simultaneous change of orientations of the knot and of the ambient space (the latter leads to the change of crossing type: we switch overcrossing to undercrossing and vice versa).*

The latter operation is called the *double involution*. The invariance under the double involution makes a significant disadvantage of the quandle. For example, the quandle does not recognize the right trefoil and the left trefoil because the orientation reversal operation on any trefoil transforms this trefoil to itself, and the ambient space orientation reversal transforms the right trefoil to the left and vice versa. A refinement of the quandle which recognizes the trefoils is constructed in [71].

Consequently, the quandle is an almost complete invariant of classical knots. On the other hand, the quandle has a natural generalization to virtual knots according to Theorem 3.2. From this observation we get Theorem 2.2.

Remark 3.4. Taking into account the fact that the quandle is not sensitive to the double involution, in order to make it complete, one has to add another invariant (*peripheral structure*) to fix the orientation of the knot and to be extendable for the case of virtual knots. This is done in [114].

The structure of the quandle has natural generalizations which appear when we pass from long arcs to arcs or to short arcs. In the case of arcs, we shall introduce the relations between arcs at virtual crossings, and when passing to short arcs we shall modify the relations at classical crossings: instead of one binary function we shall get two binary functions. We have two input short arcs and two output short arcs. Unlike the case when the invariant is constructed by using *long* arcs, in the case of arcs or short arcs the invariance under the detour move is not evident, and it has to be checked.

This leads us to the notion of virtual quandle.

3.2.3 *The virtual quandle*

The notion of a virtual quandle first appeared in [206, 208].

Definition 3.1. By a *virtual quandle* we mean a quandle $(M, \circ)$ together with an automorphism $f\colon M \to M$, i.e. such a bijection that for every $a, b \in M$ one has $f(a \circ b) = f(a) \circ f(b)$.

Analogously to quandles, virtual quandles can be generated by generators a_i and defining relations R_j, where the indices i and j run over some finite sets. First we define the set of admissible words by using operations $\circ, /, f$ and f^{-1}.

Consider the following axiomatics for constructing virtual link invariants. Given a virtual link diagram K, let $a_1, \ldots, a_n$ be the set of arcs of this diagram. We shall use the operation $\circ$ for writing relations between arcs incident to the same classical crossing just in the same way as it was done before, see (3.1), Fig. 3.4; besides, we shall introduce a formal (unary) operation f for virtual crossings (this operation will be an automorphism of the quandle), namely: Let some virtual crossing be incident to four arcs with corresponding generators $a_{j_1}, a_{j_2}, a_{j_3}, a_{j_4}$ as shown in Fig. 3.7.

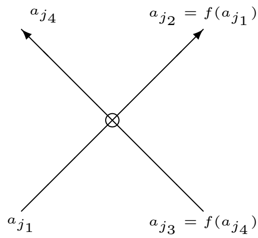


Fig. 3.7 A relation for a virtual crossing.

We shall construct a formal "free virtual quandle" with generators a_i. First, we consider all admissible words with respect to the operations $\circ, /, f, f^{-1}$, and then take its quotient modulo the virtual quandle relations. To make the definition complete, we should also include the relations $f(a\alpha b) = f(a)\alpha f(b)$ and $f(f^{-1}(a)) = f^{-1}(f(a)) = a$ into the list of elementary equivalences; here a and b are arbitrary admissible words obtained from generators by applying $\circ, /, f, f^{-1}$, and α is either $\circ$ or $/$.

Then we take the quotient by the relations:

$$a_{j_2} = f(a_{j_1}) \tag{3.2}$$

and

$$a_{j_3} = f(a_{j_4}). \tag{3.3}$$

Denote the resulting quandle by $\mathrm{V}\Gamma_f\langle a_\alpha | R_\beta\rangle$.

Now let us define the virtual quandle $\mathrm{V}\Gamma(K)$ as

$$\mathrm{V}\Gamma_f\langle a_1, \ldots, a_n | R_i, R_{j1}, R_{j2}\rangle.$$

Here the relations R_i correspond to the classical crossings with numbers i (relations of type (3.1)), and pairs of relations R_{j1}, R_{j2} correspond to virtual crossings (for every virtual crossing with the number j we have two relations of types (3.2), (3.3)).

The theorem given below was first proved in [208], see also [206].

Theorem 3.4. *The virtual quandle* $\mathrm{V}\Gamma(K)$ *is an invariant of virtual links. More precisely, let* K, K' *be two equivalent virtual diagrams. Then there exists an isomorphism between virtual quandles* $\iota\colon \mathrm{V}\Gamma(K) \to \mathrm{V}\Gamma(K')$, *compatible with the operation* f.

Proof. We shall assume that all diagrams are proper (Statement 3.1).

We have to show that when applying Reidemeister moves to a virtual diagram, its virtual quandle is transformed to some isomorphic virtual quandle.

Consider a classical, virtual or semivirtual Reidemeister move. Let K and K' be two virtual diagrams obtained from one another by applying this Reidemeister move. This move is local, i.e. it changes the diagram only in a small domain U of the plane. Let us split all arcs of the diagrams K and K' into the following three families: The common set E of exterior arcs belonging to both the diagram K and the diagram K'; the common set S of arcs intersecting the boundary circle of the disc U, and the sets I and I' of interior arcs (lying inside U) belonging to K and K', respectively. Thus, the virtual quandle $\mathrm{V}\Gamma(K)$ has the following relations.

First, we have common relations (for all virtual quandles): distributivity and idempotence. Denote this family of relations by ε.

The relations of the types (3.1), (3.2), (3.3), which originate from crossings, for diagrams K, K' are also divided into sets: the exterior relations R_E are common for K and K', but the interior relations R_I and R'_I are related to K and K', respectively. Moreover, each of these quandles has the relation ε which is common for all virtual quandles (idempotence and right self-distributivity). Further in the proof by a relation we mean a relation only at some crossing (but neither idempotence nor right self-distributivity).

It is easy to see that for each concrete Reidemeister move we can get rid of the generators I with the help of the relations R_I by expressing the generators I from S. This will lead us to an addition of some "interior" relations R_S for S. The same thing can be done for I'. Let us denote the obtained relations by R'_S. Thus, we shall transform two virtual quandles $\mathrm{V}\Gamma(K)$ and $\mathrm{V}\Gamma(K')$ to the two virtual quandles $\widetilde{\mathrm{V}\Gamma}(K)$ and $\widetilde{\mathrm{V}\Gamma}(K')$ isomorphic to $\mathrm{V}\Gamma(K)$ and $\mathrm{V}\Gamma(K')$, respectively. The quandles $\widetilde{\mathrm{V}\Gamma}(K)$ and $\widetilde{\mathrm{V}\Gamma}(K')$ are generated by the generators E, S. They have the common set of interior relations R_E (and also ε).

The only fact that we have to prove is that the relations R_S and R'_S (with the relations ε) define the same equivalence relation on S.

Let us do it for some Reidemeister moves; the other cases are fully analogous to those which will be described.

The invariance of the quandle $\mathrm{V}\Gamma$ under all classical moves can be checked in the same way as that of a "general" quandle, see [144, 242]. For the first classical Reidemeister move, we have two arcs (denote them by a' and a'') instead of one arc a (and the relation $a = a$). From the relation at the crossing it follows that $a'' = a'$ (here we applied the idempotence of $\circ$). Therefore, by adding a loop as shown in Fig. 3.8 we shall get the virtual quandle isomorphic to the initial one.

Definition 3.2. Here, by an *isomorphism* of virtual quandles (long virtual quandles) we mean an isomorphism preserving the operation f.

The remaining cases of the first Reidemeister move are considered analogously to the preceding case.

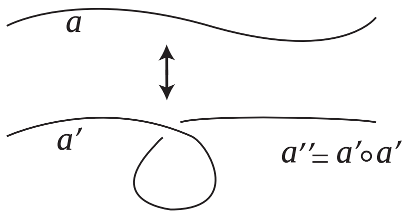


Fig. 3.8 The invariance of the quandle VΓ under Ω_1.

For the second Reidemeister move, we have four "exterior arcs" from the set E, see Fig. 3.9, namely, a, b, c, d. In the upper-left figure the relations look like $a = c$, $b = d$. In the lower-right picture the relations are the following. We have $a = c = e$, i.e. e is expressed from exterior generators. Further $b \circ a = g$, $d \circ e = g$, from this we get $b = d$ (here we use the existence of a left inverse for $\circ$).

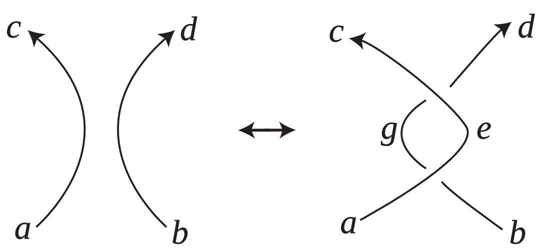


Fig. 3.9 The invariance of the quandle VΓ under Ω_2.

Analogously, for the third Reidemeister move, see Fig. 3.10, we have three incoming arcs a, b, c and three emanating arcs d, e, g. We shall not write down intermediate arcs; for the left picture we have $a = d$, $b \circ a = g$, $(c \circ a) \circ (b \circ a) = e$, when for the right one we get $a = d$, $b \circ a = g$, $(c \circ b) \circ a =$

e.

The invariance follows from the distributivity relation $(c \circ b) \circ a = (c \circ a) \circ (b \circ a)$.

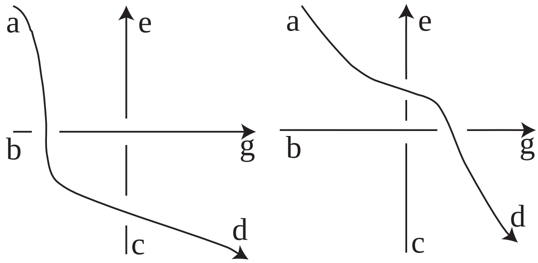


Fig. 3.10 The invariance of the quandle VΓ under Ω_3.

Let us now check the invariance of the virtual quandle VΓ under purely virtual Reidemeister moves.

The first virtual Reidemeister move is shown in Fig. 3.11. In the initial left picture we have one local generator a. Here, we just add a new generator b and two coinciding relations $a = f(b)$ and $b = f^{-1}(a)$. The new generator is expressed from given generators (namely, from the generator a); we do not impose any relations on the old generators; thus, it does not change the virtual quandle at all.

The case of the inverse orientation at the crossing gives us $b = f(a)$, that does not change the situation.

For each of the following relations, we shall check only one case of the arc orientation.

The second virtual Reidemeister move (see Fig. 3.12) adds two generators c and d and two pairs of coinciding relations $c = f(a)$, $d = f^{-1}(b)$. Thus, the virtual quandle VΓ is turned into the isomorphic virtual quandle.

In the case of the third virtual Reidemeister move, we have six "exterior arcs": three incoming (a, b, c) and three emanating (p, q, r), see Fig. 3.13. In both cases we have $p = f^2(a)$, $q = b$, $r = f^{-2}(c)$. The three interior arcs are expressed from a, b, c and give no other relations.

Let us consider the move Ω_3'. We shall check the only version of it, see Fig. 3.14.

In both pictures we have three incoming edges a, b, c and three outgoing edges p, q, r. In the first case we have the following relations: $p = f(a)$, $q = b$, $r = f^{-1}(c) \circ a$. In the second case, we have the following relations:

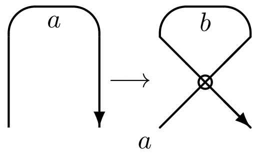


Fig. 3.11 The invariance of the quandle VΓ under the first virtual Reidemeister move.

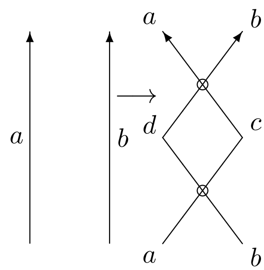


Fig. 3.12 The invariance of the quandle VΓ under the second virtual Reidemeister move.

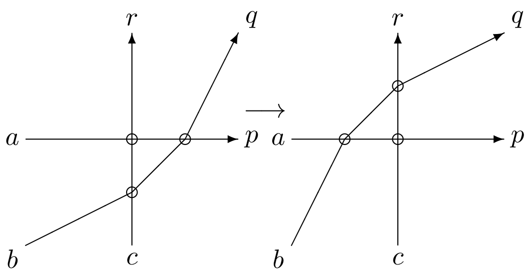


Fig. 3.13 The invariance of the quandle VΓ under the third virtual Reidemeister move.

$p = f(a)$, $q = b$, $r = f^{-1}(c \circ f(a))$.

The distributivity relation $f(x \circ y) = f(x) \circ f(y)$ implies $f^{-1}(c) \circ a = f^{-1}(c \circ f(a))$. Hence, two virtual quandles before the mixed move and after the mixed move coincide.

The other cases of the mixed move lead to other relations that are equivalent to $f(x \circ y) = f(x) \circ f(y)$.

This complete the proof of the theorem. □

Like quandles, virtual quandles are objects which are very difficult to work with. Namely, if we have two virtual diagrams K and K', and we know

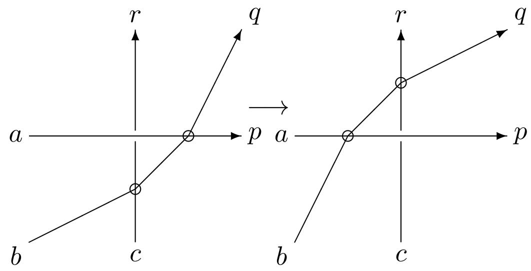


Fig. 3.14 The invariance of the quandle VΓ under the mixed move.

some presentations of their quandles (virtual quandles), then we cannot say immediately whether two corresponding quandles (virtual quandles) are isomorphic or not. The problem of recognizing whether there exists an isomorphism between two quandles given by presentations is apparently not easier than the problem of recognition of isomorphism between groups, which is, in general, not algorithmically solvable.

Therefore, it makes sense to find more simple invariants which are realized by morphisms in the category of quandles from some more simple category.

Simply speaking, we consider any category (say, the category of groups or modules) and try to find operations in it, which possess the axioms of a quandle (a virtual quandle). Having found them, we can construct more simple invariant objects, the invariance of which follows from categorical considerations.

Let us bring some examples of virtual quandles.

(1) A group with a distinguished element x is a quandle with respect to an operation $\circ$ given by $a \circ b = b^n a b^{-n}$ for some fixed natural number n, where $f(a) = xax^{-1}$.
(2) A group with a distinguished element x is a quandle with respect to the operation $\circ$ given by $a \circ b = ba^{-1}b$, $f(a) = xax^{-1}$; and also we can define $f(a)$ by $xa^{-1}x$ (with the same operation $\circ$).
(3) A module over the commutative ring $\mathbb{Q}[t, t^{-1}, s, s^{-1}]$, in which $a \circ b = ta + (1-t)b$, $f(a) = sa$, is a virtual quandle.

In these cases the axioms of virtual quandles (and, therefore, quandles) can be checked immediately.

This allows one to construct invariants of classical and virtual knots (links) in two ways: The first one is to fix a given quandle (virtual quandle)

Γ and to investigate properties of the set of morphisms from the (virtual) quandle of the link to Γ. In the case when Γ has a finite number of elements it makes sense to say that the number of such morphisms is an invariant.

On the other hand, we can associate with each knot (link) an invariant quandle (a virtual quandle) in any category. For example, in the category of groups a quandle is represented by the fundamental group, i.e. writing the relations of a quandle of a knot in the language of groups, where instead of $a \circ b$ we take bab^{-1}, we get the fundamental group of the knot.

The virtual quandle leads to the following generalization of the fundamental group of the complement to a knot.

Lemma 3.1 ([208]). *For each group G, the group $G * \{q\}$ (the free product of the group G with the infinite cyclic group generated by q) is a virtual quandle with respect to the operations $\circ$, $f(\cdot)$ defined as $a \circ b = bab^{-1}$, $f(c) = qcq^{-1}$ for all a, b, $c \in G * \{q\}$.*

Proof. Like the invertibility of the conjugation, the invertibility of the operation f is obvious. Indeed, we have to show that $f(a \circ b) = f(a) \circ f(b)$. Actually, $f(a \circ b) = f(bab^{-1}) = qbab^{-1}q^{-1} = qbq^{-1}(qaq^{-1})qb^{-1}q^{-1} = f(b)f(a)f(b^{-1}) = f(a) \circ f(b)$. □

Let us construct the *virtual group $G(K)$ of a virtual quandle* of a virtual diagram K. Denote all arcs of the diagram K by a_i, $i = 1, \ldots, n$. So, the group $G(K)$ is the group generated by $a_1, \ldots, a_n, q$ with the relations obtained from (3.1)–(3.3) by putting $f(x) = qxq^{-1}$, $y \circ z = zyz^{-1}$. Let us call the quotient group of a virtual group over the relation $q = 1$ a *classical* group.

In the case of a classical diagram, we get the Wirtinger presentation with one generator q (which does not take part in the relations); thus, the group described above will be the free product of the knot group with the infinite cyclic group generated by q.

Some virtual links (and their invariants) can possess properties which do not exist in the classical case. For example, in both the classical and virtual cases we can define upper and lower groups (in the classical case we have the fundamental group of the complement); the lower group for a virtual knot is defined as the group of the mirror image of the virtual knot. For classical knots these groups are isomorphic under geometric reasons. In the virtual case the upper and lower representations can give non-isomorphic classical groups. In the examples given below we give a group without an additional generator q. The first example of such a group was given in [114]. It has the following description.

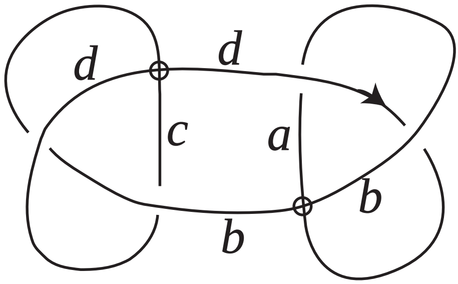


Fig. 3.15 A virtual knot with the non-isomorphic upper and lower groups.

Let us consider a virtual knot diagram shown in Fig. 3.15. Its four long arcs a, b, c, d shown in Fig. 3.15 can be chosen as generators of the group. Here a long arc goes from one undercrossing to the next undercrossing, and forms overcrossings and virtual crossings during the way. We shall get the following relations:

$$b = dad^{-1}, \qquad a = bdb^{-1}, \qquad d = bcb^{-1}, \qquad c = dbd^{-1}.$$

Thus, the generators a and c can be expressed from the generators b and d. Therefore, we get the presentation $\langle b, d \,|\, bdb = dbd \rangle$, which represents the group isomorphic to the group of the trefoil.

It is not difficult to note that the group of the mirror image of the given knot is isomorphic to the group $\mathbb{Z}$, the group of the unknot.

From the theorem of Dehn–Papakyriakopoulos [265] it follows that among non-trivial classical knots there does not exist a knot having the fundamental group of the complement isomorphic to $\mathbb{Z}$, therefore, our knot is not classical.

Theorem 3.5 ([208]). *The pair (the group $G(K)$, the element $q \in G(K)$) is an invariant of a virtual link. In other words, if diagrams K and K' give equivalent virtual links, then there exists an isomorphism $h\colon G(K) \to G(K')$ such that $h(q_K) = q_{K'}$.*

The proof of the lemma follows immediately from Theorem 3.4 and Lemma 3.1.

It is evident that for the unknot the group is $\langle a, q\rangle$; this is the free group with two generators.

From the discussion given above we get non-triviality of the virtual trefoil: Its virtual quandle is not isomorphic to the virtual quandle of the unknot by means of an isomorphism preserving the element q.

There exist other presentations of a virtual quandle in groups. For example, the presentation

$$x \circ y = y^p x y^{-p}, \qquad f(x) = qxq^{-1},$$

where p is a fixed integer number, and q is a fixed element. The proof of the invariance is the same as in the general case, only instead of abstract operations $\circ$, f we have to use their explicit group realizations.

Moreover, for obtaining invariants one can use one of the following presentations:

$$x \circ y = yx^{-1}y, \qquad f(x) = qxq^{-1},$$

or

$$x \circ y = yx^{-1}y, \qquad f(x) = qx^{-1}q.$$

Remark 3.5. Let us consider a group with generators corresponding to arcs of a classical knot and relations at crossings. The relations at crossings are obtained from the relations of a quandle by the rule $x \circ y \mapsto yx^{-1}y$. It is known that this group is isomorphic to the fundamental group of the two-sheeted covering branching over the knot.

3.2.4 *The coloring invariant*

The idea of the coloring invariant (by elements of some virtual quandle) is very simple. The coloring invariant equals the number of homomorphisms of the virtual quandle of a given link to a finite virtual quandle. More precisely, the following lemma takes place.

Lemma 3.2. *Let* $V\Gamma'$ *be a virtual quandle. Then the set of homomorphisms* $V\Gamma(K) \to V\Gamma'$ *is an invariant of the link represented by a diagram* K*. In particular, if the set of elements of the virtual quandle* $V\Gamma'$ *is finite, then the number of homomorphisms* $V\Gamma(K) \to V\Gamma'$ *is also finite and this number is an invariant under the generalized Reidemeister moves.*

Proof. The first claim is obvious by construction.

Let us prove the second claim. Consider a virtual link and its proper diagram K. In order to construct a homomorphism $h\colon V\Gamma(K) \to V\Gamma'$, we only have to define the images $h(a_i)$ of those elements of $V\Gamma(K)$ which correspond to the arcs of K. Since the number of arcs is finite, the desired number of homomorphisms is finite. □

Thus, for a finite virtual quandle $V\Gamma'$ we have an integer-valued invariant of a link, defined as the number of homomorphisms $V\Gamma(K) \to V\Gamma'$. The sense of this invariant is very simple. It represents the coloring number of arcs of K with elements of the virtual quandle $V\Gamma'$. A coloring is called *proper* if it satisfies the virtual quandle condition at classical and virtual crossings.

Here is the example of a virtual quandle with four elements. Define the operation $\circ$ by the following rule (in the intersection of the row a_i and the column a_j the element $a_i \circ a_j$ stands):

$\circ$	a_1	a_2	a_3	a_4
a_1	a_1	a_3	a_4	a_2
a_2	a_4	a_2	a_1	a_3
a_3	a_2	a_4	a_3	a_1
a_4	a_3	a_1	a_2	a_4

It is an important question: Where can we find (finite) virtual quandles? Let us generalize the constructions of quandles proposed in [242] for the case of virtual quandles. Let us consider some examples.

Let G be a finite group, $g \in G$ be a fixed element of it, and n be an integer. Then the set of all elements of G equipped with the operations $x \circ y = y^n x y^{-n}$, $f(x) = gxg^{-1}$ is a virtual quandle.

Another way for constructing virtual quandles by using groups is as follows: For a (finite) group G with a fixed element $g \in G$ we set $x \circ y = yx^{-1}y$, $f(x) = gxg^{-1}$.

Let R be a commutative ring with a unit, and let t, $s \in R$ be invertible elements.

The following claim can be checked immediately.

Theorem 3.6 ([208]**).** *The ring R equipped with operations $\circ$, $/$, f, f^{-1}, defined by rules $x \circ y = tx + (1-t)y$, $f(x) = sx$ is a virtual quandle.*

These examples give two series of integer-valued virtual link invariants.

The list of quandles of small capacity is given in the work by Carter, Kamada and Saito [51]. In this book, we also describe the theory of quandle homology which has an application in both virtual knot theory and theories of higher dimensions (for example, for constructing invariants of two-dimensional knots in $\mathbb{R}^4$). These knots are given by their projections on $\mathbb{R}^3$ in $\mathbb{R}^4$, herewith one indicates which sheet is upper and which one is lower in each intersecting line. It turns out that by using such projections one can construct invariants of quandle type: associating elements of the quandle with sheets and writing down relations in intersections of sheets. We have the same relations: The existence of an inverse element, the idempotence, the self-distributivity, in order that the constructed object is invariant.

3.2.5 *The Alexander virtual module and its generalization*

The Alexander polynomial [5] is the oldest invariant among polynomial invariants of classical knots. It can be defined by using the fundamental group of the complement to the knot or the free Fox calculus, see, e.g. [66]. There are other ways to define the polynomial, see, e.g. [316].

We shall define the Alexander polynomial by using the *Alexander module*.

The *Alexander module* (virtual or classical) of a diagram is a module over $\mathbb{Z}[t]$ generated by all long arcs of the diagram (these arcs coincide with arcs in the classical case) and the relation (3.1) at classical crossings as it is shown in Fig. 3.4, where the operation $a \circ b$ means $ta + (1 - t)b$.

These relations can be written by using the rows of square matrices which correspond to crossings and columns correspond to (long) arc of the diagram.

Let K be an oriented diagram of a link with n classical crossings. Let us construct the *Alexander matrix* $M(K)$ corresponding to K in the following manner. (We shall return to such matrices when we consider virtual knots.)

Let us enumerate all classical crossings of the diagram by natural numbers from 1 to n. In the general position, there exists precisely one long arc emanating from each crossing (unless there are separated cyclic arcs). Let us recall that we consider only proper diagrams. So, we can enumerate outgoing long arcs by natural numbers from 1 to n by associating a number i with the long arc emanating from the crossing with the number i. Now we can construct the adjacency matrix, where a crossing corresponds to a row, and an arc corresponds to a column.

Suppose that no crossing is incident twice to one and the same arc (no loops). Then, each crossing (number i) is incident precisely to three arcs: the arc passing through this crossing (number j), the incoming arc (number k) and the outgoing arc (number i), see Fig. 3.16.

In this case, the ith row of the Alexander matrix consists of the three elements at places i, j, k. If the ith crossing is positive (), then $m_{ii} = 1$, $m_{ik} = -t$, $m_{ij} = t-1$. Otherwise, we set $m_{ii} = t$, $m_{ik} = -1$, $m_{ij} = 1-t$.

Obviously, this matrix has determinant zero, because the sum of elements in each row equals zero. Therefore, we have constructed the *classical*

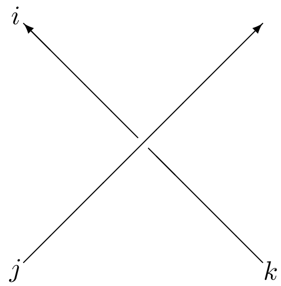


Fig. 3.16 Three arcs incident to a crossing.

Alexander matrix, the relations in it give the *classical Alexander module*; the generators of the latter correspond to long arcs of the given virtual diagram.

Denote the cofactor to m_{ij} by Δ_{ij}.

Then the following theorem holds (see, e.g. [66]).

Theorem 3.7. *All Δ_{ij} coincide with each other up to multiplication by $\pm t^l$.*

Thus, we have an element of the ring $\mathbb{Z}[t^{\pm 1}]$ defined up to multiplication by a power of the variable t. Denote this element by Δ.

It is known that the polynomial Δ can be normalized with respect to the variable t in such a way that the leading power and the lowest power differ only by a sign. Moreover, by a natural way one can define the sign such that we have an invariant polynomial of oriented links. Therefore, we get the *Alexander polynomial* which is also denoted by Δ.

The following theorem is well known (a proof can be found in [221]).

Theorem 3.8. *The function Δ defined on links satisfies the following skein relation*:

$\Delta($$) - \Delta($$) = (t^{1/2} - t^{-1/2})\Delta($$)$.

It is easy to check that for the unknot ◯ we have $\Delta(◯) = 1$.

Thus, we can conclude that the polynomial Δ coincides with the Conway polynomial [63] up to the variable change $x = t^{1/2} - t^{-1/2}$. So, it is a well-defined link invariant.

Remark 3.6. This way to derive Alexander-like polynomials from groups can be found in, e.g. [65, 66].

There are many ways of generalizing the classical Alexander module for the virtual case, but we consider only two ways. One of them (an additive way) is as follows. One adds a new generator ε in the module itself, after which the operation f looks like $x \mapsto x + \varepsilon$. The second way is a multiplicative way. Here we add a new variable to the main ring (in the case of links with many components we add new variables the number of which is equal to the number of components of the link), and the operation f is defined as the multiplication by one of these new variables. It is natural that in the case of links with many components this is a modification of the virtual quandle.

3.2.5.1 *An additive approach*

Let R be the ring of Laurent polynomials in the variable t. Sometimes it will be convenient for us to consider R to be a ring over either $\mathbb{Z}$ or $\mathbb{Q}$. This will be mentioned individually. We shall associate diagrams of virtual links with modules over the ring R, with a fixed generator ε of the module, chosen among generators of the module except of elements corresponding to arcs of the knot. This element plays the role of a distinguished additional generator. More precisely, let us consider a virtual diagram K. Denote these arcs by $a_1, \dots, a_n$. Let us construct a module $\mathcal{M}$ over the ring R generated by arc-generators $a_1, \dots, a_n$ and the generator ε. This module is defined as a virtual quandle with operations:

$$\begin{cases} x \circ y = tx + (1-t)y, \\ f(x) = x + \varepsilon. \end{cases}$$

These operations define relations in the module $\mathcal{M}$.

Thus, we assign to each virtual diagram K the pair $(\mathcal{M}(K), \varepsilon)$; the module and a fixed element in it.

From the above consideration, we have the following theorem.

Theorem 3.9. *For two equivalent diagrams K and K' there exists an isomorphism of the modules $\mathcal{M}(K) \to \mathcal{M}(K')$ sending the element $\varepsilon(K)$ to $\varepsilon(K')$.*

This module allows one to get more simple (polynomial) invariants of links. For example, one can consider the ideal $I(K)$ in the ring R which annihilates the element ε from the module $\mathcal{M}(K)$. From Theorem 3.9 it follows that the ideal $I(K)$ is also an invariant of links.

The case when the ideal $I(K)$ is a *principal* ideal is important. Let us consider the case of the Euclidean ring $R = \mathbb{Q}[t, t^{-1}]$. In this case, this ideal is generated by some polynomial which, considered up to invertible elements of the ring R, is an invariant of the link. Denote this polynomial by VA and call it the VA-*polynomial*.

From the construction we have the following theorem.

Theorem 3.10. *The* VA-*polynomial is a virtual link invariant up to multiplication by* λt^q, $q \in \mathbb{Z}$, $\lambda \in \mathbb{Q}$.

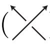
Let us pass to an explicit description of how to calculate the VA-polynomial, see also [284].

Let us consider a (proper) virtual diagram K with n classical crossings. Let us enumerate all classical crossings by natural numbers from 1 to n, where we associate each long arc with the number of that classical crossing from which it emanates.

Each long arc l begins from some arc $s(l)$. Denote the element of the module $\mathcal{M}(K)$ corresponding to the beginning arc $s(k)$ of the long arc with a number k by a_k. Then all remaining arcs of the long arc with the number k are represented in the module $\mathcal{M}(K)$ by elements $a_k + p_{ki}\varepsilon$, where p_{ki} are some integers. Therefore, the elements $\{a_1, \ldots, a_n, \varepsilon\}$ form a set of generators of the module $\mathcal{M}(K)$. In order to define the module $\mathcal{M}(K)$ we have to describe relations in classical crossings. At a classical crossing with a number i there are three meeting long arcs: The long arc with the number i emanates from it, the long arc with a number j passes through it, and the long arc with a number k comes in this crossing, see Fig. 3.16.

Moreover, three arcs which are parts of the long arcs with the numbers i, j, k are incident to the crossing. By construction, the elements a_i, $a_j + p_{ji}\varepsilon$, $a_k + p_{ki}\varepsilon$ of the quandle correspond to them. The relation at the crossing looks like

$$-a_i + t^{\pm 1}a_k + (1 - t^{\pm 1})a_j = ((t^{\pm 1} - 1)p_{ji} - t^{\pm 1}p_{ki})\varepsilon,$$

where, we have the sign $+1$ for positive crossing (), and the sign -1 for negative crossing ().

Here we have all summands containing ε on the right side.

Let us write down the relations for all classical crossings in the same manner. We get a square matrix of a non-homogeneous system of equations

$$(M) \cdot (a) = (b) \cdot \varepsilon. \tag{3.4}$$

This matrix M is called the *Alexander matrix* of the diagram K, and (a) is a column consisting of $a_1, \ldots, a_n$.

The Alexander matrix is defined up to a permutation of rows and the permutation of columns corresponding to the first permutation.

The Alexander matrix is degenerate since the sum of elements in each of its row equals zero. This means that there exists a linear combination of its rows equal to zero. From Eq. (3.4) it follows that taking the same linear combinations of elements b_i and multiplying it by ε we shall get zero. Such linear combinations form the ideal I. In the case when the ideal I is trivial, we set the VA-polynomial to equal zero.

Theorem 3.11. *For a classical link the Alexander module is split into direct sum of two summands, one of which is the module generated by the element ε. Therefore, we have* $\mathrm{VA}(K) = 0$ *for a diagram K of a classical link.*

Proof. Let us consider a diagram of the link not having virtual crossings, and write down the relation of the Alexander module for it. The element ε does not take part in them. From this fact it follows two claims of the theorem. □

Example 3.1. Consider the virtual trefoil knot. Let us calculate its virtual quandle. In Fig. 3.17, rightmost picture, we have two classical vertices: I and II. They give us a system of two relations:

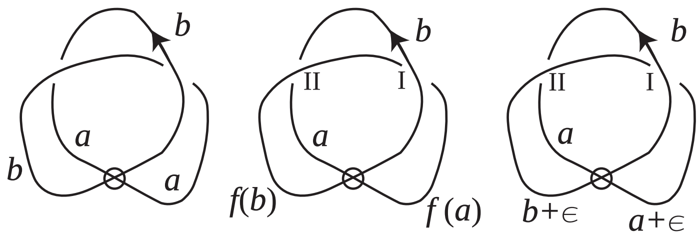


Fig. 3.17 Labeled virtual trefoil.

$$I : (a + \varepsilon)t + b(1 - t) = b + \varepsilon,$$
$$II : bt + (b + \varepsilon)(1 - t) = a,$$

or

$$at - bt = (1 - t)\varepsilon,$$
$$b - a = (t - 1)\varepsilon.$$

Multiplying the second equation by t and adding it to the first one, we get: $0 = \varepsilon(t-1)(1-t)$. It is not difficult to show that there are no relations having the form $\alpha\varepsilon = 0$ in the module. Thus, $\mathrm{VA}(K) = (1-t)^2$.

The following theorem holds.

Theorem 3.12 **([216]).** *For each virtual knot (not link) given by a diagram K, the polynomial* $\mathrm{VA}(K)$ *is divisible by* $(t-1)^2$.

Proof. Let K be a virtual knot diagram. If all crossings of the diagram are virtual, then K represents the unknot and has the VA-polynomial equal to zero.

Otherwise, choose a classical crossing v_1 of the diagram K. Let a_1 be the first arc of the long arc outgoing from v_1 (according to the knot orientation). This arc is incident to v_1. By construction, all arcs belonging to the long arc are associated with $a_1 + k\varepsilon$, $k \in \mathbb{N}$. Let the last arc of the long arc be marked by $a_1 + k_1\varepsilon$. Denote the final point of it by v_2. Now, let us take the first arc outgoing from v_2 and associate $a_2 + k_1\varepsilon$ to it, i.e. we can define a new generator a_2' "by rule" and then we can make a substitution $a_2' = a_2 + k_1\varepsilon$. Then, we set the labels $a_2 + k\varepsilon$ for all arcs belonging to the same long arc. Let the last arc have the label $a_2 + k_2\varepsilon$ and have the final point at v_3. Then, we associate $a_3 + k_2\varepsilon$ with the first arc outgoing from v_3, and so on, see Fig. 3.18.

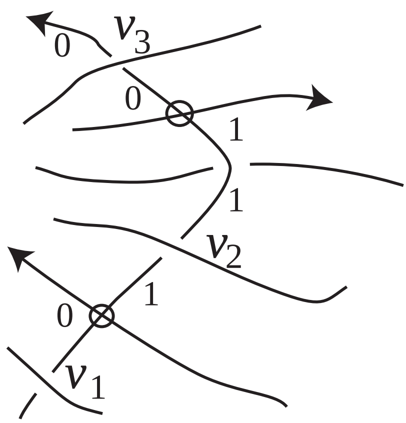


Fig. 3.18 Marks.

In what follows by mark we mean the coefficient at ε.

Finally, we shall come to v_1. Let us show that the process converges, i.e. the label of the arc coming in v_1 has the label $a_{j+1} + 0 \cdot \varepsilon$, where j is the total number of long arcs.

Indeed, let us see the ε-part of labels while walking along the diagram from v_1 to v_1 according to the orientation. In the very beginning, it is equal to zero by construction. Then, while passing through each virtual crossing, it is increased (or decreased) by one. But each virtual crossing is passed twice, thus each $+\varepsilon$ is compensated by $-\varepsilon$ and vice versa. Thus, finally we come to v_1 with $0 \cdot \varepsilon$.

Note that the process converges if we do the same, starting from any arc with arbitrary integer number as a label.

In this case, each relation of the virtual Alexander module has the right part divisible by $\varepsilon(t-1)$: the relation

$$(a_i + p\varepsilon)t + (a_j + q\varepsilon)(1-t) = (a_k + p\varepsilon)$$

is equivalent to

$$a_i t + a_j(1-t) - a_k = (t-1)(q-p)\varepsilon. \tag{3.5}$$

This proves that $\mathrm{VA}(K)$ is divisible by $(t-1)$.

Denote the summands for the ith vertex in the right of (3.5) by q_i and p_i, respectively.

Let us seek relations on rows of the virtual Alexander matrix. Each relation holds for arbitrary t, hence for $t = 1$.

Denote rows of M by M_i. So, if for the matrix M we have $\sum_{i=1}^n c_i M_i = 0$ then $\sum_{i=1}^n c_i|_{t=1} M_i|_{t=1} = 0$.

The matrix $M(K)|_{t=1}$ is very simple. Each row of it (like each column of it) consists of 1 and -1 and zeros. The relation for rows of this matrix is obvious: One should just take the sum of these rows that is equal to zero. So, for all i, $j = 1, \ldots, n$ we have $c_i|_{t=1} = c_j|_{t=1}$.

Each relation for ε looks like

$$\left(\sum_{i=1}^n c_i(q_i - p_i)\right)(t-1) = 0.$$

Since we are interested in whether this expression is divisible by $(t-1)^2$, we can easily replace c_i by $c_i|_{t=1}$. Thus, it remains to prove that $\sum_{i=1}^n (q_i - p_i) = 0$ for the given diagram K. Let us prove it by induction on the number n of classical crossings.

For $n = 0$, there is nothing to prove.

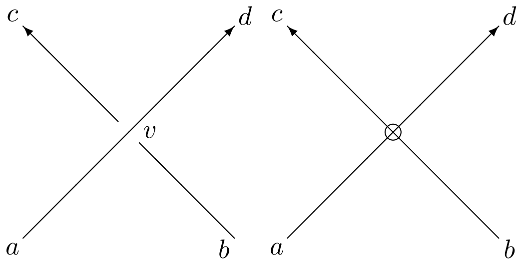


Fig. 3.19 Labels of K and K'.

Now, let K be a diagram with n classical crossings, and K' be a diagram obtained from K by replacing a classical crossing by a virtual one.

Consider the case of the positive classical crossing v (the "negative" case is completely analogous to this one), see Fig. 3.19.

Denote the lower-left arc of both diagrams by a, and other arcs by b and c, d (for K we have $a = d$), see Fig. 3.19. Assign the label 0 to the arc a of both diagrams. Let us calculate $\sum(q_i - p_i)$ for K' and K. By the induction hypothesis, for K' this sum equals 0.

Denote the label of b for the first diagram by l_{b1} and that for the second diagram by l_{b2}.

The crossing v of K has $q = l_{b1}$, $p = 0$, thus, its impact is equal to l_{b1}.

The other crossings of K (classical or virtual) are in one-to-one correspondence with those of K'. Let us calculate what the difference between the p's and q's for these two diagrams is. The difference comes from classical crossings. Their labels differ only in the part of the diagram from d to b (according to the knot orientation). While walking from d to b, we encounter classical and virtual crossings. The total algebraic number is equal to zero. The algebraic number of virtual crossings equals $-q$. Thus the algebraic number of classical crossings equals q. Each of them impacts -1 to the difference between K and K'. Thus, we have $q - q = 0$, which completes the induction step and hence, the theorem. □

3.2.5.2 *Multiplicative approach*

Let R be a ring of Laurent polynomial in two commutative variables t, s (we shall consider this ring over $\mathbb{R}$ or $\mathbb{Q}$ depending on circumstances). Let K be a virtual diagram. Associate with this diagram the module $\mathcal{M}$ over the ring R as follows. Let $a_1, \ldots, a_n$ be arcs of K (we shall use the same letters for notations of generators of the module corresponding to arcs). Let us now consider a virtual quandle over the ring R generated by $a_1, \ldots, a_n$ and the relations (3.1)–(3.3) at classical and virtual crossings according to the following formulae:

$$\begin{cases} a \circ b = ta + (1-t)b, \\ f(a) = sa. \end{cases} \tag{3.6}$$

Let us denote the obtained module by $\mathcal{M}(L)$.

Theorem 3.13. *The obtained module is an invariant of virtual links.*

This theorem follows from the much stronger Theorem 3.14, the proof of which will be given later.

In the case of a virtual link with l enumerated components (a colored virtual link) the module over the ring $R[t_1^{\pm 1}, \ldots, t_l^{\pm 1}, s_1^{\pm 1}, \ldots, s_l^{\pm 1}]$ is constructed in the same manner. Let us describe this construction more precisely. Let K be a virtual link diagram with n arcs and the components of the link are enumerated by natural numbers from 1 to l. Associate with all arcs of the diagram elements $a_1, \ldots, a_n$. Let us modify the relations between neighboring arcs at classical/virtual crossings by replacing each appearing of the generator t by one of the generators t_i in the relations (3.6) and the generator s by one of s_i. More precisely, we follow the rule. With two arcs a_j, a_k separated by a long arc A belonging to a component i in a virtual crossing we associate the relation $a_j = s_i a_k$ in the case when the arc a_j is situated on the right with respect to the orientation of the long arc A, and $a_j = s_i^{-1} a_k$, if the arc a_j is situated on the left. At classical crossings we shall use the relation (3.1) viewed as $a \circ b = t_j a + (1 - t_i)b$, where i is the number of the component corresponding an arc of the overcrossing to which the relation is related, j is the number of the component corresponding two arcs of the undercrossing. Thus, we get the module $\mathcal{M}_l(K)$.

Theorem 3.14. *The module $\mathcal{M}_l(K)$ is an invariant of colored virtual links.*

In the case of links with unordered components, we can consider a collection of modules $\mathcal{M}_l(K)$ under all possible orders of the components. This collection of modules will obvious be an invariant.

It is evident that Theorem 3.14 follows from Theorem 3.13. Since in Theorem 3.14 there is a structure which differs slightly from the structure of a quandle, namely, the structure which a set of different operations depending on link components is defined on, we have to prove independently the invariance of the corresponding module $\mathcal{M}_l$.

Let us prove Theorem 3.14.

***Proof of Theorem* 3.14.** We follow the same principle as in the proof of Theorem 3.4. The only difference is that in the given case in Reidemeister moves different link components can take part. This is expressed by the presence of different variables s_i, t_i. If we set all variables s_i to be equal to one variable s, and all variables t_i to be equal to one variable t, then the proof of the theorem immediately follows from the proof of Theorem 3.4 and category considerations.

Thus, the proof is reduced to those cases when in a Reidemeister move we have different components of the link. We have classical, virtual and semivirtual moves.

Let us consider only the most difficult case: the third classical Reidemeister move and prove it carefully.

This proof repeats the proof (known in the classical case proof) of the invariance of the analogous module for classical links (without the generators s_i).

In our case we have labels: monomials in the variables s_i on all components. Without loss of generality we may assume that all of them are equal to one.

Let us consider two diagrams shown in Fig. 3.20.

The numbers of crossings in this figure are labeled by Roman digits, the numbers of arcs are labeled by Arabic digits, the numbers of components are circled, and the monomials corresponding to incoming arcs are marked by letters P, Q, R, see Fig. 3.20.

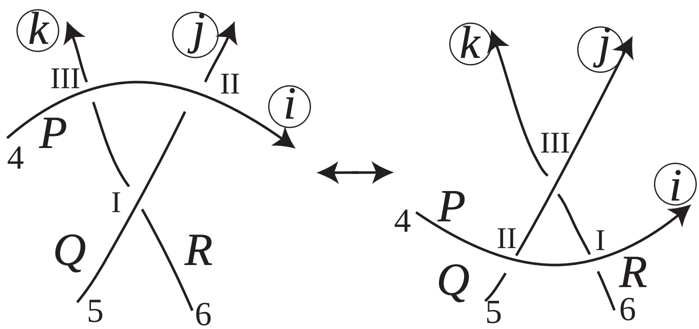


Fig. 3.20 Labels of arcs for Ω_3.

Let us consider the relation matrices corresponding to the diagram before applying the Reidemeister move and the diagram after applying the move. The matrix corresponding to the first diagram is:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & (t_k-1)Q & -t_jR & 0\dots0 \\ 0 & 1 & 0 & (t_j-1)P & -t_iQ & 0 & 0\dots0 \\ -t_i & 0 & 1 & (t_k-1)P & 0 & 0 & 0\dots0 \\ 0 & & & & & & \\ \vdots & & & W & & & \\ 0 & & & & & & \end{pmatrix}.$$

The matrix corresponding to the first diagram looks like:

$$\begin{pmatrix} 1 & 0 & 0 & (t_k-1)P & 0 & -t_iR & 0\dots0 \\ 0 & 1 & 0 & (t_j-1)P & -t_iQ & 0 & 0\dots0 \\ -t_j & (t_k-1) & 1 & 0 & 0 & 0 & 0\dots0 \\ 0 & & & & & & \\ \vdots & & & W & & & \\ 0 & & & & & & \end{pmatrix}.$$

We shall perform elementary manipulations over matrices; it is evident that these manipulations do not change the module itself.

The second rows of these matrices coincide. Let us consider their third rows. In the first matrix let us add the first row multiplied by t_i to the third row. In the second matrix let us add the first row multiplied by t_j and the second one multiplied by $(1-t_k)$ to the third row. We shall get two matrices having the same third rows. Their third row looks like

$$(0, 0, 1, (t_k-1)P, t_i(t_k-1)Q, -t_it_jR, 0, \dots, 0).$$

The first columns of the new two matrices coincide and have only one non-zero element, namely, 1 stands in the first place. Thus, by adding this column we can make the first rows of these matrices equal to each other. Therefore, the initial modules are isomorphic. □

Let us consider the following description of the module $\mathcal{M}_l(K)$. Let the diagram K have n classical crossings. Then the number of long arcs also equals n. Each long arc is divided into some number of arcs. Labels associated with these arcs are obtained from each other by multiplying by some monomials in variables s_i. Thus, we can do the following: First, we take into consideration all relations at virtual crossings, thereby we left one arc-generator for each long arc instead of all generators. Then we shall write down relations at classical crossings in which the remaining n generators take part. As a result, we have n generators and n relations which can be written in a matrix. The determinant of this matrix (up to invertible elements of the ring R) will be a link invariant. The fact of the matter is that Reidemeister moves do not change the module $\mathcal{M}_l(K)$ and do not change the system of relations defining the module. It can be expressed in the fact that each Reidemeister move leads the matrix to a matrix obtained from the initial one by elementary manipulations not changing the determinant or multiplying the determinant by some invertible element of the ring R.

Herewith we can achieve that under right normalization we get a link invariant up to multiplying by powers of variables t_i, but not s_i.

Let us describe this construction more precisely (see [207, 213]).

Let K be a proper link diagram with l components and n classical crossings representing $v_1, \ldots, v_n$. Let us associate with each component "variable-numbers" $i_1, \ldots, i_l$. We shall use variables $s_{i_1}, \ldots, s_{i_l}$ which by the substitution of indices give us the variables $s_1, \ldots, s_l$ in some order. Further, we substitute all possible permutation over l elements for $(i_1, \ldots, i_l)$, this allows us to have a collection of modules (and a collection of the corresponding polynomials) as an invariant of the link K. Let us enumerate all arcs of K and all crossings by numbers from 1 to n. Let us associate with each classical crossing the long arc outgoing from it.

We shall associate with arcs of K labels (monomials in variables $s_{i_1}^{\pm 1}, \ldots, s_{i_l}^{\pm 1}$) by the following rule. Let us choose some long arc a_i and fix its beginning arc, i.e. the arc outgoing from the crossing v_i. Let us associate the label 1 with the latter. All other arcs of the long arc will be marked by monomials as follows. While passing through the virtual crossing with i_jth component K_{i_j} of the link K we multiply the label by s_{i_j} when we pass from the left to the right or by $s_{i_j}^{-1}$ otherwise, see Fig. 3.21.

Hereby, we associated labels, monomials in $s_{i_1}^{\pm 1}, \ldots, s_{i_l}^{\pm 1}$, with all arcs of the diagram K.

Each classical crossing v_m with a number m is incident to exactly three arcs. One of them belongs to a long arc a_m with the number m and have the label equal to one by construction, the other arc belongs to the long arc a_j with a number j having the label $P(m)$, passing through the crossing v_m and lying on the i_jth component K_{i_j} of the link, the third one belongs to the long arc a_k with a number k incoming in the crossing v_m and lying on the i_mth component K_{i_m} of the link; the label of the arc equals $Q(m)$. Note that the arcs a_m, a_k belong to the one component denoted by K_{i_m}. Generally speaking, some of the numbers m, j, k can coincide. Let us now write down the *generalized Alexander matrix* $M(K)$ corresponding to K as

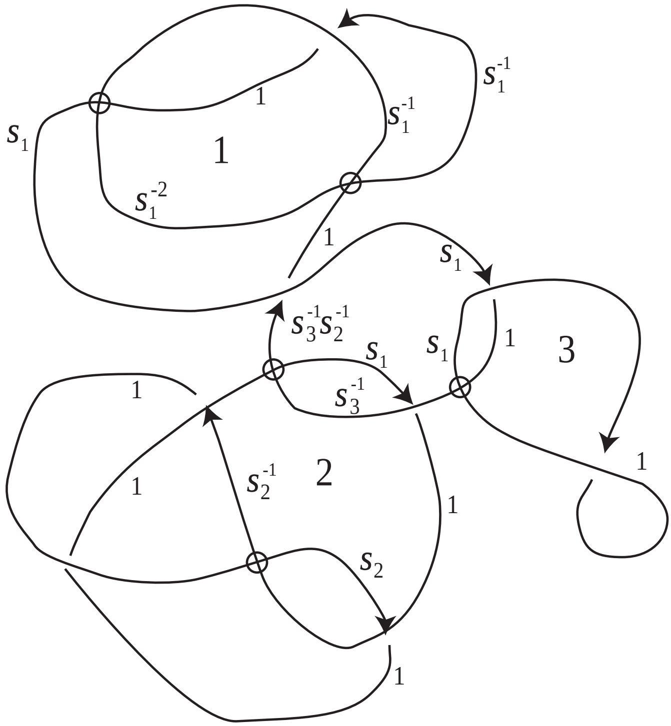


Fig. 3.21 Labeling of arcs for a virtual diagram.

follows. In each mth row at most three elements are non-zero; namely, all elements in this row except elements with the numbers m, j, k are equal to zero. This row is the sum of three rows such that one of them has exactly one non-zero element. One of them has label 1 at place m. The second one has element $-t_{i_j}^{\pm 1}Q(m)$ at place k, and the third one has element $(t_{i_m}^{\pm 1} - 1)P(m)$ at place j, herewith a sign ± 1 is equal to $+1$ if the crossing is positive, and -1 if the crossing is negative.

The matrix constructed by this way is the relation matrix of the module $\mathcal{M}_l(K)$ for a colored link (i.e. for a link with a fixed order of components).

Thus, we have constructed a collection of matrices $M(K)$ depending on a permutation $i_1, \ldots, i_l$ of components of the link K. Denote the collection of the determinants of these matrices by $\zeta(K)$.

Theorem 3.15. *The collection $\zeta(K)$ of the determinants of matrices obtained by performing all possible permutations of components of the link K*

is an invariant under the Reidemeister moves up to multiplying by monomials in variables $t_{i_1}, \ldots, t_{i_l}$.

If we fix a numeration of components of the link K, *then the obtained determinant* $\zeta(K)$ *is an invariant of a colored link up to multiplying by monomials in variables* $t_1, \ldots, t_l$.

Let us consider the case of colored links. It is evident that in the general case the collection of values of the invariant under all possible colorings of components of the link is an invariant of the link. The invariance of the determinant in the case of colored links up to multiplying by invertible elements of the given ring (i.e. monomials in $t_j^{\pm 1}$, $s_k^{\pm 1}$ with a sign ± 1) follows from the fact that the matrix is the relation matrix of the module which is invariant.

Remark 3.7. Here, by "natural" normalization we mean the normalization under which normalization factors are defined by the combinatorial of a link diagram; there is also a normalization depending on the form of the present polynomial; for example, a polynomial, which is invariant under multiplication by some monomial, can be normalized so that its leading monomial (say, under all variables in lexicographic order) is equal to a constant.

Theorem 3.15 follows from Lemma 3.3.

Lemma 3.3. *The determinant* $\zeta(K)$ *is an invariant with respect to all generalized Reidemeister moves except* Ω_1. *Applying the move* Ω_1, *it is multiplied by a monomial in variables* $t_1, \ldots, t_l, t_1^{-1}, \ldots, t_l^{-1}$ *with the sign* $+1$.

The proof of Lemma 3.3, in turn, follows from the proof of Theorem 3.14.

The polynomial ζ can be easily normalized with respect to variables $t_1, \ldots, t_l$ in order to get a virtual link invariant.

Theorem 3.16. *The polynomial* ζ *is multiplicative under a disconnected sum of links*: $\zeta(K_1 \sqcup K_2) = \zeta(K_1) \cdot \zeta(K_2)$.

Proof. Indeed, it follows from the fact that a disconnected sum of links is associated with a block-matrix the block of which are matrices corresponding to the initial links. □

Note that analogous structures appeared in works by Kauffman and Radford [174], Silver and Williams [280], and Sawollek [276]. However, in contrast to the approach of this book they used another approach: Instead of considering an additional structure at virtual crossings they used a more complicated structure at classical crossings. As a result, it turned out that the ζ-polynomial was equivalent to (up to a substitution of variables and a normalization) polynomials considered in [174, 276, 280], but this equivalence was not evident. It was proved by Fenn in [22].

In some cases different approaches to the same invariant allows one to find its different properties. For example, Sawollek [275] showed that his invariant recognized *the invertibility of virtual knots*, i.e. the non-equivalence of some oriented knots to its images obtained by reversing orientations.

Namely, for the knot K shown in Fig. 3.22 we have

$$\zeta(K) = \frac{(s^2-1)(t-1)\left(1+s-2st+s^2t(t-1)\right)}{s^2t}$$

under some orientation.

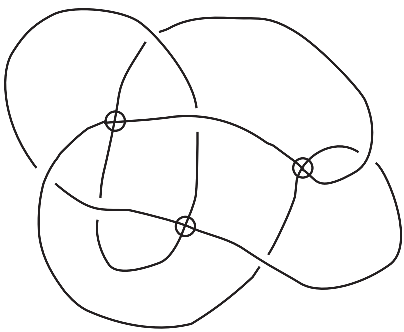


Fig. 3.22 Sawollek's example.

From the definition of the polynomial ζ, we have the following statement.

Statement 3.2. *If a knot K_2 is obtained from a knot K_1 by swapping the orientation, then for some i, j the equality $\zeta(K_1)(s,t) = s^i t^j \zeta(K_2)(s^{-1},t)$ holds.*

It is easy to see that the polynomial $\zeta(K)$ and the polynomial $\zeta(K')$ obtained from the first one by the substitution of s by s^{-1} cannot be obtained from each other by multiplying by some monomial $s^m t^l$. Therefore, the knot K is not invertible.

Moreover, in this work Sawollek showed that his polynomial could be reduced to Vassiliev invariants of virtual links (see the definition in Chap. 7), and thereby he showed that in the virtual case Vassiliev invariant could, generally speaking, distinguish knots which differed from each other by an orientation. In the classical case the question of whether there exists a Vassiliev invariant recognizing the invertibility of some knots is still open, see, e.g. [57, 73].

Note that in the case of virtual knots even Vassiliev invariants of order zero can recognize the invertibility. We shall discuss it in Chap. 7.

There are different generalizations of a quandle in which a short arcs take part, see [22, 44, 86, 90, 171].

3.3 Long virtual knots

This section contains one of the main results of the chapter. Namely, the construction of strong invariants of long virtual knots recognizing non-commutativity of long virtual knots, see [219, 221, 222].

Definition 3.3. By a *long virtual knot diagram* we mean a smooth immersion f of the oriented line $\mathbb{R}$ in the plane Oxy, such that:

(1) outside some big circle, we have $f(t) = (t, 0)$;
(2) each intersection point is double and transverse;
(3) each intersection point is endowed with a classical or virtual crossing structure.

All long virtual diagrams are supposed to be oriented according to the orientation of the line from left to right. In the depicting pictures we shall not point out the orientation.

A long virtual diagram has two distinguished *non-compact* arcs (respectively, long arcs, short arcs). *Non-compact arcs* are arcs containing "infinite places"; more precisely, an arc is called *non-compact* if it contains the image of a point $x \in \mathbb{R}$ such that the restriction of the map $\mathbb{R}^1 \to \mathbb{R}^2$ on intervals $(-\infty, -|x|]$ and $[|x|, \infty)$ is the identity embedding $\mathbb{R}^1 \to \mathbb{R}^1$.

Definition 3.4. A *long virtual knot* is an equivalence class of long virtual knot diagrams modulo generalized Reidemeister moves. A *long classical knot* is a long virtual knot having a diagram without virtual crossings. The *trivial long knot* is the long virtual knot having a diagram without crossings. This knot is unique, it has the diagram of two non-compact arcs which coincide.

Example 3.2. An example of a long virtual knot is shown in Fig. 3.23.

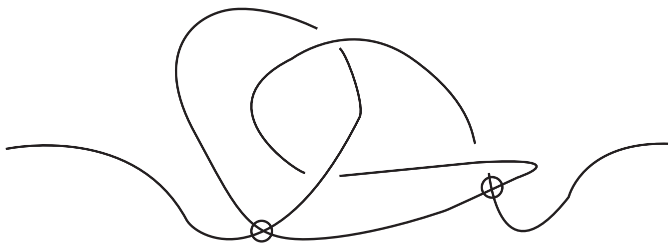


Fig. 3.23 A diagram of a long virtual knot.

Usual virtual knots (not long) are also called *compact virtual knots*.

There are two operations turning a long knot into a compact knot and vice versa. The first of them is called the *closure* of a long virtual knot, $K \mapsto \mathrm{Cl}(K)$, see Fig. 3.24.

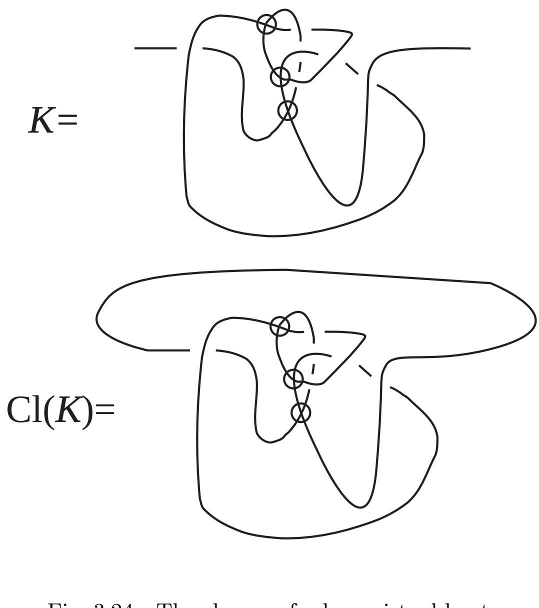


Fig. 3.24 The closure of a long virtual knot.

This operation is well defined and associates a long virtual knot to the compact virtual knot. This operation tears off two non-compact arcs in a "neighborhood of the infinity" and connects the new arcs. Herewith the obtained compact knot inherits the orientation of the long virtual knot.

The second operation, *breaking*, being the inverse operation to the first is less natural. We pick a point on a diagram of an oriented virtual knot K which is distinct from a crossing, break this diagram in the given point and pull out the ends into infinity in order to get a long virtual knot. In the case, if the ends can be pulled out into infinity so that no additional crossings appear, we shall do it in this manner. In the case if such crossings appear, all these crossings should be virtual crossings. Denote this operation by $\mathrm{Brk}\colon K \mapsto \mathrm{Brk}(K)$.

Figure 3.25 shows the procedure of breaking the diagram of a virtual knot in some points and getting long virtual diagrams.

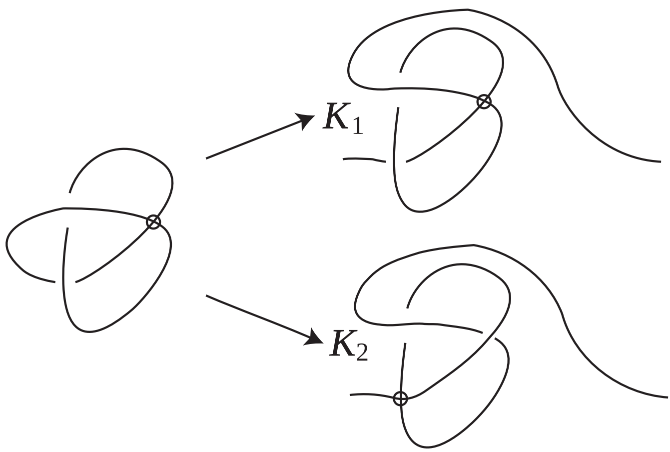


Fig. 3.25 Getting long virtual diagrams by breaking.

It is evident that the second operation is the inverse operation to the first in the sense that $\mathrm{Cl}(\mathrm{Brk}(K)) = K$. On the other hand, the operation Brk is not well defined, namely, an obtained long virtual knot (its equivalence class) depends on a selected point.

In what follows (Chap. 7) we shall show that for each compact virtual knot K there are infinite many pairwise non-equivalent long virtual knots K' such that $\mathrm{Cl}(K') = K$. It was first noted by Silver and Williams; in [221], this was proved by pursuing simple consideration: Vassiliev invariants of order zero of compact knots.

To long knots (classical and virtual) one can apply the technique constructed earlier in this chapter. Here, one should pay attention to two circumstances:

(1) A long virtual knot has the fixed beginning and ending non-compact arcs. Therefore, not only an algebraic object (a quandle or its generalization) is an invariant for long virtual knots but the algebraic object with a distinguished pair of elements is.
(2) A long virtual knot has the oriented traverse, therefore, all its classical crossings are divided into two types depending on which arc was the first: an arc forming an undercrossing or an overcrossing. In the first case we shall say that the crossing is an *early undercrossing*, in the second case we have an *early overcrossing*.

The latter allows one to construct particular delicate invariants.

It is well known that in the classical case the classifications of (compact) knots and long knots coincide. In the virtual case, as will be shown in the book, this is not true. The considerations described above can be used to construct invariants of classical (long) knots and long virtual knots. Namely, in the classical case we suggest a new approach to knot invariants by long knots with two types of crossings. In the virtual case we shall construct invariants distinguishing long virtual knots the closures of which are equivalent (compact) virtual knots.

Pursuing these considerations one can construct invariant constructions which allow one to distinguish very easily long virtual knots and find phenomena not appearing in the classical case, for example, the fact that long virtual knots, generally speaking, do not commute.

Definition 3.5. A *long quandle* is a set M with two distinguished elements a and b, equipped with two binary operations $\circ$ and $*$ and one unary operation $f(\cdot)$ such that

(1) $(M, \circ, f)$ is a virtual quandle (the reverse operation for $\circ$ is $/$);
(2) $(M, *, f)$ is a virtual quandle (the reverse operation for $*$ is $//$);
(3) the operations $\circ$, $*$, $/$, $//$ are right-distributive with respect to each other;
(4) the following relations hold. Let α, β be some operations from the list $\{\circ, *, /, //\}$, then for any x, y, $z \in M$ we have the identities (the *strange*

relations):

$$\begin{cases} x\alpha(y \circ z) = x\alpha(y * z), \\ x\alpha(y/z) = x\alpha(y//z). \end{cases}$$

The elements a and b are called *initial* and *final* elements of the long virtual quandle.

The axioms described above correspond to the construction of an invariant of long virtual knots in the following way. Let K be a long virtual knot. First, we take all arcs of it and associate elements with them, which subsequently play the role of generators of the long virtual quandle. The generators a and b are distinguished from all generators and they correspond to the initial and final non-compact arcs. At virtual crossings we write down the relations of types (3.2), (3.3), R_{k1} and R_{k2} also as in the case of a virtual quandle. At classical crossings we write down relations (analogous to the relation (3.1)) $R_{\circ,i}$ of a long quandle with respect to the operation $\circ$ in the case of an early overcrossing and $R_{*,i}$ of a long quandle with respect to the operation $*$ in the case of an early undercrossing. Namely, by *early overcrossing* we mean the situation when under the traverse of the long knot from the beginning to the end we first go along the upper branch (overcrossing) and then lower branch (undercrossing) in the crossing. Otherwise, we have *early undercrossing*, see Fig. 3.26.

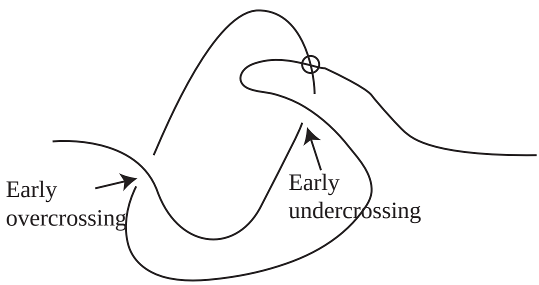


Fig. 3.26 An early overcrossing and early undercrossing.

Let us fix a common numeration $\{R_i\}$ for all relations described above. Each relation R_i looks like $r_{i1} = r_{i2}$.

As it turns out the axioms of a long virtual quandle given above allows one to get an invariant of long knots.

Let us describe it in detail. First, note that a long quandle like a virtual quandle can be defined by generators and relations. For this we have to take generators $a_1, \ldots, a_k$, and construct a set of admissible words, i.e. words obtained successively from generators by the operations $\circ$, $/$, $*$, $//$, f. All operations of the long quandle are naturally defined in the obtained set $\mathrm{Admi}(a_1, \ldots, a_k)$. Let us now define the quandle represented by generators $a_1, \ldots, a_k$ and relations $R_1, \ldots, R_l$ and denoted by $\mathrm{DG}\langle a_1, \ldots, a_k | R_1, \ldots R_l | a, b\rangle$, as the quotient set of the set $\mathrm{Admi}(a_1, \ldots, a_k)$ over the equivalence relations given by the following elementary equivalences:

(1) $A \circ A \sim A$, $A/A \sim A$, $A * A \sim A$, $A//A \sim A$ for any $A \in \mathrm{Admi}(a_1, \ldots, a_k)$;
(2) $(A \circ B)/B \sim (A/B) \circ B \sim A$, $(A * B)//B \sim (A//B) * B \sim A$ for any A, $B \in \mathrm{Admi}(a_1, \ldots, a_k)$;
(3) $(A\alpha B)\beta C \sim (A\beta C)\alpha(B\beta C)$ for any A, B, $C \in \mathrm{Admi}(a_1, \ldots, a_k)$ and any operations α, $\beta \in \{\circ, *, /, //\}$;
(4) $A\alpha(B \circ C) \sim A\alpha(B * C)$, $A\alpha(B/C) \sim A\alpha(B//C)$ for any A, B, $C \in \mathrm{Admi}(a_1, \ldots, a_k)$ and any operation $\alpha \in \{\circ, *, /, //\}$;
(5) $f(f^{-1}(A)) \sim f^{-1}(f(A)) \sim A$ for any $A \in \mathrm{Admi}(a_1, \ldots, a_k)$;
(6) $f(A\alpha B) = f(A)\alpha f(B)$ for any A, $B \in \mathrm{Admi}(a_1, \ldots, a_k)$ and any operation $\alpha \in \{\circ, *, /, //\}$;
(7) $r_{i1} \sim r_{i2}$ for any $i = 1, \ldots, l$.

The elements a and b are defined as some admissible words from the set $\mathrm{Admi}(a_1, \ldots, a_k)$.

Let K be a diagram of a long virtual knot and $a_1, \ldots, a_k$ be its arcs, herewith $a = a_1$ is the initial arc, $b = a_2$ is the final arc. Define the long quandle $\mathrm{DG}(K)$ as $\mathrm{DG}\langle a_1, \ldots, a_k | R_{\circ i}, R_{*j}, R_{k,1}, R_{l,2} | a_1, a_2\rangle$.

We have the following.

Theorem 3.17. *Let K, K' be diagrams of equivalent long virtual knots. Then there exists an isomorphism*

$$h\colon (\mathrm{DG}(K), a_K, b_K) \to (\mathrm{DG}(K'), a_{K'}, b_{K'}),$$

which respects the operations $\circ$, $*$ and f and sends the element a_K (respectively, b_K) to $a_{K'}$ (respectively, $b_{K'}$).

Proof. The proof follows line-by-line the proof of the invariance of the virtual quandle.

The invariance under purely virtual Reidemeister moves is evident. For the detour move it is sufficient for us to consider its local version in which the branch with two virtual crossings passes through a classical crossing. In this case we deal with the operation f and only one of the operations $*$, $\circ$. Further, only one of the two operations $*$, $\circ$ can appear under consideration of the first or second classical Reidemeister move.

So, the most interesting case is the third classical Reidemeister move. In fact, it is sufficient to consider the following four cases shown in Fig. 3.27 (a, b, c, d).

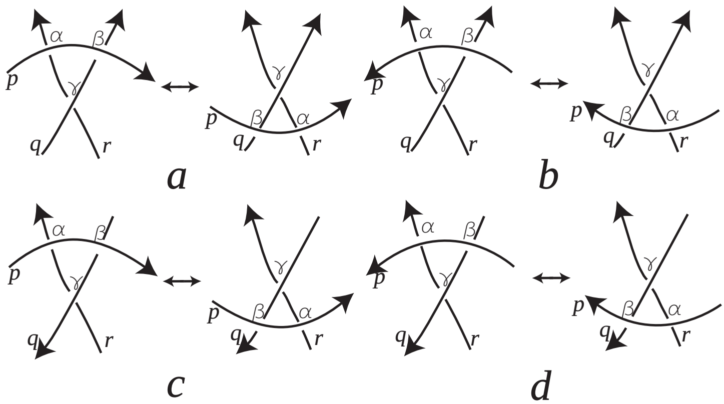


Fig. 3.27 Checking the move Ω_3.

In each of the four cases, we have three incoming arcs and three outgoing arcs. One of the arcs, marked by p, passes upper at both crossings; so the label at the end coincides with the label at the beginning. The other arc, marked by q, is seen under the arc with the label p; this action is the same in both cases corresponding to the concrete version of the move Ω_3. So, one should only check the transformation for the arc marked by r.

In each of the four cases, at each crossing we put some operation α, β or γ from the set of operations $\{\circ, *, /, //\}$. That will be applied to the arc below to obtain the corresponding arc above.

Consider the case a. Each of α, β, γ is a "multiplication" $\circ$ or $*$ (the operation / and // will be called "divisions").

Thus, in the upper-left corner we shall have: $(r\gamma q)\alpha p$ in the left picture and $(r\alpha p)\gamma(q\beta p)$ in the right picture. But, by definition, $(r\gamma q)\alpha p = (r\alpha p)\gamma(q\alpha p)$. The latter expression equals $(r\alpha p)\gamma(q\beta p)$ according to the "new relation" (because both β and α are multiplications).

Now, let us turn to the case b. Here γ is a multiplication and α, β are divisions. Thus, the same equality holds: $(r\gamma q)\alpha p = (r\alpha p)\gamma(q\alpha p) =$

$(r\alpha p)\gamma(q\beta p)$.

The same equation is true for the cases shown in pictures c and d. The only important thing is that α and β are either multiplications (as in case c) or divisions (as in case d). The remaining part of the statement follows straightforwardly. □

Like the case of quandles and virtual quandles, long quandles allow one to construct more simple invariants. For this, one can fix a long quandle (DG', a, b) and study the set of homomorphisms from long quandles of long knots to DG', respecting all the operations and sending the initial element of the long quandle to a, and the final element to b.

It may appear that the strange relations are not realizable except the case when the operation $*$ coincides with the operation $\circ$. However, it is not true. As an example of a long quandle [222] one can consider the ring $\mathbb{Z}_m$ and operations in it:

$$\begin{cases} a \circ b = pa + (1-p)b, \\ a * b = qa + (1-q)b, \\ f(a) = ka, \end{cases} \tag{3.7}$$

where k, p, q are invertible elements in the ring, and $(1-p)(p-q) = (1-q)(p-q) = 0$. The axioms of a quandle are checked straightforwardly.

Let us call long quandles of such type *linear long quandles*. As it turns out one can recognize non-triviality of some long knots having the trivial closure with the help of linear long quandles.

Let $\mathcal{R}$ be a ring with a unit, and p and q be two fixed invertible elements satisfying the equation $(p-1)(q-1) = (q-1)(p-q) = 0$. Let k be an invertible element also. For a long virtual knot K, denote by $\widetilde{M}(K)$ the module over $\mathcal{R}$ generated by the way described above (the generators are arcs, the relations at crossings are (3.7)) with two distinguished elements corresponding to the initial and final arcs.

Comparing long virtual quandles of two knots is a complicated problem. Therefore, it is useful to consider much simpler invariant objects corresponding to long virtual knots which are invariant under the same reasons as long quandles.

By construction for a fixed $\mathcal{R}$ the triple $(\widetilde{M}(K), a, b)$ is an invariant of long knots.

Indeed, the proof repeats that of Theorem 3.17 by taking into account that the operations (3.7) satisfy all axioms of a long quandle.

It is known (see, e.g. [66]) that classical long knots commute. Moreover, the following theorem is true.

Theorem 3.18. *Let a long knot K have no virtual crossings. Then for any long virtual knot K' the commutative property $K\#K' = K'\#K$ holds.*

Proof. Indeed, let us make a diagram of K very small and start pulling it through a diagram of K'. When pulling it through the virtual crossings we shall use the detour move, see Fig. 3.28, and when we have a pulling through classical crossings, we shall use the classical Reidemeister moves.

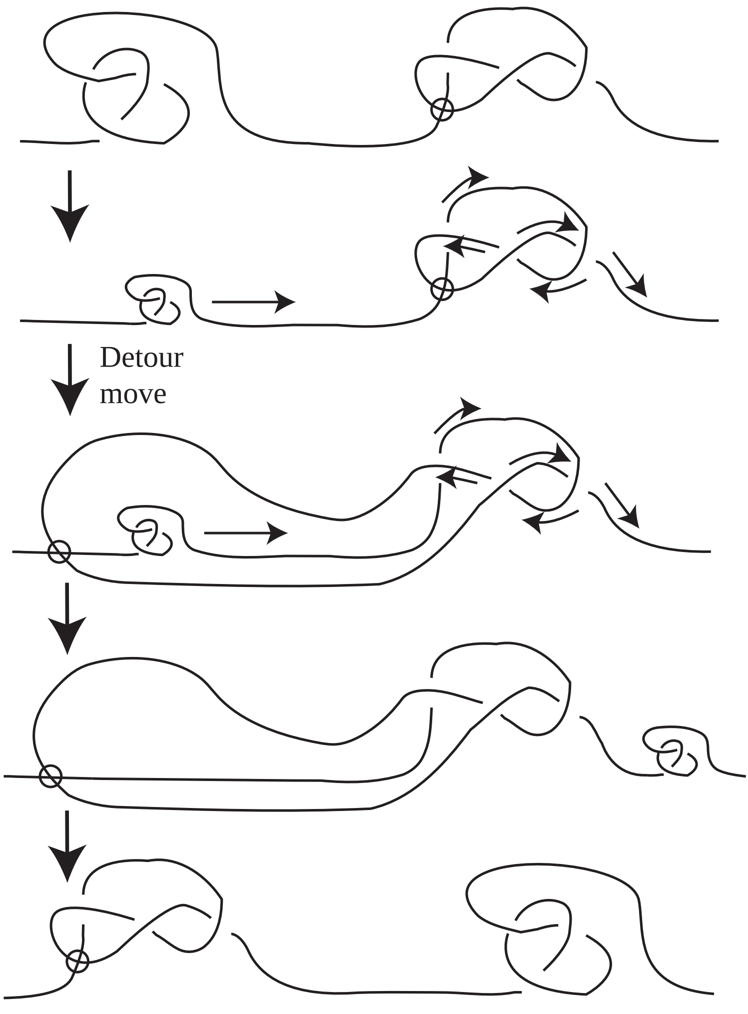


Fig. 3.28 A classical long knot commutes with any long knot.

As a result, we have the desirable equivalence. □

Remark 3.8. This proof does not work in the case when the knot K has virtual crossings, since in this case we cannot draw K through arcs of K' consisting of classical crossings. We should have used the forbidden move, see Fig. 1.16.

Let us show that there are long virtual knots which do not commute with each other. This fact was first discovered in [222].

Let us consider the long virtual knots K_1 and K_2 depicted in Fig. 3.29.

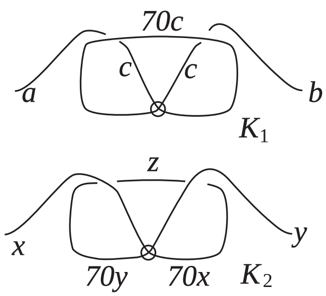


Fig. 3.29 Labeling the knots K_1 and K_2.

For $\mathcal{R}$ we take the ring $\mathbb{Z}_{11^2\cdot 19^2}$ and set $p = 20+121\cdot 19$, $q = 20$, $k = 70$. Thus, in $\widetilde{M}(K)$ the operations look like $a * b = 20a - 19b$, $a \circ b = (20 + 121 \cdot 19)a - (122 \cdot 19)b$ and $f(a) = 70a$. One notes that in the module $\widetilde{M}$ for the knot $K_1\#K_2$ the initial arc must be divided by 121, while for the module corresponding to the knot $K_2\#K_1$ there exists a homomorphism to the ring $\mathbb{Z}_{11^2\cdot 19^2}$ under which the initial arc is sent to $11 \cdot 19^2$.

Namely, for the knot K_1 (see upper part of Fig. 3.29) we have the following relations in the linear quandle $\widetilde{M}(K_1)$: $a*(70c) = c = b\circ 70c$; the first relation means that $20a = (19 \cdot 70 + 1)c = 1331c$; since the element 20 is invertible in the ring $\mathbb{Z}_{11^2\cdot 19^2}$, the elements a is divided by 121.

For the knot K_2 (see the lower part in Fig. 3.29) we have: $(70x) * y = z = (70y) \circ x$. It means that we can set $x = 11 \cdot 19^2$, $y = 0$ (the coefficient in x in the expression $(70x)*y-(70y)\circ x$ is divided by 11), i.e. we can map $\widetilde{M}(K_2)$ to the linear quandle $\mathbb{Z}_{11^2\cdot 19^2}$ with the same operations. Further, in the knot $K_2\#K_1$ we can set all remaining arcs (belonging to the long knot K_1) equal to zero; this can be done since $y = 0$ (the value z is calculated).

The other way to formulate the arguments above is: The set of homomorphisms $\mathcal{H}$ from $\mathrm{DG}(K_1\#K_2)$ to the linear quandle which is the ring $\mathbb{Z}_{11^2\cdot 19^2}$ with the operations (3.7) and $p = 20 + 121 \cdot 19$, $q = 20$, $k = 70$ such that $\mathcal{H}(a_1)$ is not divided by 121, is empty, and for $\mathrm{DG}(K_2\#K_1)$ the set of such homomorphisms is not empty.

Therefore, these knots do not commute, this fact confirms their difference, non-triviality, and also the fact that each of them is not equivalent to a classical knot.

There are two obvious cases when long virtual knots commute: either they coincide or one of them is classical. We suppose that there are, in essence, no other reasons for long virtual knots to commute, i.e. we formulate the following conjecture.

Conjecture 3.1. *If two long virtual knots K_1, K_2 commute, then there exist a long virtual knot K, non-negative integers m, n and long classical knots L_1, L_2 such that*

$$K_1 = K^m\#L_1, \qquad K_2 = K^n\#L_2,$$

where the power is understood in the sense of reiterated connected sum.

Remark 3.9. Just as in the case of compact knots the notion of a virtual quandle is generalized to the notion of a virtual biquandle [85, 125, 171], in the case of long knots the notion of a long virtual quandle is generalized to the notion of a long virtual biquandle. We are not going to develop this theme here.

3.4 Virtual knots and infinite-dimensional Lie algebras

The results of this section are also published in [220].

3.4.1 *Preliminaries*

Any group G can be transformed to a quandle if we decree:

(1) $a \circ b = b^n a b^{-n}$, where n is some fixed natural number, or
(2) $a \circ b = b a^{-1} b$,

where in the first case the structure of a virtual quandle can be defined by an arbitrary automorphism f of the group G, and in the second case f can be taken as an arbitrary (anti)automorphism of the group G. For a fixed group G we can, for example, pick out an element $q \in G$ and set $f(a) = qaq^{-1}$.

Analogously, any group can be transformed to a biquandle, virtual quandle, long quandle and so on.

For the group G, we may take a Lie group. In this case, according to the general theory, the set of morphisms from a virtual quandle of a link to a given Lie group is an invariant.

Attempts to associate with each (virtual) knot some Lie group which is invariant by the reason of the invariance of a virtual quandle lead to some difficulties.

However, Lie algebras can be defined axiomatically with the help of generators and relations. We can take a free Lie algebra generated by formal generators corresponding to arcs of the diagram. After that we have to describe appropriate relations which turn the Lie algebra into a quandle (a virtual quandle). It is possible because there exists the exponential map from a Lie algebra to the Lie group, and the multiplication in the Lie group can be lifted to the Lie algebra and can be expressed in terms of purely algebraic operations there. This is the famous Baker–Campbell–Hausdorff formula. We shall use this formula in the form proposed by Dynkin [80].

This formula expresses the logarithm for the product of the exponents of $x, y \in \mathfrak{g}$ from a Lie algebra $\mathfrak{g}$ by commutators. It looks like:

$$\log(e^x e^y) = \sum_{k=0}^{\infty} \sum_{\substack{p_i,\, q_i = 0, \\ p_i + q_i > 0}}^{\infty} \frac{1}{k} \frac{(-1)^{k-1}}{p_1! q_1! \dots p_k! q_k!} (x^{p_1} y^{q_1} \dots x^{p_k} y^{q_k})^{\circ}. \tag{3.8}$$

The summation is taken over k from 0 to ∞ and all possible sets of integer non-negative numbers $(p_1, q_1, \dots, p_k, q_k)$ connected by the relation $p_i + q_i > 0$.

Here the operation $(x) \to (x)^{\circ}$ sends the formal product of non-commutative variables x_j to the commutator according to the rule

$$(x_{i_1} x_{i_2} \dots x_{i_k})^{\circ} = \frac{1}{k} [\dots [[x_{i_1}, x_{i_2}], x_{i_3}], \dots, x_{i_k}],$$

and extended linearly for their linear combinations.

In order for the operation in question to be defined, the ground ring should contain the ring $\mathbb{Q}$ of rational numbers.

In this section, we suggest an algorithm which associates with a knot (a virtual knot), a Lie algebra or a quotient space of a Lie algebra over a subalgebra (a quotient algebra over an ideal). These objects will be invariant with respect to the (generalized) Reidemeister moves. Lie algebras being obtained will be, generally speaking, infinite-dimensional because the initial algebra (which we factorize) represents a free Lie algebra additively generated by all commutators of arbitrary orders of given generators.

The formulae obtained with the help of the Dynkin formula and the concept of virtual quandles lead us to the possibility of constructing "similar" virtual quandles which are not explicitly related with Lie groups.

Our approach is as follows. If we have a Lie algebra $\mathfrak{g}$ and the Lie group G corresponding to it, then the exponential map is a one-to-one correspondence of a neighborhood of zero of the Lie algebra to a neighborhood of the unity of the Lie group. Thus, if the formula $a \circ b = bab^{-1}$ gives a virtual quandle in the group, then the formula $a \circ b = \log(\exp(b)\exp(a)\exp(-b))$ gives a virtual quandle in the Lie algebra (at least in a neighborhood of zero). After that we can put the Dynkin formula (3.8), thereby we shall get the expression for the operation of a quandle in the Lie algebra in terms of interior operations of the Lie algebra (commutators).

The fact that this formula satisfies the quandle axioms can be checked immediately. We get some identity which is true for any Lie algebra given by formal generators and relations. Further, we can use the formula for an axiomatic definition of an invariant quandle (a virtual quandle), a Lie algebra corresponding to a classical or virtual link.

Moreover, besides the initial formula-series obtained by taking the standard quandle operation to Lie algebras by the exponential map and logarithm. It is possible to vary this formula and obtain new structures of (virtual) quandles which are not a straight generalization of the structures of quandles originated from groups.

Let us describe this construction in details.

Let K be a diagram of an oriented virtual link with n classical crossings. Pick out the first arc (according to the orientation) on the long arc with a number i and associate an element a_i with the arc. Consider a formal infinite-dimensional Lie algebra $G(n, q)$ over a field F generated by elements $a_1, \ldots, a_n$ corresponding to arcs of the diagram and a separated element q.

Using the relations at virtual crossings R_{j1}, R_{j2} (by the formulae (3.2), (3.3), interpreted according to the Baker–Campbell–Hausdorff formula (3.8)), we can define elements of the algebra $G(n, q)$ corresponding to all arcs of the diagram.

We shall now factorize the algebra $G(n, q)$ over the formal relations R_i of a quandle at classical crossings (3.1).

Herewith we shall mean the relations (3.2), (3.3) as well as the relation (3.1) in terms of the operation of the quandle defined by the Dynkin formula (3.8).

Namely, let us consider the following three cases:

(1) for any $\mathfrak{a}$, $\mathfrak{b} \in \mathfrak{g}$ we have

$$\mathfrak{a} \circ \mathfrak{b} = \log(\exp(n \cdot \mathfrak{b}) \exp(\mathfrak{a}) \exp(-n \cdot \mathfrak{b})),$$
$$f(\mathfrak{a}) = \log(\exp(q) \exp(\mathfrak{a}) \exp(-q)),$$

where n is some natural number;

(2) for any $\mathfrak{a}$, $\mathfrak{b} \in \mathfrak{g}$ we have

$$\mathfrak{a} \circ \mathfrak{b} = \log(\exp(\mathfrak{b}) \exp(-\mathfrak{a}) \exp(\mathfrak{b})),$$
$$f(\mathfrak{a}) = \log(\exp(q) \exp(\mathfrak{a}) \exp(-q));$$

(3) for any $\mathfrak{a}$, $\mathfrak{b} \in \mathfrak{g}$ we have

$$\mathfrak{a} \circ \mathfrak{b} = \log(\exp(\mathfrak{b}) \exp(-\mathfrak{a}) \exp(\mathfrak{b})),$$
$$f(\mathfrak{a}) = \log(\exp(q) \exp(-\mathfrak{a}) \exp(q)).$$

In each of them the relations at classical and virtual crossings generate the ideal in the free infinite-dimensional Lie algebra formed by commutators.

Denote the resulting quotient Lie algebras over the corresponding ideals by $\mathrm{Lie}_n(L), \mathrm{Lie}_+(L)$ and $\mathrm{Lie}_-(L)$, respectively.

Theorem 3.19. *The Lie algebras* $\mathrm{Lie}_n(L)$, $\mathrm{Lie}_+(L)$, $\mathrm{Lie}_-(L)$ *are invariants of virtual links.*

The proof of this theorem follows line-by-line the proof of Theorem 3.2 about the invariance of (virtual) quandles.

The obtained algebras are invariants of virtual knots, but they are infinite-dimensional, and it is difficult to work with them. Let us consider in detail formulae for virtual quandles used in these algebras and try to construct much simpler and convenient invariants of virtual links.

3.4.2 *Generalizations*

One of the ways to work with the constructed invariants (and to construct new ones) is, for example, to consider the group of automorphisms $\mathrm{Hom}(\mathrm{Lie}_n(L) \to A)$ to a fixed Lie group A. If this set is finite (or finite-dimensional), then its cardinality (its dimension) can be considered as an invariant of a (virtual) link.

Another way is as follows. We investigate the operation $\circ$ and its simplification.

An immediate consequence is that the factorization of the obtained Lie algebra over any "natural" ideal (for example, the ideal generated all commutators of order k) gives a link invariant. This operation leads to finite-dimensional Lie algebras, knot invariants.

However, we can forget about the nature of the operation $\circ$ and use only its formal form obtained from the Dynkin formula and modify it. As it turns out, this leads to virtual knot invariants valued in Lie algebras, finite-dimensional or infinite-dimensional, which, generally speaking, do not originate from a quandle related to groups.

Let us consider the very first term of the Baker–Campbell–Hausdorff formula:

$$\log(\exp(\mathfrak{a})\exp(\mathfrak{b})).$$

The easiest approximation gives us

$$\mathfrak{a}+\mathfrak{b}+\frac{1}{2}[\mathfrak{a},\mathfrak{b}].$$

Thus, the easiest approximation for $\mathfrak{a}\circ\mathfrak{b}$ in the form

$$\log(\exp(\mathfrak{b})\exp(\mathfrak{a})\exp(-\mathfrak{b}))$$

looks like

$$\mathfrak{a}-[\mathfrak{a},\mathfrak{b}].$$

The above formula defines the quandle operation if we take the quotient of the resulting Lie algebra by the ideal generated by all commutators $[[\mathfrak{x},\mathfrak{y}],\mathfrak{z}]$ of at least three terms. This follows from general considerations: An invariant quandle-Lie algebra is factored over an ideal which does not depend on a given link.

One can construct a more skillful example. Namely, consider the ring R of all infinite series of (positive and negative) powers in one variable ε over $\mathbb{Q}$.

Set

$$\mathfrak{a}\circ\mathfrak{b}=\mathfrak{a}+\varepsilon[\mathfrak{a},\mathfrak{b}]. \tag{3.9}$$

Theorem 3.20. *For any Lie algebra $\mathfrak{g}$ over R and its two-sided ideal $\mathfrak{h}$ generated by expressions of the form $[[\mathfrak{x},\mathfrak{z}],[\mathfrak{y},\mathfrak{z}]]$, the operation* (3.9) *is a well-defined quandle operation on the corresponding quotient algebra; the inverse operations is defined by the formula*

$$\mathfrak{a}/\mathfrak{b}=\sum_{k=0}^{\infty}(k+1)(-\varepsilon)^k(\mathfrak{a}\cdot\mathfrak{b}^k)^{\circ}. \tag{3.10}$$

Proof. The existence and uniqueness of the inverse operation follow from the formula (3.10). Indeed, solving the equation $\mathfrak{c} \circ \mathfrak{b} = \mathfrak{a}$ step-by-step (on degrees ε), we shall get:

- $\mathfrak{c} \equiv \mathfrak{a} \pmod{\varepsilon}$,
- $\mathfrak{c} \equiv \mathfrak{a} - [\mathfrak{a}, \mathfrak{b}]\varepsilon \pmod{\varepsilon^2}$,
- $\mathfrak{c} \equiv \mathfrak{a} - [\mathfrak{a}, \mathfrak{b}]\varepsilon + [[\mathfrak{a}, \mathfrak{b}], \mathfrak{b}]\varepsilon^2 \pmod{\varepsilon^3}$ and so on, and each subsequent approximation is uniquely defined from the preceding.

The idempotence is evident, since all commutators of the variable with itself are equal to zero.

We have

$$(\mathfrak{a} \circ \mathfrak{b}) \circ \mathfrak{c} = (\mathfrak{a} + 2\varepsilon(\mathfrak{a}\mathfrak{b} + \mathfrak{a}\mathfrak{c}) + 3\varepsilon^2\mathfrak{a}\mathfrak{b}\mathfrak{c})^\circ.$$

Analogously,

$$(\mathfrak{a} \circ \mathfrak{c}) \circ (\mathfrak{b} \circ \mathfrak{c}) = (\mathfrak{a} + 2\varepsilon(\mathfrak{a}\mathfrak{b} + \mathfrak{a}\mathfrak{c}) + 3\varepsilon^2(\mathfrak{a}\mathfrak{c}\mathfrak{b} - \mathfrak{b}\mathfrak{c}\mathfrak{a}))^\circ.$$

In the last equality we used the fact that $[[\mathfrak{a}, \mathfrak{c}], [\mathfrak{b}, \mathfrak{c}]] = 0$ due to the factorization.

The claim of the theorem now follows from the Jacobi identity. □

Denote the Lie algebra by $\mathrm{Li}(K)$. Thereby we obtained a map from the set of (classical and virtual) knots to the set of (infinite-dimensional) Lie algebras.

Example 3.3. Let us consider the knot invariant $\mathrm{Li}(K)$ (under construction of which we ignore all virtual crossings) factored by $\varepsilon = 1$. Note that the Lie algebra $\mathrm{Li}(K)$ is well defined: All relations in this algebra are written in the form of relations on diagram arcs with the operation $\circ$, herewith the operation $\mathfrak{a} \circ \mathfrak{b}$ is a finite sum of commutators (but not infinite series).

For majority of knots it gives an infinite-dimensional Lie algebra.

Let us show that this invariant gives finite-dimensional Lie algebras for the figure eight knot and the trefoil.

Indeed, in the case of the trefoil we have three arcs a, b, c and three crossings; for the latter we write down commutators. We get three relations: $a + [a, b] = c$, $b + [b, c] = a$, $c + [c, a] = b$. Thus, all first commutators are expressed by the generators, it follows that the algebra is finite-dimensional.

In the case of the figure eight knot we have four generators; at crossings we write down relations in which two commutators take part, see Fig. 3.30.

In the first crossing we get the relation $d \circ c = a$; the second crossing gives us $b \circ d = a$; in the third crossing we have $b \circ a = c$, and for the

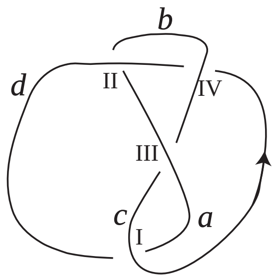


Fig. 3.30 Labeled the figure eight.

fourth one we get $d \circ b = c$. Thus, we get an expression for three of six first commutators. Therefore, $[b, d] = a - b$, $[d, b] = c - d$. It follows that the algebra is finite-dimensional.

3.5 Hierarchy of virtual knots

Algebraic and combinatorial techniques analogous to that developed in this book are successfully applied for other generalizations of knots and flat curves.

This section is devoted to hierarchy flat virtual knots. The notion of hierarchy knots was first introduced by Kauffman [86]; he formulated the classification problem of these objects. Here we give some results in this direction.

The main result of this section is Theorem 3.21.

3.5.1 *Flat virtual knots*

Algebraic structures (the Lie Goldman bracket and Lie Turaev cobracket, see [110, 298]) arising on the set of virtual knots (or virtual strings) lead us to the idea about a natural generalization of flat virtual knots. This is a motivation of the section.

Flat virtual knots (see the definition in Chap. 1) represented by framed 4-valent graphs can also be interpreted as curves in surfaces considered up to homotopy and stabilization.

Homotopy classes of curves in surfaces have the structure of Lie (Goldman) algebra and Lie (Turaev) coalgebra, the quantization of which leads us to the description of the structure of skein-module knots in thickened surfaces [298].

Flat virtual knots can be understood in the following way: We consider framed 4-valent graphs in the plane and say two graphs to be equivalent if one of them can be transformed to the other by a finite sequence of special moves which for simplicity are also called Reidemeister moves. Herewith in the case of the first Reidemeister move, the crossing can have an arbitrary type, for the second Reidemeister move we only require that two crossings are either classical or virtual; for the third Reidemeister move we allow the arc containing classical crossings passes through classical crossings, and the arc containing virtual crossings passes through classical crossings as well as virtual crossings, see Fig. 1.17.

In other words, the rule for Reidemeister moves for flat virtual knots can be formulated as follows. We do not impose any restriction on the first Reidemeister move. For the second Reidemeister move we demand that two crossings have the same type, and for the third Reidemeister move we allow the branch containing two crossings of type i passes through a crossing of a type j if and only if $i \geqslant j$, where $\geqslant$ is a binary relation (in our case it is an ordinary relation $\geqslant$ on the set $\{1, 2\}$). Here we have two types of crossings: 1 corresponds to a classical crossing, and 2 corresponds to a virtual one.

Instead of two types of crossings we can permit to note crossings from an arbitrary partially ordered set M. In this case a branch with crossings having a label i can pass through crossings with a label j if and only if $i \geqslant j$.

So, the theory of flat virtual knots has the following reformulation.

(1) Main objects of flat hierarchy are framed 4-valent graphs in the plane equipped with a special structure at crossings: classical crossings have the label 1, and virtual crossings have 2.
(2) Two objects are called *equivalent* if one of them can be transformed to the second one by a finite sequence of the following Reidemeister moves shown in Fig. 3.31.

   (a) For the first Reidemeister move the crossing can have an arbitrary label.
   (b) For the second Reidemeister move two crossings should have the same labels.
   (c) In the case of the third Reidemeister move we require that the branch γ passes through a crossing with a label x if and only if the labels of two crossings lying on γ are the same and equal to y, and $y \geqslant x$.

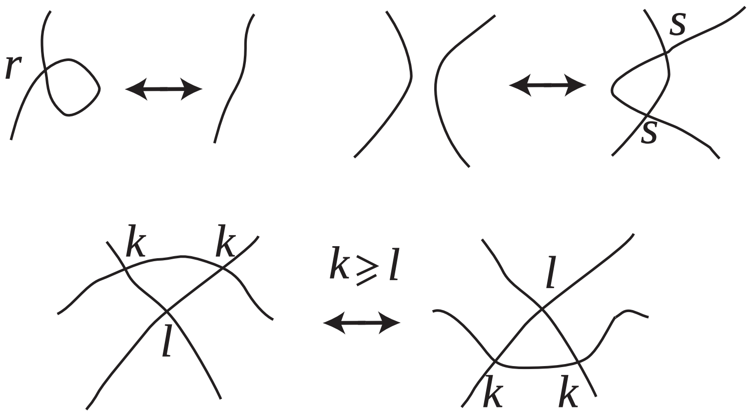


Fig. 3.31 Reidemeister moves hierarchy knots.

The last condition for the third move means that branches containing only virtual crossings can pass through classical crossings, and branches containing classical crossings cannot pass through virtual crossings.

Flat virtual knots are classified geometrically (see also Theorem 1.2). Later (Sec. 8.3.3) we shall see that virtual knots have natural hierarchy coming from *parity*.

Let us now generalize the notion of flat virtual knots (virtual strings) as follows. Let M be a set with a binary relation $\alpha \leqslant \beta$ (not necessary acyclic, i.e. from the relations $\alpha \leqslant \beta$ and $\beta \leqslant \alpha$, generally speaking, it does not follow that $\alpha = \beta$). Herewith the expression $\beta \geqslant \alpha$ is equivalent to the expression $\alpha \leqslant \beta$.

We shall consider framed 4-valent graphs and associate crossings with the labels from the set. We say that two labeled framed 4-valent graphs are equivalent if and only if one of them can be obtained from the other by a finite sequence of the Reidemeister moves with the restrictions described above.

We shall construct invariants of such combinatorial objects analogous to virtual quandles.

3.5.2 *Algebraic formalism*

Let us fix a partially ordered set M with a binary relation $\leqslant$.

Let K be a given oriented flat framed 4-valent graph all vertices of which are marked by elements from M. In what follows we shall call these graphs *M-hierarchy diagrams*. An *arc* of a M-hierarchy diagram is a connected component of the set obtained from K by deleting all vertices. Then K represents a *hierarchy diagram*.

Each hierarchy diagram represents a *hierarchy link* with some number of components. A *component of a hierarchy link* is a unicursal curve of the corresponding graph.

Let us have a hierarchy diagram of a link with n components. Enumerate components of the hierarchy link by numbers from 1 to n.

Each vertex (crossing) of this diagram has a label j and two additional numbers, the numbers of components of the link: x corresponding to the direction southwest–northeast and y corresponding to the direction southeast–northwest, see Fig. 3.32.

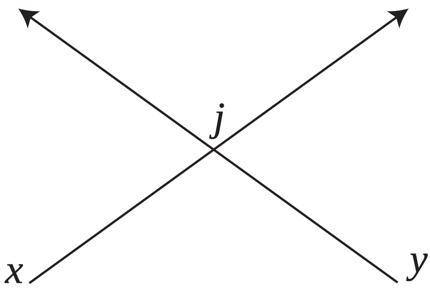


Fig. 3.32 A crossing.

Let us enumerate arcs with natural numbers (in an arbitrary way). Associate an element a_i with the arc having a number i. These elements will be generators of an algebraic set M, an invariant of a hierarchy link which we construct.

After enumerating all arcs in such a way let us define relations at crossings. They are defined as follows.

Let the crossing with a label j be related to arcs $\#x$ and $\#y$ (an order of arcs does not matter). Let the elements corresponding to the incoming arcs equal a and b (respectively, the lower-left and upper-right if the arcs are oriented from bottom to top). Then the relations look like:

$$c = b - P^j_{xy} a \tag{3.11}$$

and

$$d = a + P^j_{xy} b, \tag{3.12}$$

where c and d are emanating arcs, left and right, see Fig. 3.33.

We did not define P^j_{xy} yet. What should these elements be in order that the algebraic formalism (analogous to that developed for quandles, virtual quandles and biquandles) leads to an invariant of hierarchy links?

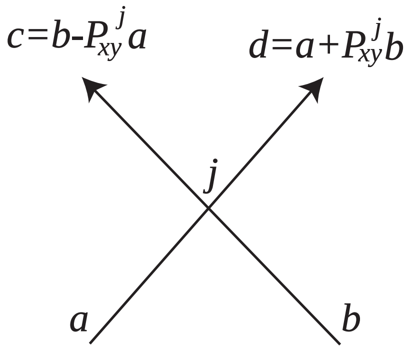


Fig. 3.33 Relations for a hierarchy knot.

Let us consider a formal non-commutative associative algebra (over an arbitrary preassigned ring of coefficients, for example, over $\mathbb{Q}$) generated by formal elements P^i_{xy}, where x, y run the set C of link components, and i runs the partially ordered set M in which the following relations hold.

For any i, x, y:

$$P^i_{xy} = P^i_{yx}. \tag{3.13}$$

For any i, $j \in M$, where $i \leqslant j$, and for any x, y, z, t:

$$P^i_{xy} P^j_{zt} = 0. \tag{3.14}$$

Denote the algebra defined by the relations (3.13), (3.14), by $\mathcal{A}(C)$.

As we shall see later, these relations are necessary for the invariance of the object which is constructed by us, under Reidemeister moves.

Associate with the M-hierarchy diagram K components of which are enumerated by elements from the set C, a module over the algebra $\mathcal{A}(C)$ as follows.

Pick out all arcs $a_1, \ldots, a_k$ of this diagram as generators over the ring $\mathcal{A}(C)$. Factorize the free module generated by arcs $a_1, \ldots, a_k$ over the relations (3.11) and (3.12), and as well as relations (3.15) which are as follows.

For any i, j, where $i > j$, for any $x, y, z, t \in C$ and for any $l, m = 1, \dots, k$ under fixed i, j all elements of the module having the form

$$P^i_{xy} P^j_{zt} a_l \tag{3.15}$$

coincide with each other. In short, we shall denote all these elements by $P^i P^j *$.

In other words, we require that pairwise products are equal to zero in some particular cases (3.14), namely, as $i \leqslant j$, which is contained in the definition of the ring $\mathcal{A}(C)$. In the case when such a product is not zero, we require that it is an element of our module, depending only on indices i and j.

Denote this module by $N(K)$.

Remark 3.10. In the case when we only have one component, we can forget about lower indices and construct an analogous ring $\mathcal{A}$ (without C) and a module corresponding to the link.

Theorem 3.21 (Main theorem [226]**).** *The module $N(K)$ is an invariant of hierarchy links.*

Proof. In order to prove the main theorem, we have to consider all Reidemeister moves. In each Reidemeister move, there is a part which is undergone by changing; this part of the diagram has incoming and emanating edges. We have to show that all output edges can be expressed in terms of inputs (like all interior edges); moreover, for two diagrams which differ by a Reidemeister move, the expression of output in terms of input is the same. Labels corresponding to "intermediate edges" can be excluded from the set of generators, since they are expressed by incoming edges (arguments are completely analogous to the proof of Theorem 3.4).

We shall consider only one case for each Reidemeister move. All the remaining cases are checked analogously.

For the first Reidemeister move we deal with only one component and one crossing; thus, it follows from the invariance check that there exists only one generator P^i_{pp} of the ring which will be denoted by P, see Fig. 3.34.

We see that the input a is connected with the middle edge x and the output edge by the following relations:

$$a + Px = x, \qquad x - Pa = b.$$

Thus, we have: $(1 - P)x = a$, where $x = (1 + P)a$. It follows from here that $b = (1 + P)a - Pa = a$, herewith x is expressed from a, it completes the proof of the invariance under the first Reidemeister move.

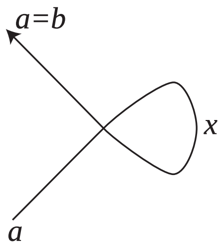


Fig. 3.34 The first Reidemeister move.

Now, let us consider the second Reidemeister move. By definition, both crossings taking part in the move have the same label, and in both cases we have the same pair of arcs with numbers k, l. Thus, we can again write P^i (without any lower indices) instead of P^i_{kl}.

The second Reidemeister move is shown in Fig. 3.35.

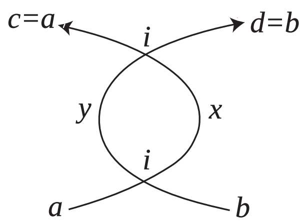


Fig. 3.35 The second Reidemeister move.

We have two input edges a, b. The remaining edges are expressed from a and b according to the following relations: $x = a + P^i b$, $y = b - P^i a$, $c = x - P^i y = a + P^i b - P^i b + P^i P^i a = a$, $d = y + P^i x = b$.

Therefore, the module is generated by the arcs a and b, as well as the exterior arcs (not depicted in the figure and identical before and after applying the second Reidemeister move). This leads us to the invariance under the second Reidemeister move.

Let us pass to the third Reidemeister move shown in Fig. 3.36.

Here we have three input edges a, b, c, labels at vertices ($r \geqslant s$), and numbers of components (i, j, k in our case). It is easy to see that all middle edges in the left case as well as the right are expressed from the input edges.

Thus, we have to check the equality of the expressions of emanating

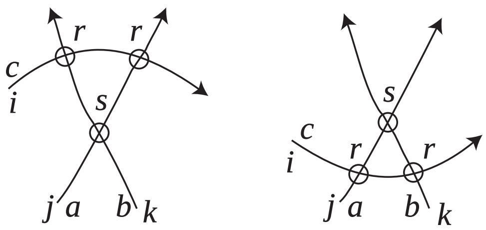


Fig. 3.36 The third Reidemeister move.

edges from the input edges for the right and left pictures (which are obtained from each other by applying the third Reidemeister move).

In the left pictures the output edges are $a+P^s_{jk}b-P^r_{ij}c$, $b-P^s_{jk}a-P^r_{ik}c$ and $c+P^r_{ik}(b-P^s_{jk}a)+P^r_{ij}(a+P^s_{jk}b)$.

In the right pictures the output edges are $a+P^r_{ij}c+P^s_{jk}b$, $b-P^r_{ik}(c+P^r_{ij}a)-P^s_{jk}(a-P^r_{ij}c)$ and $c+P^r_{ij}a+P^r_{ik}b$.

It is easy to see that the corresponding values coincide in view of the axioms (3.13), (3.14) of the ring M, and the relation (3.15). Thereby, we have completed the invariance check under the third Reidemeister move and, therefore, the proof of the theorem. □

3.5.3 *Examples*

The simplest example of a hierarchy link is the trivial link with any number of components. It is obvious that for the trivial hierarchy link with k components we get the free module with k generators over $\mathcal{A}$.

The natural shadow of the trefoil for which two of three crossings have the same labels represents a hierarchy knot reduced to the unknot, see Fig. 3.37. Indeed, first we apply the second Reidemeister move to two vertices with the same label, and then we delete the remaining loop by the first Reidemeister move.

However, if we have three distinct labels, see Fig. 3.38, then we shall have a non-trivial hierarchy knot.

Namely, denote two leftmost arcs by x and y, as shown in Fig. 3.38.

Let us show that the module corresponding to this knot is not free. Indeed, since we have only one component, we can use the notation P^j, where $j=1,2,3$, instead of P^j_{11}.

Let us factorize our module by all triple relations $P^iP^jP^k=0$. The aim is to show that even after such a factorization the module turns out to be

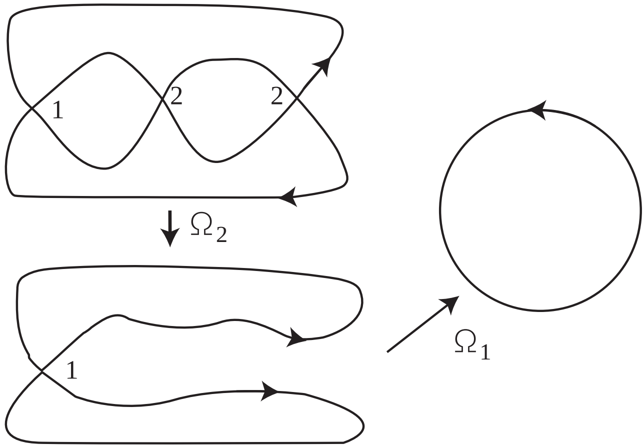


Fig. 3.37 The trivial hierarchy knot.

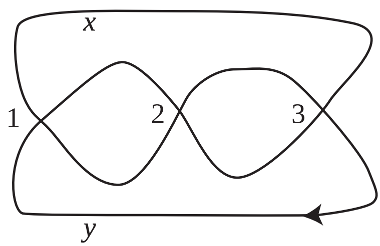


Fig. 3.38 A non-trivial hierarchy knot.

non-trivial. From this it follows that the initial module is also non-trivial.

We have the pair (x, y) of arcs which is transformed to $(y-P^1x, x+P^1y)$ (here on we first indicate the upper arc and then the lower arc).

The second crossing gives us

$$(x + (P^1 - P^2)y + P^2P^1*, y + (P^2 - P^1)x + P^2P^1*).$$

Finally, after the third crossing we get

$$\Big(y + (P^2 - P^1 - P^3)x + (P^2P^1 + P^3P^2 - P^3P^1)y,$$
$$x + (P^1 - P^2 + P^3)y + (P^2P^1 + P^3P^2 - P^3P^1)x\Big).$$

These two elements correspond to the rightmost arcs, i.e. they equal x and y, respectively.

Denote $P^1 - P^2 + P^3$ by $\mathcal{P}$ and denote $P^2P^1 + P^3P^2 - P^3P^1$ by $\mathcal{Q}$. Note that $\mathcal{P}^2 = -\mathcal{Q}$, $\mathcal{P}^3 = 0$. Then we get

$$y - \mathcal{P}x + \mathcal{Q}* = x, \qquad x + \mathcal{P}y + \mathcal{Q}* = y.$$

Thus we have $x = y - \mathcal{P}y - \mathcal{Q}*$, from which we get $y - \mathcal{P}y + \mathcal{P}^2y = y - \mathcal{P}y + \mathcal{Q}*$. These relations make our module non-trivial. Moreover, the relation does not follow from the relation by which we factorized.

Therefore, the hierarchy flat diagram depicted in Fig. 3.38 is non-trivial.

Analogously, one can prove that the "hierarchy Borromean rings" with labeling shown in Fig. 3.39 are not equivalent to the flat diagram consisting of three disjoint circles.

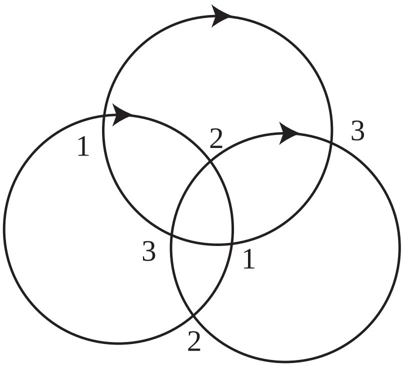


Fig. 3.39 Hierarchy Borromean rings are not equivalent to the trivial link.

Let us picture the Borromean rings in the following way (in view of the "braid closure"), see Fig. 3.40.

On the left we have three labels (a, b, c). After the first crossing (labeled by 1) they pass to the following three crossings (we write down from top to bottom): $(b - P^1a, a + P^1b, c)$.

Further, we have $(b - P^1a, c - P^3a - P^3P^1*, a + P^1b + P^3c)$, then $(c - P^3a - P^3P^1 * -P^2b + P^2P^1*, b - P^1a + P^2c, a + P^1b + P^3c)$, after that (after the crossing labeled by 1), we get: $(c - P^3a - P^3P^1 * -P^2b + P^2P^1*, a + P^3c, b + P^2c)$, further $(a + P^3P^2*, c - P^3P^1 * -P^2b + P^2P^1*, b + P^2c)$ and, finally, $(a + P^3P^2*, b, c - P^3P^1 * +P^2P^1*)$.

From this the non-trivial relations follow: $P^3P^2* = 0$ and $P^3P^1 * -P^2P^1* = 0$.

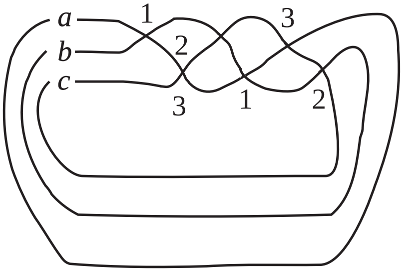


Fig. 3.40 Labels on Borromean rings.

Chapter 4

The Jones–Kauffman Polynomial: Atoms

4.1 Introduction

The Jones polynomial (in the normalized Kauffman bracket form) is a simple but at the same time fundamental construction of a classical knot invariant. Based on the Jones polynomial, several remarkable invariants of classical knots were constructed: quantum invariants, Khovanov homology, etc., as well as some invariants of 3-manifolds and knots in 3-manifolds (the Jones–Witten invariants). Here, one should also mention various generalizations of the Jones polynomial for the case of virtual knots. The original Jones' construction was first suggested in [141], the Kauffman bracket version first appeared in [155]. In the sequel, we shall also call this invariant the *Jones–Kauffman polynomial*.

In the end of 20th century, Khovanov [176] invented and developed a powerful construction generalizing the Jones polynomial. Nowadays this construction is called the *Khovanov complex* (*Khovanov homology*).

This construction is related to the Jones polynomial approximately as the cohomology ring of some topological space is related to the Euler characteristic of this space. More precisely, the Khovanov complex of a link is some formal algebraic complex whose homology groups are invariants of the link in question, and the (graded) Euler characteristic coincides with the Jones–Kauffman polynomial in some other normalization suggested by Khovanov.[1] We shall devote the next chapter to the Khovanov homology. One should note that the Khovanov homology detects the unknot, which was recently proved by Kronheimer and Mrówka, see [187]. Chapter 4 is preparatory for Chap. 5.

[1]This polynomial is obtained from the Jones polynomial by a variable change and a multiplication by a fixed polynomial factor.

In the next chapter we shall describe various ways of generalizations of the Khovanov complex for the case of virtual knots belonging to the first-named author, see [218, 221, 223]. Based on the properties of the Jones polynomial and the Khovanov homology, in Chap. 5 (see also [211]) we shall prove theorems about minimality and non-classicality of some virtual links (i.e. the absence of a classical diagram for a given virtual link), which estimate some characteristics of the links in question (the atom genus, etc.). Note that in the case of knots with non-orientable atoms (see the definition below and Chap. 8), the Khovanov construction requires serious modifications. The geometric generalizations constructed in Chap. 5 are based on geometric modifications of the initial knot.

In Chap. 5 (where the main construction of the Khovanov homology is introduced) with each diagram of an arbitrary virtual link we shall associate a chain complex whose homology groups are invariant under the Reidemeister moves. In the case of classical links and links with orientable atoms, this complex will be equivalent (quasiisomorphic) to the original Khovanov complex.

Herewith, a major part of the results of Chap. 5 related to modifications of Khovanov's construction, remains true for the new complex to be constructed in that chapter.

Besides its preliminary role, the present chapter contains results of its own interest. We construct an invariant Ξ justifying the Jones polynomial. In the end of this chapter we touch upon minimal problems.

In the last few years many new invariants refining the Jones polynomial for virtual knots and long virtual knots appeared, see, e.g. [135, 247].

4.2 Basic definitions

Given an oriented diagram K of a virtual link with n classical crossings. Consider the unoriented diagram $|K|$ obtained from K by forgetting the orientation. Each crossing of $|K|$ can be *smoothed* in one of the two ways: positive A:  $\to$  or negative B:  $\to$ . The choice of smoothings for all crossings is called a *state* of the diagram. Thus, the diagram $|K|$ has 2^n states. After smoothing the diagram with respect to some state, we get a diagram of a virtual link without classical crossings. Thus, the obtained virtual link will be trivial. To each state s we assign the following three numbers: $\alpha(s)$ is the number of A-smoothed classical crossings, $\beta(s) = (n - \alpha(s))$ is the number of B-smoothed classical crossings, and $\gamma(s)$ is the number of components of the link in the state s.

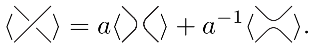
Define *the Kauffman bracket* [158] $\langle\cdot\rangle$ of a non-oriented diagram $|K|$ according to the formula:

$$\langle|K|\rangle = \sum_s a^{\alpha(s)-\beta(s)}(-a^2 - a^{-2})^{(\gamma(s)-1)}, \tag{4.1}$$

where the sum is taken over all states s of the diagram $|K|$.

The Kauffman bracket is invariant under all generalized Reidemeister moves except for the first classical Reidemeister move Ω_1.

It follows from the definition that the Kauffman bracket satisfies the relation



The Kauffman bracket which is "almost" an invariant of unoriented virtual links leads to an invariant of oriented virtual links.

To this end, the Kauffman bracket has to be normalized as follows. Let K be an oriented virtual diagram. Set

$$X(K) = (-a)^{-3w(K)}\langle|K|\rangle, \tag{4.2}$$

where $w(K)$ is the writhe number of K.

It is known [158] that in this way we can get an invariant (Laurent) polynomial for oriented links to be called *the Jones–Kauffman polynomial* [141]. Looking at the Jones–Kauffman polynomial, one can partially answer the following questions:

(1) Is a given link classical?
(2) What can we say about the atom genus of the link, see the definition in Sec. 4.2.3?
(3) Which are the lower estimates for the crossing number?

Let us describe some well-known important properties of the Jones–Kauffman polynomial.

Example 4.1. From definitions, it follows immediately that the value of the Jones polynomial on the unknot is equal to one, and on the trivial link with $(n+1)$ components it is equal to $(-a^2 - a^{-2})^n$.

Furthermore, the Jones–Kauffman polynomial is multiplicative with respect to a connected sum operation $X(K_1\#K_2) = X(K_1)\cdot X(K_2)$.

Here we mean that for *every* connected sum of virtual knots $K_1\#K_2$ the value of the invariant is equal to the product of the Jones–Kauffman polynomial on virtual links, which are summands in this connected sum.

Furthermore, for the disconnected sum (which is well defined) we have $X(K_1 \sqcup K_2) = X(K_1)\cdot X(K_2)\cdot(-a^2 - a^{-2})$.

The Jones–Kauffman polynomial, however, is not very strong invariant for virtual knots. One can easily construct, for example, non-trivial (and non-classical) knots where the Jones–Kauffman polynomial is equal to the unit.

Example 4.2. Let us consider the Kishino knot as shown in Fig. 4.1.

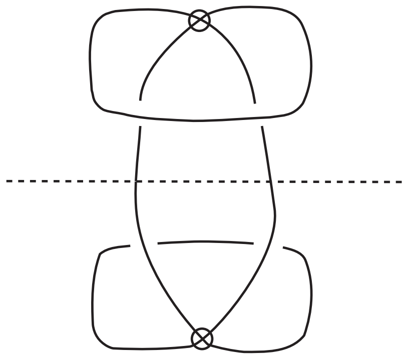


Fig. 4.1 The Kishino knot split into two long knots.

It represents a connected sum of two unknots. This yields that its Jones polynomial is equal to 1. In the sequel we shall demonstrate non-triviality of the Kishino knot by using the polynomial Ξ, a generalization of the Jones polynomial, which takes into account the geometry of virtual knots. In fact, the flat knot corresponding to the Kishino knot is not trivial either.

Each of the two parts of the Kishino knot closes to the unknot (see Fig. 3.29); however, each of these two parts is non-trivial as a long knot; if at least one of these knots were trivial, then the Kishino knot itself would be trivial.

The question about the existence of non-trivial classical knots with the trivial Jones–Kauffman polynomial is an open problem.

In this section, we shall give several results allowing one to answer partially to the questions stated above, and formulate several conjectures concerning the unknot recognition by the Jones–Kauffman polynomial.

Note that for every $n \geqslant 2$ there exists a classical n-component link sharing the Jones–Kauffman polynomial with the n-component trivial link, namely, $(-a^2 - a^{-2})^{n-1}$. These examples were constructed in the paper by Eliahou, Kauffman and Thistletwaite [82].

Moreover, in this work it is shown that for every classical n-component link K there exist infinitely many classical links with $(n+1)$ components having the Jones–Kauffman polynomial equal to $X(K) \cdot (-a^2 - a^{-2})$.

In this chapter we shall give generalizations of the Jones–Kauffman polynomial which are stronger in the case of virtual knots, see [210, 212, 214, 221].

One can easily prove the following proposition.

Proposition 4.1 (see, e.g. [149]). *For a classical n-component link K all non-zero monomials of the Jones–Kauffman polynomial $X(K)$ have exponents congruent to $2n+2$ modulo four.*

In the case of virtual links one can just say that all monomials have exponents congruent modulo two. This follows explicitly from (4.1). This immediately yields the following well-known corollary.

Corollary 4.1. *Let K be a virtual link for which the polynomial $X(K)$ has at least one non-zero monomial of degree $4k$ and at least one non-zero monomial of degree $4l+2$ for some integers k, l. Then the virtual link K is not classical.*

This is the simplest example how to prove the non-classicality of a virtual link.

The property of having exponents of monomials all congruent modulo four, belongs not only to classical links but also to virtual links having diagrams with orientable atoms, see the definition ahead, though this property is not an orientability criterion of the corresponding atom. Note that the *orientability of the corresponding atom* is a key property for defining the Khovanov homology. We shall discuss it in the next chapter.

4.2.1 *Virtualization and mutation*

There are transformations changing a virtual link diagram and the equivalence class of the virtual link, but not changing the Jones–Kauffman polynomial. These transformations are important for the study of properties of the Jones–Kauffman polynomial of classical and virtual knots and for construction of stronger invariants.

One such transformation consists of the change of a classical crossing by one classical and two virtual crossings, as it is shown in Fig. 4.2.

This transformation is called the *virtualization*. It is easy to see that

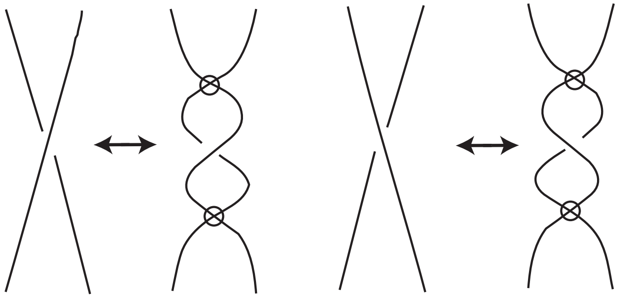


Fig. 4.2 Two variants of virtualization.

each of the two diagrams changed by a virtualization at classical crossings is expressible as a combination of the other virtualization move and the detour move.

The set of classical knots is a subset of the set of virtual knots. Then the natural question [86] arises. We want to take the quotient of the set of virtual knots in such a way that the set of classical knots remains the same (we shall talk about simplifications also in Chap. 8). Namely, we state the following conjecture.

Conjecture 4.1 (Virtualization conjecture). *If two classical knots are equivalent by means of generalized Reidemeister moves and virtualization moves, then they are isotopic.*

Note that the virtualization applied twice to the same crossing v of a diagram K (more exactly to the crossing v of the diagram K and to the crossing v' replacing it in the virtualized diagram K') leads to the diagram K'' obtained from the diagram K by detour moves. The following statement is well known.

Statement 4.1 (Kauffman [158]). *The virtualization does not change the value of the Jones–Kauffman polynomial.*

Corollary 4.2. *There are infinitely many non-trivial virtual knots with unit Jones polynomial, and that it is natural to conjecture that they are all non-classical. This problem is well known and still wide-open.*

A more interesting example of a transformation which does not change the Jones–Kauffman polynomial is a *mutation*.

It is defined as a deletion of a fragment (called a 2–2 *tangle*) from the diagram and an insertion of this tangle (i.e. a part of the diagram having four emanating ends) rotated by π to the obtained diagram, see Fig. 4.3.

Note that mutations were first considered by Conway in [63].

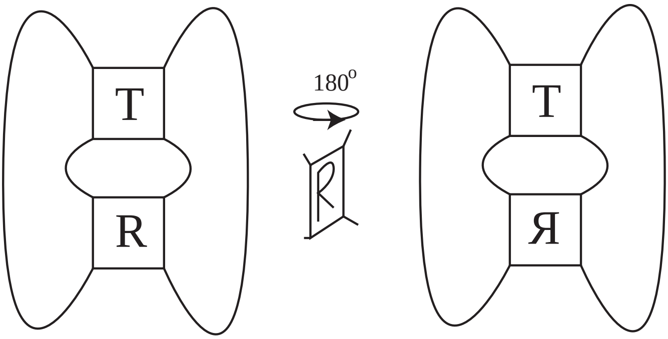


Fig. 4.3 Mutation.

Diagrams obtained from each other by applying mutations are called (mutually) *mutants*.

The notion of mutation and mutant is verbatim generalized for virtual knots and links (herewith $2-2$ tangles appearing in the view of subdiagrams can be virtual).

It is well known (the proof can be found in, e.g. [221]).

Proposition 4.2. *Mutations do not change the value of the Jones–Kauffman polynomial.*

It follows from the fact that there exists a one-to-one correspondence between states of mutants K_1 and K_2 such that the number of circles in a state s of the diagram K_1 is equal to the number of circles in the corresponding state s' of the diagram K_2. Since the writhe numbers of (oriented) mutants coincide, then the values of their Jones–Kauffman polynomials coincide, see Fig. 4.3.

Note that the Khovanov complex (the next chapter is devoted to this complex) can sometimes feel mutations in the case of links. This was first discovered by Wehrli (see [314]).

4.2.2 *Atoms and knots*

Given a classical alternating diagram K. Its shadow divides the plane into cells $C_1, \ldots, C_n$ which can be colored in the checkerboard coloring. Herewith the set of circles in the A-*state* (a state in which all classical crossings are A-smoothed) is in one-to-one correspondence with the set of cells having the same coloring, and the set of circles in the B-*state* (all crossing are B-smoothed) corresponds to the remaining circles. Thus, an alternating diagram represents a framed 4-valent graph generating the checkerboard coloring of the two-dimensional sphere (the plane with the additional infinite point).

In the case of an arbitrary classical diagram, the resulting tiling of the sphere is not checkerboard colorable. As it turns out one can associate with each virtual diagram a 2-surface divided into cells admitting a checkerboard coloring in such a way that white cells correspond to circles in the A-state and black cells correspond to circles in the B-state. A particular case of this construction is the tiling of the sphere described in the beginning of this section.

This surface (with the graph and checkerboard coloring) is called an *atom*. Having an atom we can restore more information about the link, in particular, its Jones–Kauffman polynomial and the Khovanov homology.

Important characteristics of a link diagram are the *orientability of the corresponding atom* and its *genus*.

The notion of an atom was first introduced by Fomenko in [98] (see also [43, 94–97]) for the classification purposes of integrable Hamiltonian systems of low complexity. A connection between atoms and knots was investigated by the first-named author in the series of works [203–205, 221].

In [294] Turaev was using a similar construction (hence, the atom genus is also called the *Turaev genus* [199]).

Definition 4.1. An *atom* is a pair (M, Γ): a connected closed 2-manifold M and an embedded 4-valent graph $\Gamma \subset M$ such that $M \backslash \Gamma$ is a disjoint union of cells that admits a checkerboard coloring (with black and white colors), i.e. two components of $M \backslash \Gamma$ being adjacent by an edge of Γ have always distinct colors.

The graph Γ is said to be the *frame* of the atom. A *vertex* of the atom is a vertex of this frame.

The *genus* (respectively, *Euler characteristic*) of the atom is that of its first component, i.e. the surface. An atom is called *orientable* (respectively, *connected*) if the corresponding manifold is orientable (respectively, connected). An atom of genus zero is called *spherical*, oriented atoms of genus one are called *torus*.

Remark 4.1. Atoms can be represented as a 2-surface (with boundary) on which a Morse function with distinguished critical levels with some saddle points under the condition that there are no critical points of other types (the initial definition by Fomenko [98]) is defined. In this case the frame of the atom represents a critical level, black cells are domains of values above the critical level (a *supercritical level*), and white cells are domains of values below the critical level (a *subcritical level*). Further, we shall use the term "subcritical circles" and "supercritical circles" to describe circles in states of the corresponding knot.

Remark 4.2. Atoms are usually assumed to be connected, but we admit that an atom can have more than one connected component.

The example of a torus atom is depicted in Fig. 4.4.

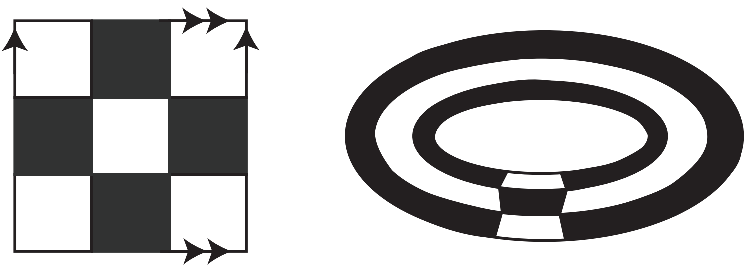


Fig. 4.4 A torus atom.

If an atom is orientable, then we can equip it with an orientation. Thus, it makes sense to say about oriented and unoriented atoms.

Atoms are considered up to the natural isomorphism. Two atoms are called *isomorphic* if there exists a one-to-one map of their first components taking frame to frame and black cells to black cells.

Each atom (more precisely, its equivalence class) can be completely restored from the following combinatorial structure:

(1) the frame (a 4-valent graph);
(2) the *A-structure* or the *structure of opposite edges* (dividing the four outgoing half-edges into two pairs, called opposite, according to their disposition on the surface); and

(3) the *B-structure* (for each vertex, we indicate some two pairs of adjacent half-edges (also: two angles) which constitute a part of the boundary of black cells).

If an atom (M, Γ) is not orientable, then we can consider its *orientable covering*, i.e. an atom, $(\widetilde{M}, \widetilde{\Gamma})$, being the preimage of the pair (M, Γ) under the two-sheeted orientable covering; in the case when the atom (M, Γ) is orientable, we define the atom $(\widetilde{M}, \widetilde{\Gamma})$ to be two disjoint copies of (M, Γ) (with a fixed isomorphism taking one atom to the other).

Atoms are very convenient for describing links and virtual links.

Among all atoms one can single out the class of *height atoms* which correspond to diagrams of classical links. According to [205], an atom is called a *height* atom if its frame can be embedded into the plane in such a way that the A-structure of the atom corresponds to the A-structure inherited by the atom from the plane, i.e. at each atom vertex opposite edges remain opposite under the embedding of the frame into the plane.

In the general case, the *height* h of an orientable atom (M, Γ) is the minimal genus of an orientable surface S, in which the frame Γ of the atom can be embedded with the A-structure preserved. From the definition of an atom, it follows that the height of an orientable atom does not exceed its genus: $h(M, \Gamma) \leqslant g(M, \Gamma)$.

The frame of each atom is a framed 4-graph: The framing (the structure of opposite edges) is inherited from the embedding of the graph into the 2-surface. Otherwise we can say that an atom is an embedding of a framed 4-graph (without cycle components) in a surface in such a way that connected components of the complement of the surface to the graph admit a checkerboard coloring.

The planarity problem of framed 4-graphs (i.e. whether a given framed 4-graph is a frame of an atom with genus zero) is considered in [224], and the problem of finding the genus of an atom having a fixed frame is considered in [232, 238].

4.2.3 *Virtual diagrams and atoms*

Given a virtual diagram K (each connected, in the sense of atoms, component of K has at least one classical crossing), let us construct the atom $\mathrm{At}(K)$ associated with K. Vertices of $\mathrm{At}(K)$ are in one-to-one correspondence with classical crossings of the diagram K. Classical crossings of K are connected to each other by branches of K which may intersect each other only at virtual crossings. At each classical crossing we have four emanating half-edges e_1, e_2, e_3, e_4 in clockwise-ordering such that the pair (e_1, e_3) forms an undercrossing and the pair (e_2, e_4) forms an overcrossing. These edges are in one-to-one correspondence with edges of $\mathrm{At}(K)$ connecting the corresponding vertices. The 1-cycles of the frame pasting black and white cells are constructed as follows. Each boundary of a 2-cell is a rotating circuit on the frame, i.e. a circuit which passes every edge at most once and switches at each vertex from an edge to an adjacent (non-opposite) one. Black cells are glued to the angles formed by (e_1, e_2) and (e_3, e_4), and white cells are glued to the angles formed by (e_2, e_3) and (e_1, e_4).

Define the *genus*, cf. [283], of a virtual link as the minimum of values $g(M, \Gamma)$ over all atoms (M, Γ) corresponding to diagrams of the link, and the *height* as the minimum of values $h(M, \Gamma)$ over all atoms (M, Γ) corresponding to diagrams of the link (see also Chap. 7).

From the definition it follows that classical links have height zero. Actually, if a link is classical, then its shadow is a framed 4-graph embedded in the plane which is the frame of the corresponding atom. On construction the A-structure of the atom coincides with the A-structure induced from the plane. Thus, *a height atom is assign to each classical diagram.*

Note that the inverse procedure of constructing a virtual diagram from an atom is not well defined, for we get a virtual diagram up to detour moves and virtualizations, see, e.g. [221]. The latter follows from the fact that the framing of the frame of an atom does not give a cyclic structure on outgoing half-edges: Knowing that the half-edges e_1, e_3 are opposite to each other, we have two possible cyclic orders for embedding a neighborhood of this vertex into the plane: e_1, e_2, e_3, e_4 and e_1, e_4, e_3, e_2. The diagrams corresponding to them differ from each other by (detour moves) and virtualizations.

Having an atom (M, Γ), we can construct an oriented diagram of a virtual link as follows. Let us consider a generic immersion of the frame Γ in the plane with the A-structure preserved. The image of this immersion will be a framed 4-graph, vertices of which are images of atom vertices and intersections of interior points of edges. The latter will be considered as virtual crossings, and images of vertices as classical crossings. For restoring the structure of classical crossings we shall use the B-structure of the atom. Namely, an edge forms an overcrossing if while moving inside an angle of the supercritical level in clockwise ordering, it moves from an undercrossing to an overcrossing edge.

Denote all possible classes of virtual links obtained from an atom At by

$L(\mathrm{At})$.

An ambiguity appears when an immersion is specified, see Theorem 4.2.

In the case of height atoms one can restrict by considering embeddings. Denote the corresponding subset of the set $L(\mathrm{At})$ by $L_{\mathrm{emb}}(\mathrm{At})$.

We have the following theorem.

Theorem 4.1 ([205]). *The isotopic type of a link from $L_{\mathrm{emb}}(\mathrm{At})$ does not depend on the embedding of the frame of the atom* At *into the plane with the A-structure preserved.*

Note that with such an approach the tautological embedding of flat atoms into the plane is associated with alternating diagrams; they divide the plane (the sphere) into black and white cells.

In the case of immersions there is no well definedness.

Theorem 4.2 ([221]). *Diagrams of links from the set $L(\mathrm{At})$ constructed from an atom* At *are obtained from each other by applying the detour move and virtualization.*

Proof. Let us first embed neighborhoods of vertices of the frame with the A-structure preserved. After that, edges connecting atoms vertices can be immersed in the plane in an arbitrary way. This corresponds to the fact that a diagram of the virtual knot is defined up to the detour move.

Here, the choice of an embedding of a neighborhood of a vertex is not unique. Namely, for each vertex v with a cyclic order of four emanating half-edges p, q, r, s on the atom, we can choose two ways of an embedding with the A-structure specified: In the first way the order of emanating half-edges in the plane is p, q, r, s in clockwise order, and in the second way we have p, s, r, q, see Fig. 4.5.

The obtained diagrams differ from each other by the virtualization of the corresponding crossing. □

4.2.4 *Chord diagrams*

An important object which is used in many problems of knot theory (in particular in studying combinatorial structures of Vassiliev invariants) is a *chord diagram*.

All considered graphs are finite, multiple edges and loops are available. Let us give formal definitions of considered graphs.

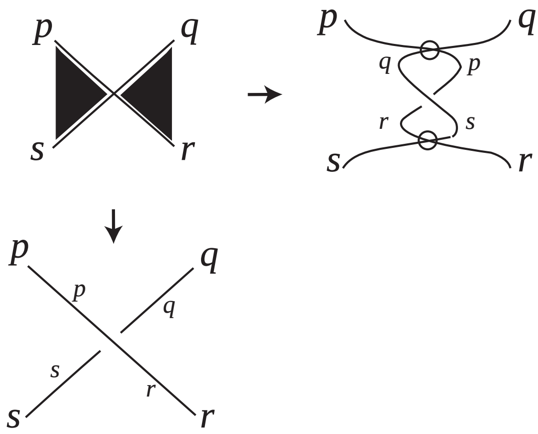


Fig. 4.5 Different embeddings of the frame of an atom and virtualization.

Definition 4.2. Let M be an oriented one-dimensional manifold (not necessarily connected). A *chord diagram* of order n on M is an embedding of unordered disconnected sum of n zero-dimensional circles $S^0 \sqcup \cdots \sqcup S^0 \to M$ considered up to an isotopy.

It is usual the manifold M is depicted as a collection of circles and lines in the plane, and pairs of points, images of zero-dimensional circles, are connected with each other by segments, chords.

Definition 4.3. A *framed* chord diagram is a chord diagram with each chord having framing 0 or 1.

If a chord diagram has no framing, then we assume that all framings are equal to zero.

In knot theory chord diagrams on one (non-oriented) circle are often considered (called just chord diagrams).

Namely, the following definition is more convenient.

Definition 4.4. By a *chord diagram* we mean a cubic graph consisting of one selected non-oriented Hamiltonian cycle (a cycle passing through all vertices of the graph exactly once) and a set of non-oriented edges (*chords*), connecting points on the cycle. Moreover, distinct chords have no common points on the cycle. We call this cycle the *core circle* of the chord diagram.

Edges of a chord diagram, not being chords, are called *arcs*.

Remark 4.3. We shall also consider *oriented chord diagrams*, i.e. chord diagrams with their core circles to be oriented. In that case we consider chord diagrams up to isomorphisms of graphs preserving the orientation of the core circle.

In what follows, by a chord diagram we mean a chord diagram on one circle unless otherwise specified.

4.2.5 *Passage from atoms to chord diagrams*

Atoms having exactly one black circle holds great interest. In this case, they can be encoded by a framed chord diagram as follows. Let us consider the sole black cell. Consider a circle lying inside this cell and passing near the boundary of the cell. This circle approaches to each atom vertex from two sides. Near each vertex let us pick out a pair of points on the circle and connect them by a chord. Define framing 0 or 1 on each chord as follows. Orient the circle in some (arbitrary) way. If neighborhoods of two points connected by a chord are oriented with respect to each other (i.e. either in clockwise order or in counterclockwise order), then we shall give this chord a framing 0, otherwise we shall give a framing 1.

Examples of two different orientations are shown in Figs. 4.6(a) and 4.6(b), respectively.

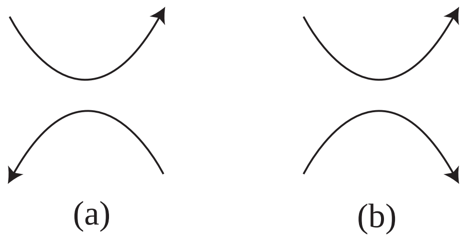


Fig. 4.6 Orientations.

If we consider a circle with chords as an abstract graph, then we shall get a *framed chord diagram*. In the case of an oriented atom, all framings of a chord diagram are equal to 0.

Definition 4.5. We say that two chords a and b of a chord diagram D are *linked* if the ends of the chord b belong to two different connected components of the complement to the ends of a in the core circle of D.

Otherwise we say that the chords are *unlinked*.

Remark 4.4. As a rule, a chord diagram is depicted on the plane as the Euclidean circle with a collection of chords connecting endpoints of chords (intersection points of chords are not, of course, vertices). If we place all chords inside the core circle, then two chords are linked if and only if they intersect each other in the plane.

We get the following theorem.

Theorem 4.3. *For each equivalence class of virtual links there exists a virtual diagram having the corresponding atom with one black cell. Herewith for classical knots this diagram can be chosen as a classical one.*

Proof. Let us consider an arbitrary diagram of a virtual diagram and the corresponding atom. Assume that the atom has k supercritical circles. If $k = 1$, then we have the claim of the theorem.

If $k > 1$, then there exists a vertex v of the atom such that in a small neighborhood of it we have two different supercritical circles.

Apply move Ω_2 to this diagram as shown in Fig. 4.7.

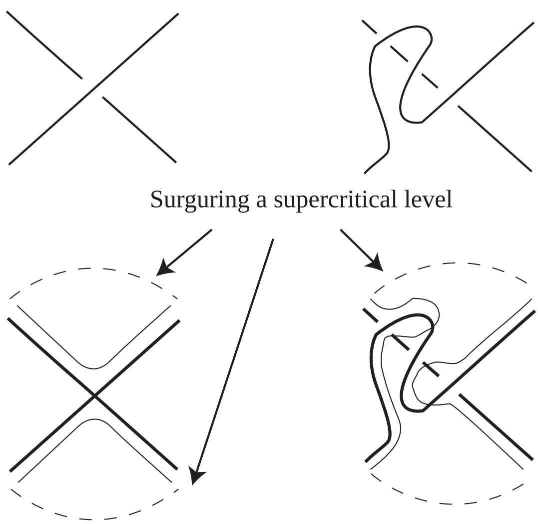


Fig. 4.7 A link diagram (upper) and supercritical circles (lower).

It is easy to see that after this application, two circles shown in Fig. 4.7 emerge as one circle: Supercritical circles are depicted in Fig. 4.7 by thick lines, and their behavior outside the transformation domain is depicted by a dash line. Thus, we decrease the number of supercritical circles by one without changing the isotopy class of our link. Iterating this operation as many times as we need, we get a diagram with precisely one supercritical circle.

Since the transformations described above are only applied to classical crossings and they create no virtual crossings, then classical diagrams will be transformed to classical ones. □

A virtual knot up to detour moves and virtualizations is defined from a framed chord diagram (on several circles, see Definition 4.3). Herewith, in the case of unframed chord diagrams we shall get an oriented atom and a virtual knot corresponding to it (up to detour moves and virtualizations).

In the sequel (see Sec. 4.5) we shall give a more explicit construction of a virtual knot from an unframed chord diagram by dividing the set of its chords into two sets. In that case, a virtual diagram will be well defined (up to detour moves), and, therefore, the virtual link will be well defined.

As it was shown, Sec. 4.2.3, by having a virtual link diagram we could restore the corresponding atom (recall that the operation of constructing the atom from a virtual link diagram is well defined), and, if the diagram is classical, then the atom is height. Indeed, in the case of a classical diagram, by definition, we have the frame of the atom, embedded in the plane with the A-structure preserved.

Corollary 4.3. *If an atom* At *corresponding to a virtual diagram K is height* (*i.e. have the height zero*), *then the diagram K is obtained from a classical diagram by detour moves and virtualizations.*

Indeed, a diagram K' corresponding to a height atom At is classical by construction. The diagrams K and K' are obtained from each other by detour moves and the virtualizations according to Theorem 4.2.

Definition 4.6. A *d-diagram* is a chord diagram such that its chords can be divided into two sets in such a way that the chords from each set are not linked with each other. Examples of chord diagrams not being d-diagrams are shown in Fig. 4.8.

A *framed d-diagram* is a d-diagram having all chords with framing 0.

If an atom is height and has exactly one black cell, then the obtained chord diagram will be a framed d-diagram. By Jordan's theorem, a circle divides the plane into two parts. Further, all chords correspond to the

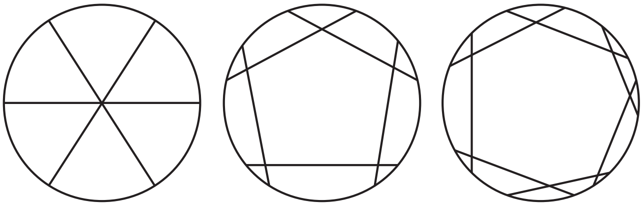


Fig. 4.8 Chord diagrams not being d-diagrams.

points where the circle embedded in $\mathbb{R}^2$ approaches itself. All these places of approach are of two sorts: interior and exterior (with respect to the circle) which correspond to two sets of chords of the d-diagram. Neighborhoods of two points connected by a chord are oriented with respect to each other, therefore, we have chords with framings equal to zero.

By a framed chord diagram the atom is restored uniquely. Indeed, a chord diagram defines the frame of the atom with the A-structure and the rule of pasting the sole black cell, i.e. the B-structure.

In the case when an atom has more than one black cells, we have to place one circle inside each cell, orient these circles arbitrarily and give framings to chords depending on coordinations of orientations of circle parts joined to a given vertex.

A passage from atoms to cubic graphs with framing was suggested by Oshemkov [261]. The constructed graph is called a *f-graph*.

An abstract f-graph is a framed chord diagram (i.e. a diagram with framing) on several circles. f-Graph is defined up to the following operation: Swapping the orientation of some circle and swapping framing of each chord connecting a vertex of the given circle with a vertex on another circle (if a chord connects the circle with itself, then the framing changes twice, i.e. it stays unchangeable). In the case when the surface is oriented, orientations of circles can be chosen accordingly such that all framings are equal to zero.

The atom is uniquely restored from the f-graph (more precisely, its equivalence class), see [261].

If orientations of circles and framings of edges are not pointed, then we assume that circles of all f-graphs are oriented accordingly and framings are zero. When depicting chord diagrams and f-graphs in the plane, the orientations of circles of chord diagrams and f-graphs will not be pointed out. If there is no framing on a diagram, then we assume that the framing is equal to zero.

When depicting one circle in the plane, we shall not point out its orientation, by assuming that it is oriented in counterclockwise order.

For oriented diagrams K, K' of virtual links there are two (formally different) equivalence relations: The general virtual equivalence and the *oriented equivalence*, in which the existence of a chain of oriented diagrams $K = K_0, \ldots, K_n = K'$ is required, where two neighboring diagrams differ from each other by one generalized Reidemeister move.

In [86] the following theorem was formulated as the conjecture of orientability.

Theorem 4.4 ([86]). *These two equivalence relations coincide.*

The proof of Theorem 4.4 follows from Kuperberg theorem (Theorem 2.1), and we shall also prove it in Chaps. 8 and 9.

4.2.6 *Spanning tree for the Kauffman bracket polynomial*

Let us give a fast way for computing the Kauffman bracket polynomial suggested by Thistlethwaite [285] for classical knots. For virtual knots, it can be generalized straightforwardly. In the sequel, this method will be especially useful for working with the Khovanov homology.

Given a virtual link diagram K. The relation

allows one to smooth classical crossings step-by-step. If having chosen some order of a smoothing of all crossings, we smooth our diagram completely, then we shall get a collection of circles having no classical crossings (i.e. a collection of immersed circles having intersection points and self-intersections at virtual crossings).

In other words, by this method we reduce the Kauffman bracket polynomial of the initial link diagram to a linear combination of values of the Kauffman bracket polynomial for simplest diagrams of trivial links which are equal to $(-a^2 - a^{-2})^{(\gamma-1)}$, where γ is the number of circles in the corresponding state. Thus, we get the formula (4.1), where the sum is taken over all states.

The idea of Thistlethwaite is as follows. We smooth a link diagram not to a diagram without classical crossings but to some intermediate diagrams which still have classical crossings and are trivial links.

This construction works for virtual knots. We smooth classical crossings and as a result, we have a collection of diagrams of virtual links with classical and virtual crossings.

Let us describe this construction in more detail. Let Q be the *state cube* of K, i.e. the set of states written as the discrete cube $\{0,1\}^n$, where each zero (respectively, one) corresponds to the A-smoothing (respectively, B-smoothing) of the corresponding crossing.

Let us call a collection of states obtained by fixing values of some coordinates, zero or one, a *subcube* of the state cube. Let $Q_1, \ldots, Q_k$ be subcubes of Q not sharing common vertices such that the set of vertices of all Q_i gives the set of vertices of Q, see Fig. 4.9.

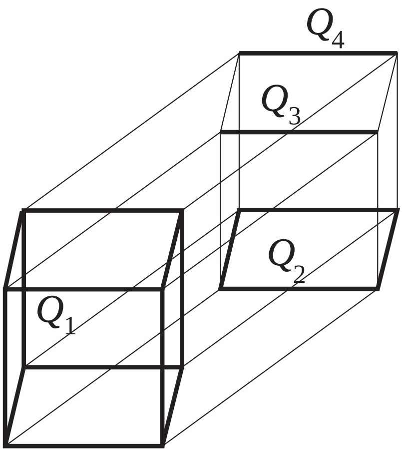


Fig. 4.9 Splitting a four-dimensional cube into subcubes.

Each subcube is obtained from Q by specifying the value for some set of coordinates, i.e. a choice of smoothings in all vertices of this set. Each subcube Q_i gives the link K_{Q_i} obtained from the initial link by smoothing those crossings of the diagram, to which coordinates being constant on all vertices of Q_i correspond. Further, we shall write $\langle Q_i \rangle$, keeping in mind $\langle K_{Q_i} \rangle$.

Herewith, each smoothing of K lies in some cubes of Q_i. Then by definition of the Kauffman bracket polynomial we have:

$$\langle K \rangle = \langle Q \rangle = \sum a^{f(i)} \langle Q_i \rangle,$$

where $f(i)$ means the sum of signs (the sign $+1$ if we have the A-smoothing, and the sign -1 if we have the B-smoothing) of those crossings the smoothings of which correspond to the ith cube.

Let us consider the cube Q and denote by $\mathcal{V}_1$ the set of states, in which the number of circles is equal to one.

Let us now enumerate all classical crossings of K by numbers from one to n. The general process of a smoothing is as follows. At the ith step we smooth the classical crossing with the number i in two ways. At length we come to all 2^n states.

Instead of this, in some steps we shall not smooth classical crossings but we shall make them unchangeable. Thus, at the ith step each diagram either splits into two diagrams at the ith crossing or passes to the next level with non-smoothed ith crossing. More precisely, at the kth step we shall get a collection of diagrams where each of them has a smoothing of some crossings with numbers from the set $\{1, \ldots, k\}$.

In each step some subcube corresponds to each such diagram. Herewith, the union of subcubes at each level gives the initial state cube.

Note that the following formulae and the algorithm *depend* on the numeration of crossings, but in the end we shall get a formula for computing the Kauffman bracket polynomial of the initial link, i.e. the final result will be the same.

Definition 4.7. A *splitting point* of a connected virtual diagram is a crossing of it such that after removal of the vertex of the frame of the corresponding atom $\mathrm{At}(K)$, which corresponds to this crossing, this frame decomposes into more than one connected components.

The rule for crossings with or without smoothings is as follows. We shall smooth those crossings which are not splitting points.

Thus, in the end we shall get some collection of "middle smoothings"(diagrams) which will be denoted by $\mathcal{S}_K$. Each of these smoothings gives some virtual link, here the subcubes corresponding to these virtual diagrams exhaust the whole state cube. To each middle smoothing $S \in \mathcal{S}_K$ corresponds a diagram. This diagram represents a link K_S. According to [285], we assert that:

(1) each of the links K_S is the unknot;
(2) elements of the set $\mathcal{S}_K$ is in one-to-one correspondence with elements of the set $\mathcal{V}_1$.

Indeed, each diagram corresponding to a smoothing from $\mathcal{S}_K$ possesses the property that at each classical crossing one of the smoothings of this diagram leads to a tiling. From this it follows that each such diagram can be reduced to the trivial diagram by applying only first decreasing Reidemeister moves (and the detour move at virtual crossings).

If we smooth the diagram K_S further, then a "splitting" smoothing at any crossing leads to the splitting of the diagram. Therefore, from all states of the diagram K_S, only one will correspond to the state with one circle. Namely, this will be the state obtained by smoothing all crossings by "non-splitting" ways. It is easy to see that in this case we actually have exactly one circle.

Note that the Thistlethwaite's construction also works when we divide the state cube into subcubes corresponding to trivial diagrams. Whatever tiling of the state cube into subcubes we take, we shall get an analogous decomposition of the Kauffman bracket polynomial.

Since all states of the diagram K are obtained by further smoothings of K_S, we shall get the required correspondence.

For a state s from $\mathcal{V}_1$ denote by K_s the diagram corresponding to a smoothing from the set $\mathcal{S}_K$, a further smoothing of which leads to the state s.

Thus, by computing the Kauffman bracket polynomial of the initial diagram K, we can summarize all states from $\mathcal{V}_1$ by meaning herewith the corresponding diagrams from the set $\mathcal{S}_K$. Both diagrams in states of $\mathcal{V}_1$ and diagrams from $\mathcal{S}_K$ give the unknot (the one-component link). Here, the diagram K_s corresponding to a state $s \in \mathcal{V}_1$ has some (not necessarily equal to zero) writhe number $w(K_s)$, which leads us to the additional term $(-a)^{3w(K_s)}$ when we compute the Kauffman bracket polynomial. Besides this, when we pass from the diagram K to the diagram K_s, we smooth some number of crossings positively, and some number of crossings negatively. Denote the corresponding difference by $r(s)$.

In the end we get the formula for computing the Kauffman bracket polynomial:

$$\langle K \rangle = \sum_{s \in \mathcal{V}_1} (-a)^{3w(K_s)} a^{r(s)}.$$

An example of evaluating the Kauffman bracket polynomial by Thistlethwaite's method for the trefoil is shown in Fig. 4.10.

From this picture we get:

$$\langle K \rangle = (-a)^3 a^2 + (-a)^{-3} + (-a)^{-6} a^{-1} = -a^5 - a^{-3} + a^{-7},$$

the value of the Kauffman bracket polynomial in the right trefoil.

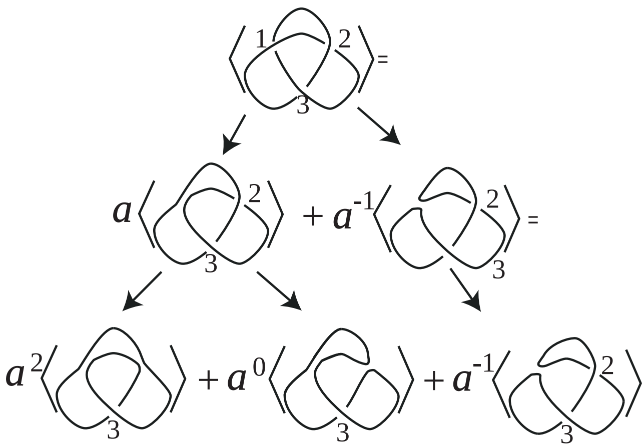


Fig. 4.10 Evaluating the Kauffman bracket polynomial by Thistlethwaite's method.

4.3 The polynomial Ξ: minimality problems

Definition 4.8. A *reduced* diagram is a connected diagram of a virtual link without splitting points.

In this section, all diagrams are assumed to be connected, and link diagrams are assumed to be reduced, unless otherwise specified.

Definition 4.9. The *span* of a Laurent polynomial $P(x)$ in one variable x is the difference between its leading degree and lower degree occurring in it with non-zero coefficients. Notation: $\operatorname{span} P(x)$.

Theorem 4.5. *Let K be a reduced diagram of a virtual link with n classical crossings. Then* $\operatorname{span}\langle K\rangle \leqslant 4n + 2(\chi - 2)$, *where χ is the Euler characteristic of the corresponding atom.*

Proof. We have to estimate the difference between its leading degree and lower degree occurring in the Kauffman bracket polynomial with non-zero coefficients.

Consider the A- and B-states of the diagram K: In the first state, all crossings are smoothed positively, and in the second, all crossings are smoothed negatively. Let us denote the numbers of circles in these states by $\gamma_{\max}$ and $\gamma_{\min}$, respectively. It is easy to see that no state can contain a monomial with the degree more than the degree of the leading term in the A-state ($\alpha = n$, $\beta = 0$), and no state can contain a monomial with the degree less than the degree of the lowest term in the B-state (the opposite state to the A-state at all crossings).

In the maximal state let us consider the leading term with the degree $n + 2(\gamma_{\max} - 1)$, and in the minimal state the lowest term with the degree $-n - 2(\gamma_{\min} - 1)$.

It is evident that the span of the Kauffman bracket polynomial $\langle K \rangle$ does not exceed $2n + 2(\gamma_{\max} + \gamma_{\min} - 2)$.

Taking into account $\chi = n - 2n + (\gamma_{\max} + \gamma_{\min})$, we shall obtain the desired claim. □

Definition 4.10. We call a virtual diagram K with n crossings 1-*complete*, if span $\langle K \rangle = 4n + 2(\chi - 2)$.

An orientable diagram is 1-*complete* if so is the non-orientable underlying diagram.

Definition 4.11. A *minimal* diagram of a virtual link with n classical crossings is a diagram such that any diagram of this link has at least n classical crossings.

Remark 4.5. Note that on the set of classical links there are two formal different definitions of a minimal diagram. One of them means the minimality in the classical category, the second one in the virtual category. If a classical diagram is minimal in the virtual category, then it is minimal in the classical category by definition. The question whether the inverse statement is true is still unsolved. This question is one of the problems formulated in [86].

From Theorem 4.5 one can get a series of corollaries concerning the estimations of the number of classical crossings.

Let K be a virtual diagram with n classical crossings, and let the corresponding atom have the Euler characteristic χ, and span $\langle K \rangle = 4n + 2(\chi - 2)$.

Then if there exists a diagram K' with n' classical crossings and $n' < n$, then the atom corresponding to the diagram K' must have the Euler characteristic more than the Euler characteristic of the atom corresponding to K. From this the minimality of many diagrams having non-negative Euler characteristics (for example, diagrams lying in the sphere, torus, Klein bottle, projective plane) follows. A particular case of this is the famous Kauffman–Murasugi–Thistlethwaite theorem about the minimality alternating diagrams of classical links.

This theorem was independently proved by Kauffman, Murasugi [253] and Thistlethwaite [285] in the year 1987.

Theorem 4.6 (Kauffman–Murasugi–Thistlethwaite). *The span of the Jones–Kauffman polynomial for links with connected shadow and n classical crossings is less than or equal to $4n$. The equality holds only for alternating diagrams without splitting points or connected sums of such diagrams.*

Kauffman–Murasugi–Thistlethwaite theorem was proved for classical links. Here we give the proof of this theorem for virtual links.

However, for some reason or other the span turns out to be less than $4n+2(\chi-2)$ (the leading and the lowest terms cancel). In this case we can appeal to stronger invariants: the polynomial Ξ or the Khovanov complex which will be defined further.

On the other hand, the leading term of the Kauffman bracket polynomial is interesting by itself. We shall say about it in Chap. 7 devoted to graphs and Vassiliev invariants and flat curves.

Let us give an example of a "simple" minimality theorem.

Theorem 4.7 ([221]**).** *Let K be a virtual diagram with n classical crossings representing a virtual link not having unconnected diagrams. Let one of the following two conditions hold*:

(1) $\mathrm{span}\,\langle K\rangle = 4n-2$;
(2) $\mathrm{span}\,\langle K\rangle \geqslant 4n-4$, *herewith in $\langle K\rangle$ we have monomials having degrees not congruent to each other modulo four.*

Then the diagram K is minimal (in the class of non-split diagrams).

Proof. Let us assume that there exists a diagram K' with m classical crossings and $m \leqslant (n-1)$.

In the first case let us consider the atom corresponding to the diagram K'. Since K' is connected, the atom corresponding to it is connected and has the Euler characteristic not exceeding two. From Theorem 4.5 we get $\mathrm{span}\,\langle K\rangle = \mathrm{span}\,\langle K'\rangle \leqslant 4(n-1)$, a contradiction.

In the second case any atom corresponding to a diagram of the link generated by the diagram K' is unoriented, since $\langle K'\rangle = \langle K\rangle$ has monomials with degrees not congruent to each other modulo four. Therefore, the Euler characteristic χ of this atom does not exceed one (as we have a non-split diagram). According to Theorem 4.5, we have $\operatorname{span}\langle K\rangle = \operatorname{span}\langle K'\rangle \leqslant 4(n-1)-2$, a contradiction. □

Definition 4.12. A *quasialternating* diagram is a diagram of a virtual link obtained from an alternating diagram of the classical link by virtualizations and detour moves, see Fig. 4.11.

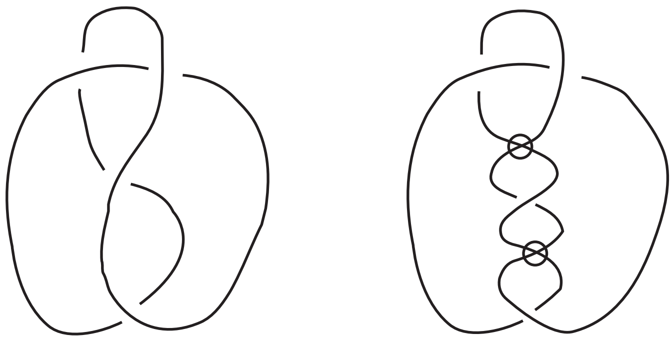


Fig. 4.11 An alternating classical diagram (the figure eight knot) and a quasialternating diagram.

From the definition of spherical atom and Theorem 4.2, we have the following proposition.

Proposition 4.3. *An atom corresponding to a connected diagram of a virtual link is spherical if and only if the diagram is quasialternating.*

In terms of atoms, the definition of a connected diagram can be reformulated as follows.

Definition 4.13. A virtual diagram is called *connected* if the frame of the corresponding atom is connected.

For the estimate from Theorem 4.5 to be sharp, it is sufficient to have the coefficients of the leading term with degree $n+2(\gamma_{\max}-1)$ and the lowest term with degree $(-n-2(\gamma_{\min}-1))$ not to be equal to zero. This condition is easily formulated in terms of atoms. Moreover, from non-cancellation of the leading term and lowest term one can obtain the non-cancellation of the leading term and lowest term for *cables*, see definition below.

Definition 4.14. A virtual diagram K is called *adequate* [285], if every vertex of the corresponding atom $\mathrm{At}(K)$ is incident to exactly four different cells. In other words, no cell touches itself at a vertex.

The adequacy is a sufficient (but not necessary) condition for the leading monomial in the A-state of the Kauffman bracket not to cancel with other monomials and the lowest monomial in the B-state not to cancel with other monomials. This follows from the fact that in the expansion (4.1) none of the states other than the A-state gives monomials having degree equal to the leading degree. Analogously, none of the states other than the B-state gives monomials having degree equal to the lowest degree.

This property is one of the key properties for detecting minimality of some diagrams.

Let us fix one of the two colors, black or white.

Definition 4.15. A *semiadequate* (*white* or *black*) diagram is a diagram for every vertex of which the two adjacent white (respectively, black) cells are different. Adequate diagram is white-semiadequate and black-semiadequate at the same time.

An important role in classical and virtual knot theory, and in three-dimensional topology is played by the notion of a framed link.

By a framed link (see Definition 1.1) we mean the equivalence class of virtual knots modulo generalized Reidemeister moves, where instead of the first classical Reidemeister move Ω_1 we consider the "double move" Ω_1^2, shown in Fig. 4.12.

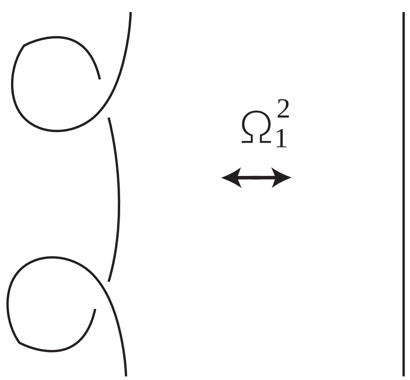


Fig. 4.12 The double loop.

We call Ω_1^2, Ω_2, Ω_3 and the detour move the (*generalized*) *framed Reidemeister moves.*

Framed classical diagrams have the following geometric meaning. With each link component of such a diagram one can associate a number called the *self-linking number*. This number will be invariant under all framed Reidemeister moves.

For each virtual link diagram K with n classical and m virtual crossings one can define its *cables*: the double diagram $D_2(K)$, the triple diagram $D_3(K)$ and so on. This is done as follows. Consider the diagram K and draw $(p-1)$ parallel copies of it near it. One can assume, for example, that every point on a branch of the diagram is transported by small intervals of lengths $\varepsilon, 2\varepsilon, \ldots, (p-1)\varepsilon$ along the normal vector on the plane. The resulting diagram $D_p(K)$ will have p^2n classical and p^2m virtual crossings; instead of each crossing we get p^2 new ones; if at a classical crossing of K some branch a forms an overcrossing when it intersects another branch b, then in the diagram $D_p(K)$ each of the p "parallel" branches corresponding to a will form an overcrossing with each of the p branches parallel to b; the double of the simplest trefoil diagram is shown in Fig. 4.13.

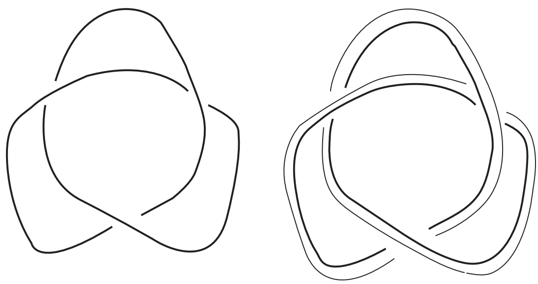


Fig. 4.13 Doubling the trefoil: the parallel copy is shown by a thin line.

Let us give a geometric sense why we forbid the first Reidemeister move for framed links. Consider a classical knot diagram K and its double $D_2(K)$. We can view the latter as a locally flat band in the three-space, whose boundaries are the two parallel knots K and K'. The band in the space is uniquely defined by the knot (one boundary component) and the linking index between the boundary components.

Thus, the first Reidemeister move Ω_1 does not change the knot isotopy class but it does change the isotopy class of the band since it switches by one the linking coefficient for the boundary components of this band. The move Ω_1^2 does not change the linking coefficient.

Analogously, in the case of virtual knots the replacement of Ω_1 with Ω_1^2 leads to framed virtual knots given by bands in stabilized 3-manifolds of the form $S_g \times I$.

If we apply a detour move to the diagram K, the diagrams $D_p(K)$ are operated by a composition of detour moves. If we apply to K the second or third classical Reidemeister move, then the diagram $D_p(K)$ is also transformed to an equivalent diagram. Consequently, for an arbitrary natural p the p-cabling operation $K \mapsto D_p(K)$ is a well-defined operation for framed virtual links.

As mentioned above, this operation is not well defined for virtual links. When applying the move Ω_1 to the diagram K, the two "parallel" components $D_2(K)$ change its linking coefficient.

Lemma 4.1. *Assume that a virtual diagram K with n classical crossing is adequate. Then for every p, the diagram $D_p(K)$ is adequate, i.e. the following equality holds*:

$$\operatorname{span}\langle D_p(K)\rangle = 4p^2 n + 2(\chi_p - 2),$$

where χ_p is the Euler characteristic of the atom corresponding to $D_p(K)$.

Proof. Every crossing v of the diagram $D_p(K)$ originates from some crossing v' of the diagram K: to the latter, one assigns p^2 crossings of $D_p(K)$.

We shall show that for every p the diagram $D_p(K)$ is adequate. Indeed, if there is a crossing v of $D_p(K)$ that some white circle attaches from two sides, then the crossing v' is also incident to one white circle from two sides. Thus, the diagram K would be inadequate, too. Furthermore, the equality described in the lemma follows from the count of the difference between the leading and lowest monomials of the Kauffman bracket $\langle K\rangle$ of K. □

From Lemma 4.1 we get the following theorem.

Theorem 4.8. *Let a (non-split) diagram K of a framed virtual link be adequate. Then the framed link has no (non-split) diagrams with number of crossings strictly less than that of the diagram K.*

In the case of classical knots this theorem was known to Thistlethwaite [286]; this statement also follows immediately from Khovanov homology arguments. We shall give an elementary proof of this statement based on the cabling techniques.

***Proof of Theorem* 4.8.** Let us consider diagrams $D_2(K)$, $D_3(K), \ldots$. With them we can associate framed links, cables of our framed link: a double diagram, a triple diagram and so on. We shall investigate the span of $\langle D_p(K)\rangle$ which is an invariant of K depending on the parameter p.

Let the atom corresponding to the diagram K have the Euler characteristic χ. It means that the number Γ of cells of the atom is equal to $\Gamma = \chi + n$. Let us consider the atom corresponding to the diagram $D_p(L)$. We have p^2n vertices, $2p^2n$ edges and $p\Gamma$ 2-cells (for each vertex of the initial atom we have p^2 vertices for the cable, the number of edges is twice more than the number of vertices, and the number of sub- and supercritical circles under cabling increases by p times). Thus, the Euler characteristic of the atom equals

$$\chi_p = -p^2n + pn + \chi.$$

From Lemma 4.1 it follows that the leading and lowest degrees of the Kauffman bracket polynomial of the diagram D_p do not cancel (all cables of the initial diagram are adequate diagrams). Therefore, according to Theorem 4.5, the span of the Kauffman bracket polynomial of the diagram D_p equals

$$\operatorname{span}\langle D_p(K)\rangle = 4p^2n + 2(\chi_p - 2) = 2(p^2+p)n + 2p\chi - 4.$$

Assume that the framed link has a diagram K' having n' classical crossings such that the corresponding atom has the Euler characteristic χ' and $n' < n$. Then according to Theorem 4.5, arguing by the same reason, we have

$$\operatorname{span}\langle D_p(K')\rangle = 2(p^2+p)n' + 2p\chi' - 4.$$

Therefore,

$$(2p+2)(n'-n) = 2(\chi - \chi').$$

Since $n' - n \leqslant (-1)$, and p can be chosen to be arbitrary large, we shall get that the Euler characteristic χ' is bigger than an arbitrary preassigned number, a contradiction. □

Remark 4.6. Note that in the proof it is sufficient for us that the adequate property is true only for some set of diagrams $D_{p_i}(K)$, $i \in \mathbb{N}$. The diagram K can be inadequate at the same time.

Minimality Theorems 4.7 and 4.8 can be strengthened after introducing more powerful invariants generalizing the Jones polynomial: a Ξ polynomial, the Khovanov homology.

For this reason it is sufficient for us to replace the span of the Kauffman bracket polynomial with the span (with respect to the variable a) of the polynomial Ξ or with $2(h-2)$, where h is the height of the Khovanov homology, see Definition 5.3.

4.3.1 *The leading and lowest terms of the Kauffman bracket polynomial*

Let K be a virtual diagram. Let us find the lowest term in the Kauffman bracket polynomial $\langle K \rangle$ according to the formula (4.1). In the case of quasialternating diagrams of links (without splitting points) it is equal to one since a supercritical circle of the corresponding atom (lying in the sphere) cannot touch itself; by the same reason the term is equal to one for adequate diagrams.

We are interested in cases when it is not equal to zero (but not necessary equal to one).

In the general case, the atom $\mathrm{At}(K)$ corresponds to a virtual diagram K, this atom is not necessarily spherical. If a diagram is not adequate, then some supercritical circle can approach itself without forming splitting points in the atom. The way of approaching the circle to itself is naturally described by a chord diagram (which in the classical case is a framed d-diagram). If we have exactly one supercritical circle, then the lowest term of the Jones–Kauffman polynomial is described by some function on such d-diagrams which after natural extending to arbitrary chord diagrams gives symbols of some Vassiliev invariants.

The description of this function is given in [14, 15, 201]. This function is related to Vassiliev invariants. This theme will be discussed in Chap. 7.

Let K be a diagram of a link with n classical crossings, $\mathrm{At}(K)$ be the atom corresponding to it, having n vertices. Let us consider supercritical circles $c_1, c_2, \dots, c_l$ of $\mathrm{At}(K)$. Let these circles correspond to the B-state of $\mathrm{At}(K)$. The impact of this state in the lowest degree of the expansion (4.1) of the Kauffman bracket polynomial equals $(-1)^{l-1}a^{-n-2l+2}$. Let us calculate the coefficient at this degree in $\langle K \rangle$.

Remark 4.7. This coefficient, generally speaking, is not a knot invariant. The problem is that saying the "lowest" term of a polynomial we mean its lowest non-zero coefficient. However, it turns out that the "lowest" coefficient in the expansion of the Kauffman bracket polynomial is equal to zero, and at the same time for some other diagram (for example, a diagram with less number of crossings) the analogous expansion gives a "real", i.e. non-zero, lowest coefficient.

A state s' at which m crossings are A-smoothed gives the minimal possible degree of a monomial if and only if in this state the number of circles equals $l+m$. This means that under passing from the B-state to the state s' by successively swapping smoothings at classical crossings we have to make transformations "splitting" one circle into two at each step. Let us connect by bridges arcs of circles in the state s' which pass in neighborhoods of classical crossings (atom vertices). We get a chord diagram on several circles.

Let us consider crossings in which the state s' differs from the B-state, and the chords corresponding to these crossings. It is clear that each such chord connects points in the same circle, otherwise having changed the smoothing in the corresponding crossing we should have obtained $l-1$ circles in the first step. From this it follows that in the mth step we cannot obtain more than $l+m-2$ circles.

Let us consider one circle C in the B-state and one chord c connecting a pair of points in the circle C. If we rebuild the B-state along the chord c, we shall get two circles. Further, in order that after rebuilding along the chord the obtained state gives a monomial with possible minimal degree both ends of each successive chord d in the diagram C have to lie on the same side from the chord c, since otherwise we can change the state s' with chords c and d and we get l circles but not $l+2$ circles, therefore, any changing along m chords, amongst which there are c and d, gives no more $l+m-2$ circles. We can conclude by continuing arguing in the same manner. To each circle of the B-state a chord diagram is assigned.

Definition 4.16. An *admissible*, i.e. leading to minimal degree of the monomial, collection of chords is a collection of chords in which any two chords corresponding to the same circle (of the chord diagram) are not linked.

Having an admissible collection of chords consisting of a_1 chords on the first circle, a_2 chords on the second circle and so on, a_k chords on the kth circle we shall get a state which gives the impact to the lowest coefficient $a^{-n-2k+2}$ equal to $(-1)^{\sum_i a_i}(-1)^{k-1}$. Factorizing $(-1)^{k-1}$ (further, we shall not take into account this coefficient), we get that the leading coefficient equals the sum $\sum_{l=0}^{n}(-1)^l N_l$, where N_l is the number of admissible collections with l chords. Thus, for calculating the lowest coefficient of the Kauffman bracket polynomial, chords connecting different circles do not matter and the problem is reduced to considering a collection of chord diagrams not connecting with each other. It is obvious that the final coefficient $M_1(D)$ is equal to the product of coefficients over all chord diagrams, where for each chord diagram it is equal to

$$\sum_{i=0}^{n}(-1)^i(\text{the number of collections of } i \text{ non-intersectiong chords}).$$

Similarly the leading term of the Kauffman bracket polynomial is found.

4.3.2 *The polynomial* Ξ

It turns out that in the formula (4.2) for the Kauffman bracket polynomial we can add a "geometry of virtual knots", which leads to a much stronger invariant of virtual links, see, e.g. [76, 248].

This is done by the following way. Let us consider the set S of all pairs (a smooth orientable surface, an unordered finite system of closed curves immersed in the surface). Here by a surface we mean an oriented closed manifold with a finite number of connected components (if the surface is not connected, then each of its components is assumed to be oriented). Some curves can also be oriented. Let us define an equivalence relation on S partially described in Chap. 1, but with some specifications (herewith we preserve the same notations: S and $\mathcal{S}$).

We shall consider elements from S up to equivalence relation generated by the following elementary equivalences:

(1) an orientation-preserving homeomorphism of surfaces sending curves to curves with the orientation preserved;
(2) stabilization (an addition of handles in such a way that respects the surface orientation and does not intersect curves) and destabilization (inverse to stabilization);
(3) a free homotopy of curves on the surface;
(4) an addition/deletion of simple unoriented curves bounding discs on the surface and not intersecting with other curves.

Denote the set of equivalence classes on S by $\mathcal{S}$. It is evident that this set is countable.

The question about whether two elements from S generate the same equivalence class in $\mathcal{S}$ is recognized algorithmically, since in the fundamental group of a 2-surface the problem of conjugacy is solved.

The basic idea of this invariant is the following. We construct a function on oriented virtual links valued in $\mathcal{S}\mathbb{Z}[a, a^{-1}]$; values of this function should be linear combinations of elements from $\mathcal{S}$ with coefficients from $\mathbb{Z}[a, a^{-1}]$, here, coefficients are constructed like monomials in the expansion of the Kauffman bracket polynomial, and elements from $\mathcal{S}$ represent a "geometry" of virtual knots.

Let K be a virtual link diagram. Let us consider the surface presentation of the virtual diagram as a diagram in an oriented 2-surface M, described in Chap. 1.

More precisely, each virtual link can be represented like a link in some thickened surface $S_g \times I$. Under the projection in S_g along I it represents a collection of curves, the number of curves is equal to the number of components of the link.

In this case all smoothings of the diagram can be performed on the surface M. Let us fix the shadow of the link K on M. It is a collection δ of oriented closed curves. Further, to each state s of the unoriented diagram $|K|$ (which can also be considered on the surface M) some collection $\delta'(s)$ of unoriented closed curves on M is assigned.

Thus, we get a collection of curves (some of them are oriented and the other part is not oriented) $p(s) = \delta \sqcup \delta'(s)$, which can be treated as an element from the set S and, therefore, as an element from the set $\mathcal{S}$. Let us consider the formal free module $\mathcal{M} = \mathcal{S}\mathbb{Z}[a, a^{-1}]$ over the ring of Laurent polynomials in a; generators of this module will be elements from $\mathcal{S}$.

Let us assign an element $\Xi(K) \in \mathcal{M}$ to the diagram K by the following formula:

$$\Xi(K) = (-a)^{-3w(s)} \sum_s p(s) a^{\alpha(s)-\beta(s)} (-a^2 - a^{-2})^{(\gamma(s)-1)}. \tag{4.3}$$

Theorem 4.9. *The function* Ξ *defined by the formula* (4.3) *is an invariant of virtual links under the generalized Reidemeister moves.*

Proof. It is obvious that purely virtual Reidemeister moves and the semivirtual move (i.e. detour moves) applied to K do not change $\Xi(K)$ at all: by construction all terms of (4.3) stay the same.

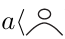
The proof of the invariance of $\Xi(K)$ under the first and third classical Reidemeister moves is quite analogous to the same procedure for the classical Jones–Kauffman polynomial, see, e.g. [221]; one should accurately check that the corresponding elements of $\mathcal{S}$ coincide and, therefore, they cancel each other in the formula for $\Xi(K)$.

In fact, if K and K' are two diagrams obtained one from the other by some first or third Reidemeister move, then for the diagrams $|K|$ and $|K'|$, the corresponding surfaces $M'(K)$ and $M'(K')$ with boundaries are homeomorphic, and the behavior of the system of curves γ for $M(K)$ and $M(K')$ differs only inside the small domain where the Reidemeister moves take place.

For the first move Ω_1, the two situations (corresponding to the twisted curls with local writhe number $+1$ or -1) are considered quite analogously.

Let K be a diagram and K' be the diagram obtained from K by adding such a curl . To each state s of $|K|$ there naturally correspond two states of $|K'|$. Fix one of them and denote it by s'. Let $K' = K \sqcup \bigcirc$ be the disconnected sum of K and a small circle. Then we have: $p(s) = p(s')$.

Indeed, both surfaces for $|K|$ and $|K'|$ are homeomorphic to each other and the only possible difference between the corresponding curve systems is one added circle (the elementary equivalence (4)). So, we have to compare terms with the same coefficients from $\mathcal{S}$. The comparison procedure coincides with that for the classical Jones–Kauffman polynomial. Namely,

$\langle$  $\rangle = a\langle$  $\rangle + a^{-1}\langle$  $\rangle = (-a)^3\langle$  $\rangle$,

herewith $(-a)^3$ is compensated by the corresponding factor $(-a)^{-3}$ originating from the writhe number.

Now, if we consider two diagrams K and K' obtained one from the other by using the third Reidemeister move, we see again that their surfaces coincide. Let us select the three vertices P, Q, R of the diagram K and the corresponding vertices P', Q', R' of the diagram K', as shown in the upper part of Fig. 4.14.

So, both diagrams K, K' differ only inside a small disc D in the plane. The same can be said about the surfaces $M(K)$ and $M(K')$ corresponding to the diagrams. They differ only inside a small disc D_M in M. Thus, one can indicate six points on the boundary ∂D such that all diagrams of smoothings (in M) of K, K' pass through these and only these points of ∂D.

Consider the three possibilities X, Y, Z of connecting these points shown in the lower part of Fig. 4.14. In fact, there are other possibilities to do it but only these will play a significant role in the future calculations of the expression (4.3).

We shall need the following three elements from $\mathcal{S}$ represented by K_X, K_Y and K_Z, see Fig. 4.15. The element K_X contains the three lines of the third Reidemeister move (with fixed six endpoints) inside D_M. It

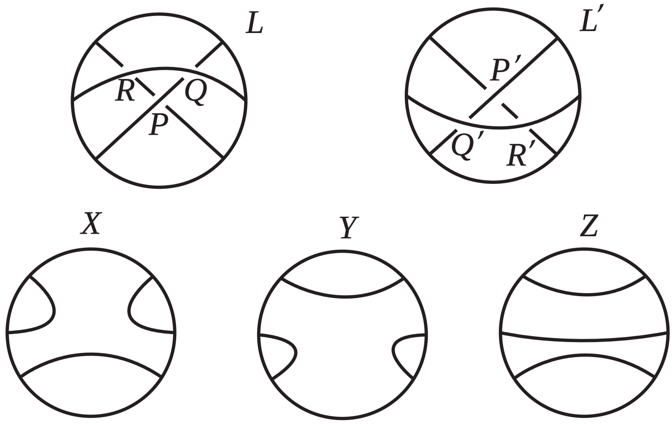


Fig. 4.14 Diagrams and lines after smoothings.

also contains X. Analogously, K_Y contains the three lines and Y, and K_Z contains the three lines and Z. The only thing we need to know about the behaviors of K_X, K_Y and K_Z outside D_M is that they coincide.

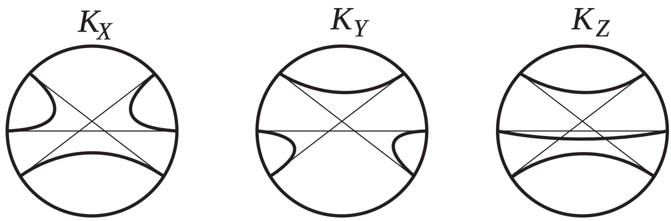


Fig. 4.15 Parts of diagrams K_X, K_Y, K_Z.

We have to prove that $\Xi(K) = \Xi(K')$. Obviously, we have $w(K) = w(K')$. So, we have to compare the terms of (4.3) for $|K|$ and $|K'|$. With each state of K, one can naturally associate the state of K'. We associate the crossings P, Q, R with the crossings P', Q', R' and in a natural way we identify the others. For each state of $|K|$ having the crossing P in position A (A-smoothing), the corresponding state of $|K'|$ gives just the same contribution to (4.3) as $|K|$ since diagrams $|K|$ and $|K'|$ coincide after smoothing P (respectively, P') in position A.

So, we have to compare all terms of (4.3) corresponding to the smoothing of P in position B. We shall combine these terms (for $|K|$ and $|K'|$) in fours, the terms from each "four" differ only in the way of smoothing the vertices Q and R. Now, let us fix the way of smoothing for $|K|$ and $|K'|$ outside D in the same way and compare the corresponding four terms. If we delete the interior of the disc D and insert there X, Y or Z, we obtain a system of curves in the surface M. Denote the numbers of curves in these three systems by ν_X, ν_Y and ν_Z, respectively.

Now, the four terms for $|K|$ give us the following:

$$\begin{aligned} &aK_X(-a^2 - a^{-2})^{(\nu_X - 1)} + a^{-1}\big(K_Z(-a^2 - a^{-2})^{(\nu_Z - 1)} \\ &+ K_X(-a^2 - a^{-2})^{\nu_X}\big) + a^{-3}K_X(-a^2 - a^{-2})^{(\nu_X - 1)} \\ &= a^{-1}(-a^2 - a^{-2})^{(\nu_Z - 1)}K_Z. \end{aligned}$$

Analogously, for $|K'|$ we have a similar formula with terms containing K_Z and K_Y. The latter terms are reduced, so we obtain the same expression:

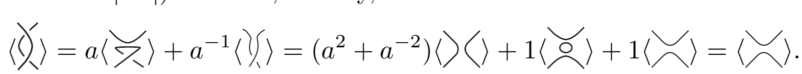
$$a^{-1}(-a^2 - a^{-2})^{(\nu_Z - 1)}K_Z.$$

Let us now check the invariance of Ξ under the second classical Reidemeister move. Let K' be the diagram obtained from K by applying the second classical Reidemeister move by adding two classical crossings. Obviously, $w(K) = w(K')$.

Consider the manifold $M(K)$. The image of K divides it into connected components. We have two possibilities. In one of them, the Reidemeister move is applied to one and the same connected component. More precisely, in this case on the surface $M(K)$ there exists a connected component $\widetilde{M}$ (as a result of dividing the surface by the link diagram) such that two projection branches of K taking part in the Reidemeister move are parts of the boundary of the component $\widetilde{M}$, herewith under the second Reidemeister move both branches are directed inside $\widetilde{M}$.

Then $M(K')$ is homeomorphic to $M(K)$, and curves from the sets $\delta(K)$ and $\delta(K')$ represent the same homotopy type, but under the application of the move we get two more crossings. In this case the proof of the equality $\Xi(K) = \Xi(K')$ is just the same as in the classical case (the reduction here treats no polynomials but elements from $\mathcal{S}$ with polynomial coefficients). Moreover, the proof is even simpler than that for the third move: We have to consider the sum of four summands for $|K|$ and $|K'|$.

In each case, three of them vanish, and the remaining ones (one for $|K|$ and one for $|K'|$) coincide, namely,



Taking into consideration that $w(K) = w(K')$, we get the desired result.

Finally, let us consider the case of the second Reidemeister move, where $M(K')$ is obtained from $M(K)$ by adding a handle. On this handle, two extra points P and Q appear, see Fig. 4.16.

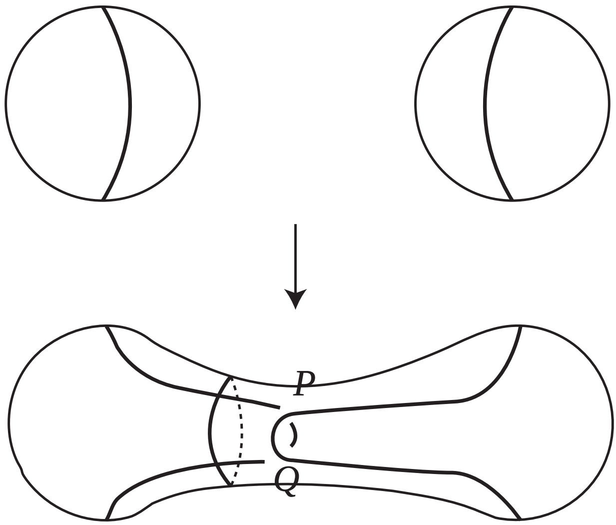


Fig. 4.16 Adding a handle while performing Ω_2.

Consider all states of the diagram $|K'|$. They can be split into four types depending on smoothing types of the crossings P and Q. Thus, each state s of $|K|$ generates four states s_{++}, s_{--}, s_{-+} and s_{+-} of $|K'|$. Note that $p(s) = p(s_{+-})$ (because of handle removal, see Fig. 4.16), and $p(s_{++}) = p(s_{--}) = p(s_{-+})$.

Besides, for each s, we have the following equalities:

$$\alpha(s) - \beta(s) = \alpha(s_{+-}) - \beta(s_{+-}), \gamma(s) = \gamma(s_{+-}),$$
$$\gamma(s_{++}) = \gamma(s_{--}) = \gamma(s_{-+}) - 1.$$

Thus, all terms of (4.3) for K' corresponding to s_{--}, s_{++} and s_{-+} will be reduced because of the identity $a^2 + a^{-2} + (-a^2 - a^{-2}) = 0$. The terms corresponding to s_{+-} give just the same as (4.3) for K. □

It is important to note that this invariant is constructive, since elements of the set $\mathcal{S}$ are algorithmically recognizable.

4.3.3 *Examples of applications of the polynomial* Ξ

On the one hand, the polynomial Ξ was constructed by using Vassiliev invariants of order zero for virtual knots.

On the other hand, it is a generalization of the Jones–Kauffman polynomial. We shall give examples showing that the polynomial Ξ is stronger than Vassiliev invariants of order zero and the Jones–Kauffman polynomial taken one with another, i.e. we give examples of links which cannot be distinguished from each other by Vassiliev invariants of order zero and the Jones–Kauffman polynomial, but the polynomial Ξ takes different values on them.

Moreover, the invariant Ξ gives an *obstruction to destabilization.*

Statement 4.2. *If a virtual link K can be represented by a link on the surface of genus g, herewith in $\Xi(K)$ there exists an element $s \in \mathcal{S}$ with non-zero coefficient such that the minimal representative of it has the genus g, then the link K has the underlying genus g.*

This statement follows from the definition of the polynomial Ξ.

Example 4.3. Let $P \in \mathcal{S}$ be the element represented by the sphere without curves. It is obvious that for each classical link K, we gave $\Xi(K) = P{\cdot}V(K)$. So, for the two-component trivial link $\widetilde{K}$ we have $\Xi(\widetilde{K}) = \cdot(-a^2 - a^{-2})$.

It is known that the two-component trivial link and the link K shown in Fig. 4.17 have the same Jones polynomial. Indeed, the link K is obtained from $\widetilde{K}$ by virtualizations.

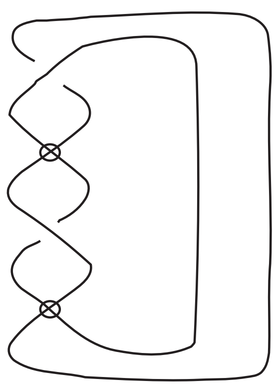


Fig. 4.17 A link with the trivial Jones–Kauffman link polynomial.

Consider the following two elements from $\mathcal{S}$ (for the sake of simplicity, we shall draw the elements of S), see Fig. 4.18. Here we consider the torus as the square with identified opposite sides.

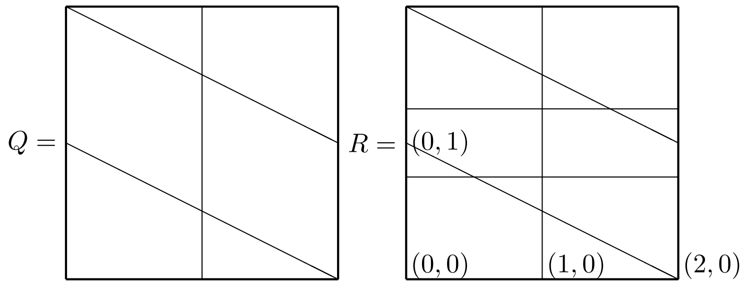


Fig. 4.18 Two elements from S.

The element $Q \in \mathcal{S}$ is initially represented by the same diagram shown in Fig. 4.18 with two additional circles (which appear under constructing Q from the diagram and are not shown in the figure) which can be removed by the equivalence (4) from Sec. 4.3.2. For the sake of simplicity we shall not distinguish elements from $\mathcal{S}$ with the collection of curves from S representing it.

Let us show that $Q \neq P$, $R \neq P$ and $Q \neq R$ in $\mathcal{S}$. Actually, $Q \neq P$ because Q has two curves with non-zero intersection ($+2$ or -2 according to the orientation); thus, none of these curves can be removed by the equivalences described above. So, $R \neq P$ either. Besides, $R \neq Q$ because R contains three different curves on the torus (in coordinates from Fig. 4.18 they are $(0, 1)$, $(1, 0)$, and $(2, -1)$); each two of them has non-zero intersection. Thus, none of them can be removed. So, the simplest diagram representing the class $[R]$ in $\mathcal{S}$ cannot have less than three curves.

Now, for the link K, we have

$$\Xi(K) = Qa^2 + 2R(-a^2 - a^{-2}) + Qa^{-2} = (2R - Q)(-a^2 - a^{-2}).$$

Thus, $\Xi(K) \neq \Xi(\widetilde{K})$.

Discussion. The invariant Ξ has the following disadvantage. The fact of the matter is that for each state s the element $p(s)$ in the formula (4.3) meets only once, i.e. curves $\delta'(s)$ are considered disjoint from curves $\delta'(\tilde{s})$ of any state. This can lead to the fact that each element $p(s)$ is individually undergone by simplifying equivalence (destabilization) in $\mathcal{S}$ while the initial knot cannot be destabilized.

For example, the knot K shown in the upper part of Fig. 4.19 has the polynomial Ξ reduced to the Jones polynomial, i.e. equals $X(K) \cdot P$.

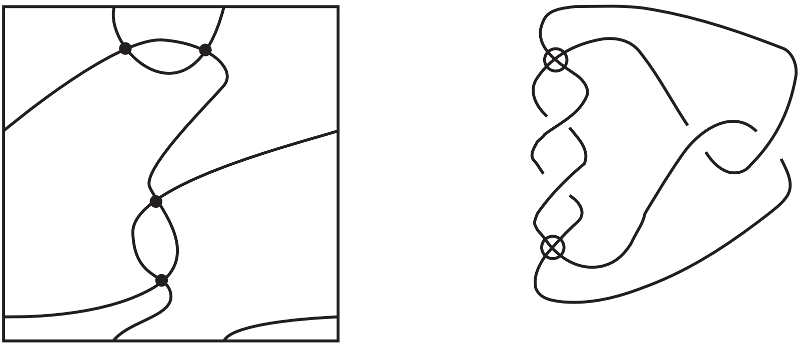


Fig. 4.19 A non-trivial virtual knot having the trivial polynomial Ξ.

Indeed it suffices to check that for the knot K in each state s the element $p(s)$ is trivial. Note that the element $p(s)$ consists of two collections of curves, one of them is $\delta'(s)$ consisting of non-intersecting arcs with each other; therefore, in $\mathcal{S}$ the element $\delta'(s)$ is trivial. Thus, if δ consists of contractible circles, then, whatever curves are in states s, the polynomial Ξ will have the "classical" view $X(K) \cdot P$.

The triviality of the collection δ in the given concrete case is easily checked. From this it follows that $\Xi(K) = X(K) \cdot P$.

Thus, we have the degeneration of the polynomial Ξ whenever the collection of curves $\delta'(s)$ is contractible on the surface.

Therefore, with the help of the polynomial Ξ we cannot say whether the knot K is equivalent to a classical knot (for example, the trefoil knot) or not.

The fact that the knot K is not classical will be shown in the sequel: For this we have to construct an invariant taking into consideration *all collections $\delta(s)$ for different s simultaneously* and, thereby, being more stable to such degenerations. Curves from $\delta(s)$ do not intersect each other for the same s, but curves from $\delta(s)$ and $\delta(s')$ can intersect each other as $s' \neq s$. By means of new ideas the latter can give an obstruction to destabilization.

Therefore, the following problem is actual: How does one use the information about curves $\delta'(s)$ of all states simultaneously? This idea belongs to Dye and Kauffman [76]. Further in this chapter we shall give a generalization which combines the idea with an idea of constructing the invariant Ξ, thereby, leading to a stronger invariant.

4.3.4 *A surface bracket and the invariant* Ξ

Let $g \geqslant 0$ be an integer. We begin with the construction of a knot invariant in sickened surfaces $S_g \times I$, where S_g is a closed oriented 2-surface of fixed genus g.

Let us consider the following module SurfM over the ring $\mathbb{Z}[a, a^{-1}]$. Generators are homotopy classes of unordered sets of unoriented curves in S_g (the empty set is called the unity) with the unique relation: $\alpha = (-a^2 - a^{-2})\beta$ if β is obtained from α by deleting a closed curve homotopic to the trivial one. In this section we shall construct a knot invariant in thickened surfaces valued in SurfM.

Oriented links in these thickened surfaces and oriented links in $\mathbb{R}^3$ are represented by their diagrams, herewith the equivalence relation is defined by the Reidemeister moves. Diagrams of these links are framed 4-valent graphs in the surface S_g with the over/undercrossing structure specified at each crossing and an orientation for each unicursal component is given. The Reidemeister moves are local moves and look like the usual (classical) Reidemeister moves.

For unoriented links one can define the surface bracket polynomial [76] as follows. Let $|K|$ be a diagram of an unoriented link in the surface S_g with n crossings (all crossings are classical; there are no virtual crossings on the surface). Define its state in the same manner as in the classical case. At each state we have $\alpha(s)$ crossings smoothed positively, and $\beta(s) = n - \alpha(s)$ crossings smoothed negatively. For each state s we have some collection $C(s)$ of closed curves (for the polynomial Ξ this collection is denoted by $\delta'(s)$) corresponding to this state. Some of these curves are contractible into a point in the surface S_g, and the others are not contractible. Let us pick out from $C(s)$ all null-homotopic circles; replace each of them with the factor $(-a^2 - a^{-2})$. Summing up over all states we shall get an element of the module SurfM which is denoted by $\widetilde{C}(s)$. Let us now set (for the sake of simplicity we shall use the brackets $\langle \cdot \rangle$ for denoting the new invariant different from the Kauffman bracket polynomial; the Kauffman bracket polynomial will not meet till the end of this section)

$$\langle |K| \rangle = \sum_s a^{\alpha(s)-\beta(s)} \widetilde{C}(s).$$

After that for the oriented link K we can define the *surface Kauffman* SurfK with the help of the same formula as for (4.2):

$$\text{SurfK}(K) = (-a)^{-3w(K)}\langle|K|\rangle.$$

The obtained function SurfK is a link invariant in thickened surfaces. It can be slightly simplified by considering $C(s)$ (and, therefore, $\widetilde{C}(s)$) *homologically*, but not homotopically, herewith we have to replace all null-homological circles with $(-a^2 - a^{-2})$.

The main disadvantage of this function is that it cannot be treated as an *invariant of virtual links*, since it is not invariant with respect to stabilization/destabilization and surfaces homeomorphisms. Having a concrete surface, we can define some coordinates on it (for example, we can choose basis in the linear space of one-dimensional homology), and write values of this invariant SurfK in these coordinates. But if we apply the stabilization and surface automorphisms many times, then it will be very difficult to keep track of these coordinates.

An advantage of this function is that we can consider it as an *obstruction to destabilization* (the polynomial Ξ has the same advantage).

Let us consider the set of minimal realizations of virtual links (in the sense of Chap. 2). On this set the surface bracket evaluated on such minimal surfaces is an invariant of virtual links by virtue of Theorem 2.1.

Let K be a virtual link having a realization on the surface $S_g \times I$. Let SurfK(K) look like $\sum \alpha_i\gamma_i$ in this realization, where γ_i is a set of non-trivial curves (obtained from $\widetilde{C}(s)$), and α_i are non-zero coefficients from $\mathbb{Z}[a, a^{-1}]$. Let $\Psi = \bigcup_i \gamma_i$ be the collection of all curves from all sets γ_i.

The main result of the paper [76] is the following theorem.

Theorem 4.10. *If a stabilization of $K \subset S_g \times I$ is possible, then there exist a curve ψ on S_g not homotopic to the trivial curve and representatives of homotopy classes of all elements from Ψ: $\psi_1, \ldots, \psi_N$ such that ψ intersects with no curve ψ_i.*

In contrast to the polynomial Ξ which is also an obstruction to destabilization, the polynomial SurfK considers simultaneously all non-trivial curves of states.

Based on these ideas (the polynomial Ξ and the surface bracket) let us construct a link *invariant* (the surface bracket is not that) as follows.

Let T be a set of collections: $(M, \alpha_1, \ldots, \alpha_n, \delta, \delta_1, \ldots, \delta_n)$, where M is an oriented closed 2-surface, $\alpha_1, \ldots, \alpha_n$ are elements from the ring of Laurent polynomials $\mathbb{Z}[a, a^{-1}]$, δ is a (finite) collection of oriented closed curves in M, and $\delta_1, \dots, \delta_k$ are collections of closed unoriented curves in M, with which Laurent polynomials α_i are associated. The surface M is closed and has a finite number of connected components.

Let us define the set $\mathcal{T}$ of equivalence classes on the set T by means of the following elementary equivalences:

(1) Renumbering of the collections δ_i with the corresponding renumbering of the coefficients α_i.
(2) Manifold homeomorphism sending the corresponding collections of curves to the corresponding collections of curves and preserving the manifold orientations and curve orientations.
(3) Free homotopy of curves in the manifold M.
(4) Removal of a null-homotopic curve from some collection δ_i and the multiplication of each element α_i by $(-a^2 - a^{-2})$.
(5) The operation inverse to the preceding one.
(6) Let two collections δ_i and δ_j be homotopic to each other (under coincidence of homotopy classes of all curves from the collections with multiplicities of curves taken into account). Let us delete these two collections and remain only one with the coefficient $\alpha_i + \alpha_j$ associated to it, and renumbering the corresponding indices (in an arbitrary order). Assume that the obtained collection is equivalent to the initial one.
(7) Removal (an addition) of a collection with zero coefficients.
(8) An addition to the surface (a removal from the surface) M handles which do not intersect any curve of any collection.

Let K be a diagram of a virtual link with n classical crossings. Let us consider a realization of K on the 2-surface M (described in the first chapter, see Fig. 1.20). The surface M inherits the orientation from the plane, and the image K' of the link diagram K in the surface is a finite collection of oriented curves. This collection will play the role of δ. Further, for each state s_i, $i = 1, \dots, 2^n$, we have a collection of curves δ_i. Set by definition

$$\alpha_i = (-a)^{-3w(s_i)} a^{\alpha(s_i) - \beta(s_i)},$$

$$\kappa(K) = (M, \alpha_1, \dots, \alpha_{2^n}, \delta, \delta_1, \dots, \delta_{2^n}).$$

As well as values of Ξ, we shall call values of the function κ by polynomials.

Analogously to Theorem 4.9, the following theorem can be proved.

Theorem 4.11. *The polynomial κ is an invariant of virtual links.*

For the detour move κ does not change by construction. The proof of the invariance of κ under the classical Reidemeister moves is analogous to the proof of Theorem 4.9.

Thus, the invariant κ is stronger than the polynomial Ξ and with it, is not weaker than the surface bracket as an obstruction to the reduction of the underlying genus.

4.4 Rigid virtual knots

Quantum invariants play an important role in the theory of knot invariants. These invariants are obtained from knots by putting special 4-valent vertices at crossings of a knot diagram, 2-valent tensors at local maxima and local minima (with respect to some height function on the plane) of the knot diagram and contracting these tensors along edges.

The invariance under the third Reidemeister move corresponds to the so-called Yang–Baxter equation, see, e.g. [259].

An important question is the question of extending quantum invariants of classical knots to invariants of virtual knots. A famous theorem due to Kauffman [158] claims that this is possible to do not for virtual knots themselves, but for another version of knots, called *rigid virtual knots* (and links); rigid virtual knots are equivalence classes of virtual knot diagrams modulo all generalized Reidemeister moves (local versions) except the first virtual Reidemeister moves. Rigid virtual knots were first introduced by Kauffman in [158]. For virtual crossings, we just put the permutation tensor. The reason behind the impossibility of extending quantum knot invariants is the following. If we allow both the first classical and first virtual Reidemeister moves, it will lead to three possibilities of "capping" a couple of vertical strands: We can either add a usual cap without crossings, or add a classical crossing and then a cap, or add a virtual crossing and then a cap. After contracting the tensors, this should lead to the same tensor. In the classical case this is not quite the same, because contracting the R-matrix by a "cap" does not give quite the same as the cap itself. However, in the classical case this is handled by adding a normalization factor corresponding to the writhe number of the knot. But if we try to look at rigid knots, we shall have to handle the similarity of a classical crossing and a virtual crossing at the level of similarity for the corresponding 2–2 tensors, which is generally impossible.

There is no such problem when we forbid the first virtual Reidemeister move.

A natural question arises: Which quantum invariants extend to virtual knots themselves without restrictions? Certainly, there are such ones, e.g. the Kauffman bracket polynomial and the Jones polynomial.

Thus, rigid virtual knots deserve special attention, since one can extend a large and important class of invariants to them. Besides, since we diminished the class of moves (deleted the first virtual Reidemeister moves), we did not change the equivalence relation for classical knots: If two classical diagrams are equivalent in the class of rigid knots, then they are equivalent in the class of virtual knots, and consequently, they are equivalent in the class of classical knots.

Below, we construct a refinement for the Kauffman bracket for rigid knots.

4.4.1 *The Kauffman bracket for rigid knots*

Below, we present a slightly modified idea due to Kauffman on how to construct the bracket for rigid knots.

As in the case of usual virtual knots, for rigid knots we can define a bracket invariant (we shall denote it by $\langle\langle\cdot\rangle\rangle$); axiomatically, this bracket is defined according to the rules

$$\begin{cases} \langle\langle \diagdown\!\!\!\!\diagup \rangle\rangle = a\langle\langle \rangle\langle \rangle\rangle + a^{-1}\langle\langle \smile\!\!\!\!\frown \rangle\rangle, \\ \langle\langle L \sqcup \bigcirc \rangle\rangle = (-a^2 - a^{-2})\langle\langle L \rangle\rangle. \end{cases} \tag{4.4}$$

These equations are responsible for the invariance of the bracket to be constructed, under the second and third classical Reidemeister moves (we do not require the invariance under the detour move!). By using the relation (4.4), we shall represent the bracket of every virtual link diagram as a linear combination of the bracket for diagrams without classical crossings. In the case of usual virtual knots, when we dealt with all generalized Reidemeister moves, every diagram without classical crossings represents a trivial link. In our case it is not so: We may "pull away" the components of the link K, however, it is impossible to reduce every component to the unknot. Namely, if we have a curve on the plane without classical crossing, and allow one to apply virtual moves Ω_2' and Ω_3', Ω_3'' to it, but not Ω_1', then the classifying invariant for such curves is the *Whitney index*. This invariant indicates the total rotation angle of the tangent vector when walking along the curve (the angle is a multiple of 2π). Since the curve in question corresponds to some state of the link diagram K, it is not oriented. Thus, the Whitney index is defined only up to a sign $\pm$.

In the case of classical links, all curves turn out to be non-intersecting, thus, the curve may have the Whitney index equal to one. In the virtual case it may be any arbitrary integer. Curves with Whitney index equal to one play a special role for us. They appear in the second equation of (4.4), and the value of the bracket $\langle\langle \bigcirc \rangle\rangle$ for them has to be equal to $(-a^2 - a^{-2})$. In the case of a curve γ_i with arbitrary Whitney index we have $\langle\langle K \sqcup \gamma_i \rangle\rangle = A_i(-a^2 - a^{-2}) \cdot \langle\langle K \rangle\rangle$.

Now let us define the *rigid virtual knot Kauffman bracket* according to the rule

$$\langle\langle |K| \rangle\rangle = \sum_s a^{\alpha(s)-\beta(s)}(-a^2 - a^{-2})^{(\gamma(s)-1)} \prod_{j=1}^{\gamma(s)} A_{(s,j)}, \tag{4.5}$$

where the sum is taken over all states s, and the product is taken over all circles (the number of circles is $\gamma(s)$) for a given state; for each of them we write down the factor $A_{(s,j)}$, where (s,j) denotes the Whitney index of the corresponding circle. Herewith, A_1 is set to be equal to one.

We end up with a function defined on unoriented links and valued in $\mathbb{Z}[a, a^{-1}, A_0, A_2, A_3, \ldots]$.

A straightforward check shows that this bracket is invariant under all generalized Reidemeister moves except the first virtual Reidemeister move. This yields the following lemma.

Lemma 4.2. *The bracket $\langle\langle \cdot \rangle\rangle$ is an invariant under all generalized Reidemeister moves, except for the first classical one and the first virtual one.*

The normalization of the bracket with respect to the first classical Reidemeister move is the same as in the case of the usual Kauffman bracket. Namely, let K be an oriented virtual diagram with writhe number $w(K)$, and let $|K|$ be the diagram obtained from K by "forgetting the orientation". We set

$$X\langle\langle K \rangle\rangle = (-a)^{-3w(K)} \langle\langle |K| \rangle\rangle.$$

Then the following theorem takes place.

Theorem 4.12. *The (Laurent) polynomial $X\langle\langle \cdot \rangle\rangle$ is an invariant of rigid virtual links.*

The following statement is evident.

Statement 4.3. *For every rigid virtual link diagram we have $X\langle\langle K \rangle\rangle\big|_{A_0=A_2=\cdots=1} = X(K)$.*

4.4.2 *Minimality properties*

The behavior of the polynomial $X\langle\langle\cdot\rangle\rangle$ allows one to estimate the number of virtual crossings of some rigid virtual knots from below by looking at powers of variables A_0, A_2, A_3, $A_4 \dots$.

We define the *span* of the polynomial $X\langle\langle K\rangle\rangle$ (in variable a) as the difference of the leading degree and the lowest degree (with respect to a) of monomials in $X\langle\langle K\rangle\rangle$. Notation: $\operatorname{span}_a X\langle\langle L\rangle\rangle$. For every monomial $A_0^{i_0}A_2^{i_2}A_3^{i_3}\dots A_n^{i_n}$ we define its *width* as the sum $i_0+i_2+2i_3+\dots+(n-1)i_n$. We define the *width* of the polynomial $X\langle\langle K\rangle\rangle$ as the maximum of all widths of non-zero monomials of it. Notation: $\operatorname{wid} X\langle\langle K\rangle\rangle$.

Then the following statement holds.

Proposition 4.4 (Kauffman). *Assume for a rigid link diagram K one has* $\operatorname{wid} X\langle\langle K\rangle\rangle = m$. *Then there are no planar diagrams of the rigid link generated by K with less than m virtual crossings.*

Proof. Indeed, for every diagram K of the rigid link, we have $\operatorname{wid} X\langle\langle K\rangle\rangle = m$, or, in other words, $\operatorname{wid}\langle\langle |K|\rangle\rangle = m$. From (4.5), we see that there exists a Kauffman state s of the diagram $|K|$, for which $\sum_{j=1}^{\gamma(s)} |j-1| = l \geqslant m$. Assume in this state we have a_0 circles of the Whitney index zero, a_2 circles of the Whitney index two, a_3 circles of the Whitney index three, etc. Let us count the number of virtual crossings of the diagram K in the state s (which is equal to the total number of crossings in K). Every circle of index zero or index two has at least one self-intersection; every circle of index three has at least two intersections; every circle of index four has at least three intersections, etc. This means that the total number of virtual crossings of the diagram K cannot be less than l. □

The following statement is evident.

Statement 4.4. *For every virtual diagram K the following equality takes place*: $\operatorname{span}_a X\langle\langle K\rangle\rangle \geqslant \operatorname{span} X(K)$.

Thus, the width allows one to make an estimate for the minimal number of virtual crossings. On the other hand, the span of the polynomial $X\langle\langle\cdot\rangle\rangle$ allows one, just as in the case of classical or usual virtual knots, to estimate the minimal number of classical crossings of the given rigid link.

Example 4.4. Consider the two-component "trivial" link $\widetilde{K}$, shown in Fig. 4.20.

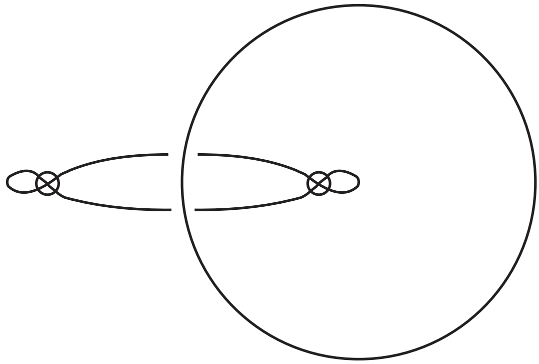


Fig. 4.20 A virtual diagram which is non-trivial in the rigid category.

Let us show that this virtual link is non-trivial in the *rigid* category. The bracket of this link looks like:

$$\langle\langle\widetilde{K}\rangle\rangle = a^2 + (A_0^2 + A_0 A_2)(-a^2 - a^{-2}) + a^{-2}. \tag{4.6}$$

Thus, we see that the *first virtual Reidemeister move* Ω_1' *is not expressible in terms of the remaining Reidemeister moves.*

If in the formula (4.6) we set $A_0 = A_2 = A_3 = \cdots = 1$, then we get the usual Kauffman bracket of the two-component trivial link, namely, $(-a^2 - a^{-2})$.

From $\langle\langle\widetilde{K}\rangle\rangle = (-a^2 - a^{-2})(A_0^2 + A_0 A_2 - 1)$ it follows that the number of virtual crossings of the rigid diagram $\widetilde{K}$ cannot be less than two. On the other hand, the number of classical crossings of this link cannot be less than two either. The latter follows from the fact that this rigid link is not representable as the disconnected sum of two trivial links with indices i and j (in the latter case we would get the value of the bracket equal to $A_i A_j(-a^2 - a^{-2})$).

4.5 Minimal diagrams of long virtual knots

In the case of classical knot theory, Theorem 4.8 leads us to the proof of the classical Thistlethwaite theorem stating the minimality of the adequate diagrams of classical links. In order to get rid of the framing condition in Theorem 4.8, we may pass to the connected sum of the link with its mirror image (for details, see ahead). Thus, the techniques of detecting minimality of knot diagrams described above in Theorems 4.5, 4.7, 4.8 work in the case of long virtual knots because for long virtual knots the connected sum operation is well defined. Unlike the case of classical knots, where only *adequate* diagrams satisfy this criterion, the minimality was proved by Thistlethwaite by using the Kauffman polynomial, where we establish new series of minimal diagrams of long virtual knots by using the same methods.

Let us touch upon these questions in more detail. Consider long virtual knots and their closures (see Definition 3.4 and Fig. 3.24). Note that some results concerning the minimality problem for long virtual knots were established in [2] by using a generalization of the Alexander polynomial.

Recall that the cable diagram $D_p(K)$ was defined in Fig. 4.13.

Then the following theorem takes place.

Theorem 4.13. *Let K be a long virtual knot diagram such that for every $p > 1$ the virtual diagram $D_p(\mathrm{Cl}(K))$ is 1-complete. Then the diagram K is minimal with respect to the number of classical crossings.*

Proof. We shall prove the theorem analogously to Theorem 4.8. Let $L = K\#K'$ be the connected sum of the long virtual knot diagram with its mirror image. By definition we set $\mathrm{Cl}(D_p(K)) = D_p(\mathrm{Cl}(K))$. Obviously, we have $D_p(L) = D_p(K)\#D_p(K')$. Moreover, with each long virtual knot diagram K one can uniquely associate the link diagram L with zero framing. Thus, the operation $K \mapsto D_p(L)$ is well defined.

Furthermore, one may check that 1-completeness of the diagram $\mathrm{Cl}(D_p(K))$ yields 1-completeness of $\mathrm{Cl}(D_p(L))$. This follows from the fact that every circle in the A-state of the diagram $\mathrm{Cl}(D_p(L))$ leads to circles from the A-state of the diagram $\mathrm{Cl}(D_p(K))$ and circles from the B-state of the diagram $\mathrm{Cl}(D_p(K'))$. By virtue of 1-completeness of B and B', the self-incidence (i.e. the incidence with itself at some crossing) for each of these diagrams gives zero coefficient (M_1, see Sec. 4.3.1), consequently, the leading coefficient in the Kauffman bracket extension $\langle \mathrm{Cl}(D_p(K'))\rangle$ is non-zero.

Arguing as in the proof of Theorem 4.8, we see that we can decrease the number of classical crossings of the diagram K only at the expense of the genus of the atom corresponding to $\mathrm{Cl}(L)$, by any prefixed integer. The contradiction completes the proof of the theorem. □

Now let us construct concrete examples of such long virtual knots. For example, in the case of classical knots the 1-completeness for $D_p(K)$ may take place only in the case when the initial diagram is adequate and all diagrams $D_p(K)$ are adequate as well (see Definition 4.14). In the case considered below we do not assume 1-completeness for the initial diagram, but the diagrams $D_p(K)$ may be 1-complete but *not adequate* and even *not semiadequate*.

The point is that the "leading" coefficient of the Kauffman bracket polynomial (as well as its "lowest" coefficient) for all diagrams $D_p(K)$ for all $p > 1$ is calculated from the same formula. Here we mean the leading coefficient of the expansion (4.1); in the case when $D_2(K)$ is not 1-complete, i.e. this coefficient is equal to zero, the "real" leading coefficient is much more difficult to calculate.

Thus, it is important to find good combinatorial formulae for the leading and lowest coefficients of the Kauffman bracket (4.1) for cables $D_p(K)$.

This coefficient is closely connected to the function M_1 (see Sec. 4.3.1). More precisely, let D be a chord diagram on several circles. Let us split the set of chords of the diagram D into two sets; consider the chord diagrams D' and D'' composed of the chords from these sets; let us calculate the coefficients $M_1(D')$ and $M_1(D'')$. This leads us to the expansion (4.1) of the leading coefficients for all diagrams $D_p(K)$ for all $p > 1$ simultaneously.

The latter easily follows from the statement (first proved in [201]) that if a chord diagram D' is obtained from a chord diagram D by adding a parallel chord to some chord (never mind, which), then $M_1(D') = M_1(D)$. This yields, in particular, the fact that the value of the function M_1 will not change if instead of every chord we take a set of m parallel chords.

When we take the pth cable $D_p(K)$, in the A-state we get two circles whose chord diagrams are diagrams obtained from D' and D'' by taking p parallel copies of every chord. The latter is shown in Fig. 4.22: The triple cable for the figure eight knot at every crossing gives three chords either for the innermost circle or for the outermost circle.

In the classical case the situation is the following: The initial chord diagram is a d-diagram, and the resulting splitting of the set of chords into two families leads to two families of pairwise disjoint chords. Thus, if the initial diagram is non-empty (at least one of the two sets is non-empty) then none of the diagrams $D_p(K)$, $p > 1$, will be 1-complete. From this we see that in the classical case the 1-completeness of $D_p(K)$ yields the adequacy of the diagram K.

Let us concentrate ourselves on the case when the atom has exactly one subcritical (respectively, one supercritical) circle.

In the case of virtual diagrams this way of splitting can be arbitrary for a given chord diagram D. The construction of diagrams corresponding to arbitrary splittings is obtained by using virtualization. Namely, when applying a virtualization of some crossing in the initial diagram K, the chord corresponding to this crossing in the diagram $D_p(K)$ is moved from one family to the other.

We shall use the following auxiliary construction. In this chapter, all atoms corresponding to diagrams of virtual links are assumed oriented.

Let D be a chord diagram on one circle having n chords, and let $\mathrm{At}(D)$ be the corresponding atom with one black cell and n vertices. Assume the frame of the atom $\mathrm{At}(D)$ has one unicursal component, i.e. for every projection of the frame to the plane, preserving the A-structure, we get a knot but not a multicomponent link.

Now let D' and D'' be two chord diagrams obtained by splitting the set of chords of the diagram D into two subsets.

Let us define the virtual diagram $K_{D;D',D''}$ in the following way (the resulting diagram will be defined up to detour moves). For the atom defining this diagram we take the atom $\mathrm{At}(D)$; from this atom, the diagram is restored uniquely up to virtualizations and detour moves. To avoid the ambiguity coming from virtualization, let us use the splitting of D into subdiagrams D' and D''.

Namely, let us immerse the chord diagram D in $\mathbb{R}^2$ in the following way. We embed the core circle of this diagram in $\mathbb{R}^2$ in a standard way and locate those chords coming from D' inside the circle, and those chords coming from D'' outside the circle (e.g. we may draw the inner chords by straight intervals and the outer chords by images of straight segments under the immersion). Certainly, some chords of the diagram D' may be linked with each other (as well as chords of D''). After that, we replace a neighborhood of each of the embedded chords by two arcs lying close to each other forming a classical crossing. In a neighborhood of every intersection point of the initial chords, we shall get four virtual crossings; we mark all such crossings as virtual, see Fig. 4.21.

It is clear that the obtained diagram (up to detour moves) is defined by positions of chords of the chord diagrams D', D''.

We denote the obtained diagram by $K_{D;D',D''}$. We shall use this construction for constructing minimal diagrams of some virtual links.

Consider the nth cable $D_n(K_{D;D',D''})$.

Then the following lemma takes place.

Lemma 4.3. *For every $n > 1$, for the diagram $D_n(K_{D;D',D''})$ the lowest coefficient in the Kauffman bracket expansion* (4.1) *is equal to* $\pm M_1(D') \cdot M_1(D'')$.

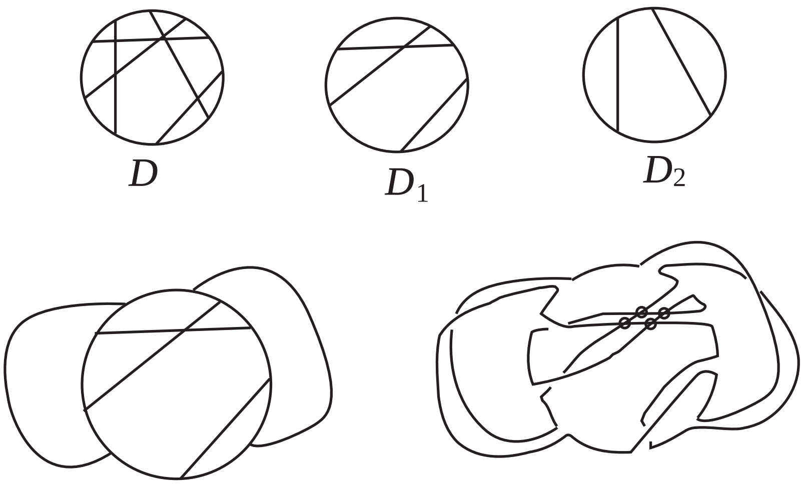


Fig. 4.21 Constructing the knot diagram $K_{D;D',D''}$.

We say that in some Kauffman state a state circle touches itself at a crossing, if this circle is incident to this crossing twice.

Lemma 4.3 follows from the following arguments: The cable $D_n(K_{D;D',D''})$ has n circles in the B-state corresponding to every circle of the B-state of $K_{D;D',D''}$. Two of these circles, the "innermost" and the "outermost" may touch themselves at some crossings; the others are "clutched" between these two, see Fig. 4.22. One of these outer circles touches itself according to the chord diagram D', and the other touches itself according to D''. Thus we get the desired statement of Lemma 4.3.

By virtue of Lemma 4.3, it is to find a way of splitting the diagram D into two diagrams D' and D'' such that $M_1(D') \cdot M_1(D'') \neq 0$. This happens often even in the case when $M_1(D) = 0$.

A natural problem is to construct such examples of virtual knot diagrams for which both the leading and the lowest term in the expansion (4.1) are non-zero.

In order to handle both coefficients, let us assume that we deal with an atom with one subcritical circle and one supercritical circle.

Besides, we restrict ourselves for the case of orientable atoms. Moreover, we shall make one more constraint. Let D be an oriented chord diagram with the corresponding atom $\mathrm{At}(D)$ with one supercritical circle (according to the diagram D) and one subcritical circle. From the point of view of

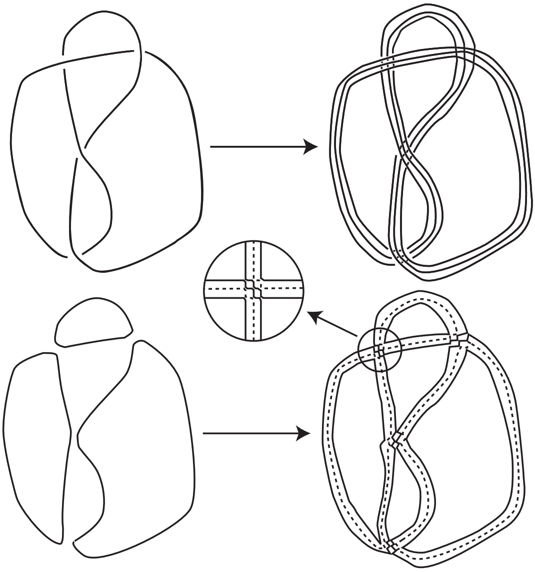


Fig. 4.22 B-state for the triple cable of the figure eight knot.

the subcritical circle, the atom can also be encoded by the chord diagram coming from the atom $\mathrm{At}^*(D)$, which differs from the atom $\mathrm{At}(D)$ by colors of the cells.

We shall call this diagram *dual* to the diagram D and denote it by D_*. Then there exists a bijective correspondence between chords of D and chords of the diagram D_* (both the former chords and the latter chords correspond to vertices of the atom $\mathrm{At}(D)$). Thus, every way of splitting of the chord diagram D into two subdiagrams D', D'' generates a splitting of D_* into D'_*, D''_*.

From the arguments above we get the following lemma.

Lemma 4.4. *Assume the atom corresponding to the chord diagram D has one black cell and one white cell. Then for the diagram $D_n(K_{D;D',D''})$ the leading coefficient in the Kauffman bracket extension* (4.1) *is equal to $M_1(D'_*) \cdot M_1(D''_*)$.*

Thus, we are led to the following theorem.

Theorem 4.14. *Let D be a chord diagram corresponding to an oriented atom with one black cell and one white cell, and assume that there is some* *way of splitting of the corresponding set of chords into two sets leading to two chord diagrams D', D''. Assume that the diagram D generates a knot (not a link) and $M_1(D') \cdot M_1(D'') \cdot M_1(D'_*) \cdot M_1(D''_*) \neq 0$. Then every diagram of a long knot obtained by breaking the diagram $K_{D;D',D''}$ at any point is minimal with respect to the number of classical crossings.*

This theorem is convenient for constructing examples for Theorem 4.13.

Note that if the diagram D generates an atom with one black cell and one white cell, then the diagram $K_{D;D',D''}$ is neither adequate nor semiadequate (if the set of chords of D is not empty).

Let $n \in \mathbb{N}$. Consider the chord diagram D consisting of $(4+6n)+4$ chords, constructed as follows. The first $(4+6n)$ chords form a non-closed chain where each chord with a number i is linked with the chords having adjacent numbers $i-1$ (if $i > 1$) and $i+1$ (if $i < 4+6n$). The remaining four chords are denoted by letters a, b, c, d. The whole diagram is shown in Fig. 4.23.

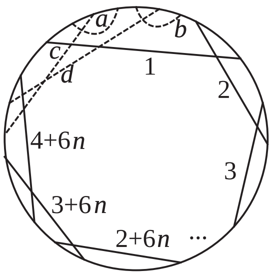


Fig. 4.23 A chord diagram for which the corresponding long knot diagram is minimal.

Remark 4.8. The chords a, b, c, d are drawn by dashed lines for visualization purposes. They all have framing 0, as well as all the other chords of this chord diagram.

Now, we are ready to split chords of D into two families in the following way. The chords a, d go to the family D', the chords b, c go to the subdiagram D''. For the remaining chords we have: the initial three chords 1, 2, 3 are in D', then the chords 4, 5 are in D'', the next two chords are in D', then we alternatively put pairs of adjacent chords to one or another family, until we are left with three chords. The chords $2+4n$, $3+4n$, $4+6n$ will also be in one family (different from the family the preceding two chords belong to).

It is not difficult to see that $M_1(D') \cdot M_1(D'') \cdot M_1(D'_*) \cdot M_1(D''_*) = \pm 1$.

Thus we have constructed an infinite sequence of examples of minimal diagrams for long virtual knots, which are neither adequate nor semiadequate. It can also be easily verified that the critical diagonals of the Khovanov homology for such series of diagrams are empty (see Chap. 5), which means that the minimality of these diagrams cannot be established just by looking at the difference between the top and bottom heights of the Khovanov homology, as in Lemma 5.11.

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Chapter 5

Khovanov Homology

5.1 Introduction

One of the most outstanding achievements of knot theory of past years is the construction suggested by Khovanov [176], which associates with each diagram of an oriented virtual link some bigraded chain complex. The homology of this complex is an invariant of the link. One of the most striking applications of the Khovanov homology was Rasmussen's combinatorial proof of the Milnor conjecture, see [269].

Khovanov's idea is to replace the Kauffman bracket $\langle K \rangle$ of a link diagram K with a chain complex of graded vector spaces (*Khovanov complex*) whose graded Euler characteristic is $\langle K \rangle$. The chain complex is defined by the axioms:

$$[[\emptyset]] = 0 \to \mathbb{Z} \to 0, \qquad [[\bigcirc K]] = V \otimes [[K]],$$
$$[[\times]] = \mathcal{F}\left(0 \to [[\asymp]] \xrightarrow{d} [[)(]]\{1\} \to 0\right).$$

Here V is a vector space of graded dimension $q + q^{-1}$ (see below), the operator $\{1\}$ is the operation of grading shift by 1, $\mathcal{F}$ is the flatten operation which sets a double complex to a single complex by taking direct sums along diagonals, and d is a differential. The Khovanov invariant is the homology of a renormalization of the Khovanov complex. The Khovanov invariant is indeed a link invariant and its graded Euler characteristic is the unnormalized Jones polynomial.

This passage from polynomials to (bi)graded complexes is also called *categorification*: Complexes form a category in which there are natural morphisms generated, for example, by cobordisms.

This theory has many generalizations and led to solutions of many problems in classical knot theory (for example, a simple proof of Milnor's conjecture about the Seifert genus of torus links, [269]). It was also shown that the Khovanov homology detected the unknot, see [187]. For more details see [13, 19, 20, 101, 176–182, 262, 269, 278, 279, 309, 315] and references therein.

As it follows from the paper by Bar-Natan [20], the Khovanov complex can be considered as a complex of cobordisms: The chain space consists of framed states (circles with labels $\pm$) instead of abstract graded modules, and differentials are linear combinations of cobordisms. This consideration led Bar-Natan to the "universal topological" theory of the Khovanov homology. Note that chains in the universal topological theory, generally speaking, are not chains of a cell complex; they just admit a combinatorial description by means of topology of two-dimensional cobordisms connected with knots. Note that the question about a "real" topological interpretation, the spectrification of the Khovanov homology (to find a topological space related to the knot such that the homology of this space coincides with the Khovanov homology of the knot), is still open. Partial attempts to answer this question belong to Seidel and Smith [278].

In [178] Khovanov constructed the universal algebraic theory by using ideas similar to those of Bar-Natan. In this theory, two rings $\mathcal{R}$ and $\mathcal{A}$ are considered, where $\mathcal{A}$ has a structure of the Hopf algebra over $\mathcal{R}$. It turns out that under some natural algebraic assumptions on the structure of the ring $\mathcal{A}$ one can construct an extraordinary theory of link homology in which $\mathcal{R}$ is the ring of coefficients, and $\mathcal{A}$ is the homology ring of the unknot.

An important generalization in the theory of extraordinary homology of links was the construction of categorification for a set of polynomials of type HOMFLY, made by Khovanov and Rozansky [180, 181]. Polynomials of type HOMFLY have more difficult relations and the problem of categorification for them was elegantly solved by means of instruments of *matrix factorizations* and *Koszul complex*. Khovanov and Rozansky [182] devoted their paper to the categorification of the so(N)-type Kauffman polynomial in which virtual knots are also used besides matrix factorizations.

The Khovanov homology possesses important properties coming from algebraic topology: the (projective) functoriality. In the given case, the morphisms are cobordisms of knots. Thus, the Khovanov homology is extended to invariants of knot cobordisms representing two-dimensional surfaces with boundary in $\mathbb{R}^3 \times I$. The projective functoriality (i.e. functoriality up to the overall minus sign) was first established by Jacobsson [138], see also [20].

The functoriality allows one to construct invariants of cobordisms of two-dimensional surfaces in $\mathbb{R}^4$ from the Khovanov complex; a particular case of cobordisms is the cobordism between two links consisting of an empty set of components. In the case of projective functoriality, a cobordism invariant is defined up to an inverse element of the main ring. In this case the Khovanov construction gives an invariant of two-dimensional knots, and two-dimensional surfaces embedded in $\mathbb{R}^3 \times I \subset \mathbb{R}^4$. The accurate functoriality was established in [61], see also [35].

In [20] an interesting construction was suggested. This construction describes a *topological* Khovanov complex, a formal chain complex, in which linear combinations of labeled sets of circles in the plane play the role of chains, where linear combinations of cobordisms are differentials. For invariance, some relations originating from the topology of two-dimensional complexes are imposed on such cobordism complexes. The general (algebraic) Khovanov complex is obtained from the geometrical one by "substitution" of concrete graded spaces for sets of circles, and concrete maps for elementary cobordisms; and it also requires that the relations originating from the relations on cobordisms hold.

Note also the paper [179] where Khovanov constructed a homology theory for colored links, connected with the Jones polynomial for cables, by using the cobordism theory and combinatorial analysis of the representation of the Lie algebra $\mathfrak{sl}_2$.

One of the most natural problems in the theory of virtual knots is the problem of generalization of the Khovanov complex for virtual knots. An immediate attempt to generalize the theory leads to an algebraic difficulty: By writing down all necessary equations for the Khovanov complex to be invariant, we conclude that the main ring of coefficients should be the two-element ring. The indicated generalization was done in [218]. Some difficulties of the immediate approach can be avoided by using geometrical constructions connected with atoms.

The main goal of this chapter is the construction of a chain complex for a virtual diagram with the homology being invariant under the generalized Reidemeister moves.

Note that the Khovanov homology for knots in thickened surfaces and in bundles over surfaces S_g whose fiber is an interval (by using some additional gradings for curves in a *given* surface) was also constructed by Asaeda, Przytycki and Sikora [13]. This homology does not lead to the Khovanov homology for virtual knots, since it depends on a concrete surface S_g and is not invariant under destabilizations and homeomorphisms of the surfaces onto itself.

A further development of the Khovanov homology theory for virtual knots representing a generalization of the paper [13], and the results of this chapter, are given in [230, 231], see also [79]. In these papers topological and combinatorial coefficients at terms in the Kauffman bracket polynomial are "lifted" to new gradings in the Khovanov homology.

Note also the paper by Turaev and Turner [303], in which a "topological" theory of cobordisms for virtual knots is constructed (with different restrictions). Namely, in [303] a homology theory with coefficients from $\mathbb{Z}_2$ was constructed. Different theories with coefficients from $\mathbb{Z}$ in which one of the gradings disappeared were also constructed.

Besides Khovanov homology, there are other link homology theories: The theory [263, 268] which categorifies the Alexander polynomial, diagramless link homology [244], etc.

In this chapter, we shall first describe four ways of constructing the Khovanov complex for virtual knots with some restrictions. First, we construct the Khovanov complex for arbitrary virtual knots with coefficients from $\mathbb{Z}_2$; in the second case, we show how one can construct the Khovanov complex for framed virtual links (by means of double diagrams) with coefficients from an arbitrary ring; in the third case, we construct the Khovanov complex of two-sheeted coverings over virtual knots (in the sense of atoms) with coefficients from an arbitrary field. The fourth way is connected with the projective map, see Chap. 8 and Definition 8.11, which sends all virtual knots to knots with orientable atoms and remains knots with orientable atoms. This projective map allows one to "lift" all invariants defined for virtual knots with orientable atoms to all virtual knots.

In the second part of the chapter with each ring of each diagram of a virtual link we associate a complex with the homology being invariant under the generalized Reidemeister moves (this construction first appeared in [228, 229]). Moreover, in the classical case, the complex has the same homology as the ordinary Khovanov complex, and the particular cases constructed in this chapter give the complexes with the homology being isomorphic to the homology constructed for all virtual knots. The graded Euler characteristic of this complex coincides with the Jones polynomial $\widehat{J}$ of the virtual link. Proceeding with this construction and using the parity arguments, see Chap. 8, we get the invariance of the Khovanov homology of two-sheeted coverings over virtual knots with coefficients from an arbitrary ring.

The main difficulty in constructing a Khovanov homology for virtual knots is how to define the differential for complexes corresponding to arbitrary virtual knots. Here one must consider many more cases than for classical knots (the corresponding atoms are considered in Sec. 5.9.1). This difficulty is overcome by means of a construction of a new complex having the same homology as the usual Khovanov complex. The first key idea is to change the basis of the Frobenius algebra representing the Khovanov homology of the unknot (it is connected with a choice of a local orientation of the corresponding circle originating from the crossing) as we pass from one crossing of the knot diagram to another. The second key idea is to replace the usual tensor product (corresponding to several circles in a given state) by the *exterior product* of the corresponding graded spaces. This enables us to avoid the "artificial" procedure of transforming the commutative cube into an anticommutative one, as was done in [17, 176] and in the first part of the chapter.

We mention some important properties of this construction.

(1) The construction of the complex uses *atoms*. The complex is invariant under virtualization. This is proved in Lemma 5.6.
(2) There is a natural map from the set of "twisted virtual knots" in the sense of Bourgoin and Viro [42, 310] (see below) to the set of virtual knots modulo virtualization. Therefore, our approach yields invariants of twisted virtual knots. The set of twisted virtual knots (knots in oriented thickenings of non-orientable two-dimensional surfaces up to stabilization) contains all knots in the punctured three-dimensional projective space. A particular case of this theory is the theory of knots in the three-dimensional projective space $\mathbb{R}P^3$. Note that the Kauffman bracket polynomial for knots in $\mathbb{R}P^3$ was constructed by Drobotukhina in [72]. Moreover, this theory admits different generalizations constructed for the ordinary Khovanov homology: Lee's theory [193, 194], Wehrli's and Champanerkar–Kofman spanning tree expansion [54, 315], etc.
(3) For the coefficient field $\mathbb{Z}_2$, our complex coincides precisely with the complex constructed in Sec. 5.3.
(4) For orientable atoms (in particular, for classical knots), the homology of our complex is the same as the homology of the complex constructed in Sec. 5.4.
(5) The proof of invariance of the homology is local. It repeats the proof of the invariance in the classical case (see, e.g. [17]). The main difficulty is in defining the differential: How can one choose signs that make the cube anticommutative? We overcome this difficulty by constructing a new complex which is homotopy equivalent to Khovanov's original

complex.

In the next part of the chapter we show that the approach using atoms can be applied for the general Khovanov homology theory (Frobenius extensions) [223], and we describe algebraic equations and structures which appear under the attempt of generalizing the universal theory of the Khovanov homology directly (as opposed to the simplest (initial) Khovanov complex it turns out that this theory is richer).

An important question in the theory of classical and virtual knots is the problem of defining minimal diagrams of links, diagrams with minimal number of classical crossings in a given class.

At the end of the chapter, we give the construction of a spanning tree for the Khovanov complex, first suggested by Wehrli [315] for classical knots (much stronger result in the classical case was obtained by Champanerkar and Kofman [54]) and the first-named author [227] (for virtual knots). We show how one can establish the minimality of link diagrams by using the Khovanov complex. Different minimality theorem [221, 227] will be formulated which are based on the Jones polynomial as well as the Khovanov complex.

5.2 Basic constructions: The Jones polynomial $\widehat{J}$

In the sequel, we shall deal with bigraded complexes $\mathcal{C} = \bigoplus_{i,j} \mathcal{C}^{i,j}$, where i is called the *height*, and j is called the (*quantum*) *grading*. The differential in the complex does not change the grading and increases the height by one. Sometimes the height is also called the *homological grading*.

Since the differential increases the height, it would possibly be more appropriate to talk about *cohomology*, but *Khovanov homology* is a well-established terminology, so we shall use the terms *chains, cycles, boundaries* instead of *cochains, cocycles, coboundaries*.

Let a linear space M (or a free module M over a ring $\mathcal{R}$) have a preferred *quantum* grading q. Then one has the following decomposition $M = \bigoplus_i M_i$, where M_i is the homogeneous component of grading i. By the *graded dimension* of the space M we mean the polynomial $\text{qdim}\, M = \sum_i q^i \dim M_i$.

For such complexes there are naturally defined operations of the *height shift* $\mathcal{C} \mapsto \mathcal{C}[k]$ and the *grading shift* $\mathcal{C} \mapsto \mathcal{C}\{l\}$ defined according to the following rules: $(\mathcal{C}[k])^{i,j} = \mathcal{C}^{i-k,j}$; $(\mathcal{C}\{l\})^{i,j} = \mathcal{C}^{i,j-l}$. In the first case, together with chains, all differentials are shifted accordingly (i.e. the differential ∂_i, which was acting from $\mathcal{C}^{i,*}$ to $\mathcal{C}^{i+1,*}$, will now act in the same way from

$\mathcal{C}^{i-k,*}$ to $\mathcal{C}^{i+1-k,*}$). By the *graded Euler characteristic* of the complex $\mathcal{C}^{i,j}$ we mean the alternating sum of the graded dimensions of the chain spaces, or, which is the same, the graded dimension of the homology groups. For chain spaces, we have:

$$\chi(\mathcal{C}^{i,j}) = \sum_i (-1)^i \operatorname{qdim} \mathcal{C}^i = \sum_{i,j} (-1)^i q^j \dim \mathcal{C}^{i,j}.$$

For such complexes, for every bigraded dimension (i, j) there is the (co)homology group $H^{ij}(\mathcal{C})$ which is defined as the quotient module of the corresponding module of cycles by the submodule of boundaries.

Definition 5.1. Two graded (respectively, bigraded) complexes $\mathcal{C}$ and $\mathcal{C}'$ are called *quasiisotopic*, if there exist two bigrading preserving maps $f: \mathcal{C} \to \mathcal{C}'$, $g: \mathcal{C}' \to \mathcal{C}$ together with a map u decreasing the height by one and preserving the second grading if such exists, such that $f \circ g = \mathrm{Id}_{\mathcal{C}'}$, and $g \circ f - \mathrm{Id}_{\mathcal{C}} = d \circ u + u \circ d$. Here, $\mathrm{Id}_{\mathcal{C}}: \mathcal{C} \to \mathcal{C}$, $\mathrm{Id}_{\mathcal{C}'}: \mathcal{C}' \to \mathcal{C}'$ denote the corresponding identity maps.

Homology groups of quasiisomorphic complexes are isomorphic.

Let us describe the normalization of the Jones polynomial suggested by Khovanov which differs from the one described in the previous chapter.

Let us make the substitution $a = \sqrt{(-q^{-1})}$ in the Kauffman bracket. Then, instead of the Jones polynomial we shall get its modified version J. Let us consider the polynomial $\widehat{J} = J \cdot (q + q^{-1})$. More precisely, $\widehat{J}$ is defined as follows. Let K be an oriented virtual diagram, and let $|K|$ be the corresponding unoriented virtual diagram obtained from K by forgetting the orientation, let n_+ and n_- be the numbers of positive and negative classical crossings of K, and $n = n_+ + n_-$ be the total number of crossings. We set:

$$\widehat{J}(K) = (-1)^{n_-} q^{n_+ - 2n_-} [K],$$

where $[K]$ is the modified Kauffman bracket defined according to the rule $[\bigcirc] = (q + q^{-1})$, $[K \sqcup \bigcirc] = (q + q^{-1}) \cdot [K]$, $[\times] = [\rangle\langle] - q[\asymp]$.

The polynomial $\widehat{J}$ has the following conceptually important description in terms of the state cube. Taking away the normalizing factor $(-1)^{n_-} q^{n_+ - 2n_-}$, we get a (slightly modified) Kauffman bracket $\sum_s (-q)^{\beta(s)} (q + q^{-1})^{\gamma(s)}$. This means that we take the sum over all vertices of the cube, of the following products $(-q)^h \times (q + q^{-1})^{\#\bigcirc}$, where h is the height of the vertex, and $\#\bigcirc$ is the number of circles in the state corresponding to the given vertex of the cube.

Thus, in order to compute the polynomial, one has to associate with every circle the Laurent polynomial $(q + q^{-1})$, and then multiply these polynomials taken with some coefficients of the form $\pm q^k$, and take the sum of the obtained polynomials over all vertices of the cube.

Consequently, the Jones polynomial can be restored from the information about the *number* of circles in each of the Kauffman states. If we also take into account the way *how these circles interfere* when passing from one state to another, we would be able to construct the Khovanov complex.

5.3 Khovanov homology with $\mathbb{Z}_2$-coefficients

Let K be an oriented diagram of a virtual link with n classical crossings.

Definition 5.2. By *bifurcation cube* we understand the cube $\{0,1\}^n$ where each vertex is assigned the number of circles (as in the state cube), and each edge indicates which circles bifurcate when passing from a state to an adjacent one. The *height* of a state (a vertex of the cube) is the number of B-smoothings.

We orient the edges of the cube as the sum of coordinates increases (i.e. from an A-smoothing to a B-smoothing). With each circle in each state we associate the linear space V over the field $\mathbb{Z}_2$ generated by two vectors v_+ and v_- where vectors $v_\pm$ have grading ± 1 respectively. Thus, $\operatorname{qdim} V = (q + q^{-1})$. For each vertex $s = \{a_1, \ldots, a_n\}$ of the cube, we have a certain number of circles to be denoted by $\gamma(s)$. With such a vertex, we associate the vector space $V^{\otimes\gamma(s)}\{\sum_{i=0}^{n} a_i\}$ obtained from the tensor power of the space V by a grading shift.

Remark 5.1. In the sequel, we shall use the same notation V for the two-dimensional free module generated by the elements v_+, v_- of grading ± 1 considered over an arbitrary ring of coefficients.

Remark 5.2. In the first part of the chapter, we consider the symmetric tensor product for which for elements $x_i \in V_i$, $i = 1, \ldots, n$, the following equality $x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)} = x_1 \otimes \cdots \otimes x_n$ holds for any arbitrary permutation σ. We shall also call this product *unordered.* In the second part, we shall consider the tensor product where the sign is the sign of the permutation when identifying products in different orders (this is also called *signed tensor product*).

The replacement of $(q + q^{-1})$ with the space V such that $\operatorname{qdim} V = (q + q^{-1})$ is an important step towards categorification. We define the chain space of chains of height k as the direct sum of spaces corresponding to all vertices of height k of the bifurcation cube.

We have defined the chain groups of our graded complex. This yields that whatever differentials we take for this complex (provided that $\partial^2 = 0$), the Euler characteristic of this complex will not depend on them. Namely, $\chi(\mathrm{Kh}(K)) = \widehat{J}(K)$, where $\mathrm{Kh}(K)$ denote the bigraded homology of the complex we are going to construct.

Let us define the *partial differentials* between the chain groups, acting along the edges of the cube according to the edge directions, i.e. from a smoothing of type A to a smoothing of type B, in the following way. Let an edge of the bifurcation cube correspond to a passage from a state s to a state s' in such a way that l circles are not incident to the crossing in question. These circles do not change when passing from s to s'. At the crossing of $|K|$, corresponding to the edge either one circle splits into two circles or two circles merge into one. In the first two cases, we shall define the partial differential as it was defined in the case of classical knots [19], namely, on an edge increasing the number of circles we set $\Delta \otimes \mathrm{Id}^{\otimes l}\{1\}$ and on an edge decreasing the number of circles we set $m \otimes \mathrm{Id}^{\otimes l}\{1\}$. Here the identical mapping Id is referred to the circles which are not incident to the crossing in question, and the maps $m\colon V \otimes V \to V$ and $\Delta\colon V \to V \otimes V$ are defined according to the rule.

The map m:

$$\begin{cases} v_+ \otimes v_+ \mapsto v_+, & v_+ \otimes v_- \mapsto v_-, \\ v_- \otimes v_+ \mapsto v_-, & v_- \otimes v_- \mapsto 0. \end{cases}$$

The map Δ:

$$\begin{cases} v_+ \mapsto v_+ \otimes v_- + v_- \otimes v_+, \\ v_- \mapsto v_- \otimes v_-. \end{cases}$$

For those chains corresponding to the fixed vertex of the cube, the differential ∂ is a sum of all partial differentials (each to be denoted by ∂', possibly, with an index indicating to the edge along which the partial differential acts) along all edges emanating from the given vertex of the cube (oriented in a way increasing the sum of coordinates).

In the general case, the main problem is to define the differential of type $(1 \to 1)$ in a way compatible with differentials of types $(1 \to 2)$ and $(2 \to 1)$ to make the cube anticommutative. For coefficients from $\mathbb{Z}_2$ this difficulty is easy to overcome.

Namely, in the case of bifurcation of type $(1 \to 1)$ we define the partial differential on the edge as the map taking the whole space to zero. Thus, we get the *bifurcation cube*, where in comparison with the state cube we additionally indicate how the partial differentials ∂' act. Denote the obtained set of the bigraded groups (the cube) by $[[K]]$. In order for the differential to be well defined, the cube has to be *anticommutative*, i.e. for every two-dimensional face of the cube, the composition of the maps corresponding to one pair of consecutive edges is equal to minus the composition of the maps corresponding to the other pair of consecutive edges connecting the same pair of points. Note that in this case (for the field $\mathbb{Z}_2$) the anticommutativity and commutativity are the same.

Let us define the differential ∂ as the sum of all differentials ∂'.

Lemma 5.1. *The cube* $[[K]]$ *defined above is commutative.*

This statement is verified by a routine check analogous to that from Bar-Natan's paper [19]. Namely, we check the anticommutativity for every face of the cube.

Here we give an example of such a check (the most interesting one), see Fig. 5.1.

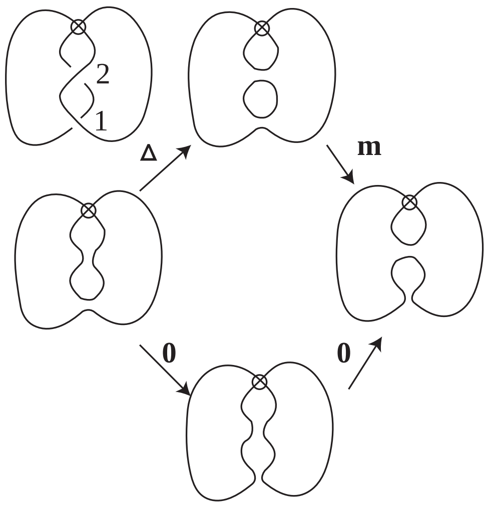


Fig. 5.1 The commutativity check for a 2-face of the cube.

Later on, we shall see (Secs. 5.4 and 5.7.2) that every 2-face of the cube generates a certain atom.

In the present case (Fig. 5.1), it is necessary to check that the map $m \circ \Delta : V \to V$ takes the whole space V to zero. Indeed, for such a map we have: $v_- \mapsto v_- \otimes v_- \mapsto 0$, $v_+ \mapsto v_+ \otimes v_- + v_- \otimes v_+ \mapsto 2v_- = 0$ over $\mathbb{Z}_2$.

Note that this case is the only essential "non-classical" case where a bifurcation of type $1 \to 1$ takes place. Indeed, from the parity arguments it follows that on every 2-face of the cube the number of $1 \to 1$-bifurcations is either equal to zero or it is at least two. For more details see Sec. 5.7.2. If there are no such bifurcations, then the problem is reduced to one of the classical cases (all such cases were considered by Bar-Natan).

If we consider the case when there are two or four such bifurcations, then in the 2-face of the cube in question,

$$\begin{array}{ccc} V^{\otimes a}\{1\} & \xrightarrow{s} & V^{\otimes b}\{2\} \\ r\uparrow & & \uparrow t \\ V^{\otimes c} & \xrightarrow{p} & V^{\otimes d}\{1\}, \end{array}$$

either each of the compositions $t \circ p$ and $s \circ r$ contains a zero map corresponding to $1 \to 1$-bifurcation (for example, in the case $a = b$, $c = d$ the maps p and s are both zero) or the above case takes place.

We set (cf. [176]), $\mathcal{C}(K) = [[K]]\{n_+ - 2n_-\}[-n_-]$. In this case $\mathcal{C}(K)$ is a well-defined chain complex. Denote the homology groups of the complex $\mathcal{C}(K)$ by $\mathrm{Kh}(K)$ (or by $\mathrm{Kh}_{\mathbb{Z}_2}(K)$ in the case when we have to emphasize that the Khovanov complex is considered over the field $\mathbb{Z}_2$).

Theorem 5.1 ([218, 221]). *The graded homology* $\mathrm{Kh}(K)$ *is an invariant of the link* K; *the graded Euler characteristic* $\chi(\mathrm{Kh}(K))$ *is equal to the Jones polynomial.*

The second statement of the theorem follows from the fact that the Euler characteristic defined as the alternating sum of (graded) dimensions of homology groups is equal to the alternating sum of the graded dimensions of chain spaces.

The proof for the homology to be invariant under the Reidemeister moves just repeats the proof of the Khovanov homology for the case of classical links.

To prove the invariance, Bar-Natan used the so-called *cancellation principle* which means the following.

Let $\mathcal{C}$ be a (bigraded) chain complex, and let $\mathcal{C}'$ be a subcomplex of the complex $\mathcal{C}$.

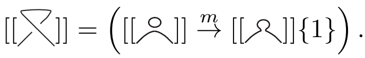
Then,

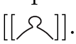
- if the complex $\mathcal{C}'$ is acyclic, then $H(\mathcal{C}) = H(\mathcal{C}/\mathcal{C}')$;
- if the complex $\mathcal{C}/\mathcal{C}'$ is acyclic, then $H(\mathcal{C}) = H(\mathcal{C}')$.

Here H stays for homology groups. Later on, for any of Reidemeister moves, the bifurcation cubes (corresponding to the diagram before the move is applied and to the diagram after the move is applied) are split into subcubes; the set of vertices of the whole cube is the disjoint union of sets of vertices of these subcubes; the subcubes refer to those sets of vertices the corresponding Reidemeister move is applied to. Thus, for example, in the case of the move Ω_3 the cube of dimension n is split into eight subcubes, each of dimension $(n-3)$. Then, applying the cancellation principle and taking into account the "locality" of partial differentials (corresponding to the maps Δ and m), we may reduce the homology of the cube of dimension n to a simpler form. It turns out that this simpler form is the same for the diagrams before the Reidemeister move and after the Reidemeister move.

Herewith, in the proof we *never use the global information about the differentials*, i.e. we never say that some edge of the bifurcation cube is a comultiplication (Δ) or multiplication (m) if it is clear from the picture.

The invariance under the first classical Reidemeister move repeats Bar-Natan's arguments from [17] verbatim. The complex corresponding to the diagram with an added curl looks like:

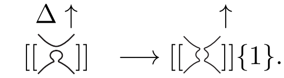


The map m is surjective; herewith the subcomplex of the complex  in the left part, where the small circle is marked by $\pm v_+$, is mapped by m to the whole complex  in the right part. Thus, we see that the whole complex has the same homology as its quotient complex by the acyclic part described above. This quotient complex has the same homology as .

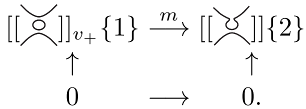
Analogously, one can consider the other curl; in that case one has to use the injectivity of Δ.

Like the case of the first Reidemeister move, the invariance proof under the second Reidemeister move repeats Bar-Natan's proof in the classical case. We shall give it here because it will be useful for us when we prove the invariance under the third Reidemeister move.

The initial complex $\mathcal{C}$ looks like:



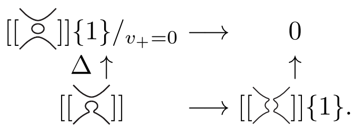
This complex contains the following subcomplex $\mathcal{C}'$:

$\mathcal{C}' =$ 

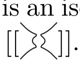
Here and further, the lower index v_+ (in the leftmost part) denotes the mark on the small circle.

The acyclicity of the complex $\mathcal{C}'$ is evident.

Taking the quotient of $\mathcal{C}$ modulo $\mathcal{C}'$, we get:



In the upper-left angle, we have $v_+ = 0$, which means that the space $V = \{v_+, v_-\}$ corresponding to the small circle is taken modulo the relation $v_+ = 0$, i.e. in the corresponding (signed) tensor product instead of the usual two-dimensional space we take just the one-dimensional space generated by v_-.

In the latter complex, the arrow Δ directed upwards, is an isomorphism. Consequently, this complex has the same homology as . This completes the invariance proof under Ω_2 (up to a height shift and a grading shift).

The transformation of the Khovanov complex under the third Reidemeister move is shown in Fig. 5.2. Here on each of the two cubes we have one map of type Δ and one map of type m; we do not make assumptions about the other maps.

The invariance of the Jones polynomial in one variable under the third Reidemeister move Ω_3 is usually proved as follows. We smooth one crossing for two diagrams, after that we see that the resulted diagrams either coincide or are obtained from each other by applying a second Reidemeister move Ω_2. We shall act in the same manner: Let us consider the three-dimensional cube (each vertex of it is a cube of the codimension three); two of these cubes, which are distinguished from each other by the third Reidemeister move, have the same low levels (subcubes) and different upper levels by a second Reidemeister move.

Let us consider the special case of the second Reidemeister move Ω_2 related to the upper level in Fig. 5.2. The corresponding complex $\mathcal{C}/\mathcal{C}'$

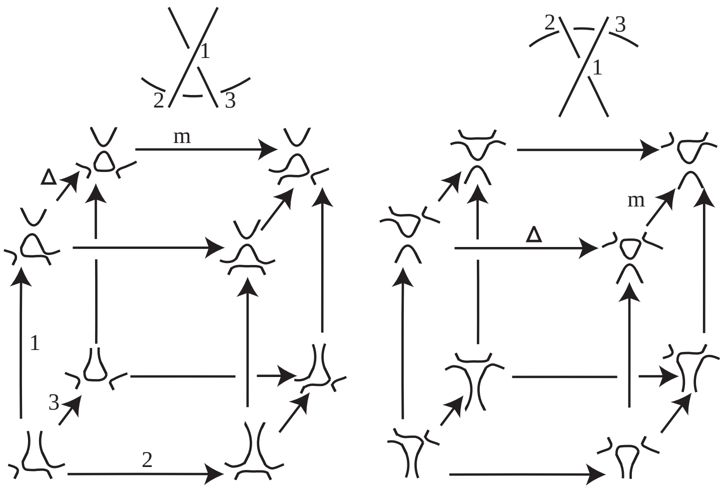


Fig. 5.2 The alteration of the Khovanov complex under Ω_3.

contains the following subcomplex:

$$\mathcal{C}''' = \begin{array}{ccc} \beta & \longrightarrow & 0 \\ \Delta\uparrow & \overset{\tau=d_{*0}\Delta^{-1}}{\searrow} & \uparrow \\ \alpha & \xrightarrow{d_{*0}} & \tau\beta. \end{array}$$

The latter is acyclic since the map Δ is a monomorphism.

Remark 5.3. Here the arrow τ is not a partial differential. Later on, an arrow of the form $\beta \to \pm\tau\beta$ on the diagonal means that we identify two elements of the cube. A sign $\pm$ depends on where the additional sign minus stands on the two-dimensional face. In the case of coefficients from $\mathbb{Z}_2$ the presence of this sign is inessential.

In the case of coefficients from a ring with characteristics different from two, a sign can appear; all arguments are analogous to those of Bar-Natan. We shall consider the problem with signs under the third Reidemeister move in Sec. 5.7.

After that we see that the complex $\mathcal{C}$ looks like:

$$(\mathcal{C}/\mathcal{C}')/\mathcal{C}''' = \begin{array}{ccc} \beta & \longrightarrow & 0 \\ \uparrow & \searrow & \uparrow \\ 0 & \longrightarrow & \gamma. \end{array}$$

By virtue of the cancellation principe, we can apply this operation (factorizing the upper face of the three-dimensional cube over the corresponding complexes $\mathcal{C}'$ and $\mathcal{C}'''$) for two cubes shown in Fig. 5.2 (only for their upper faces). As a result, we shall get the cubes shown in Fig. 5.3.

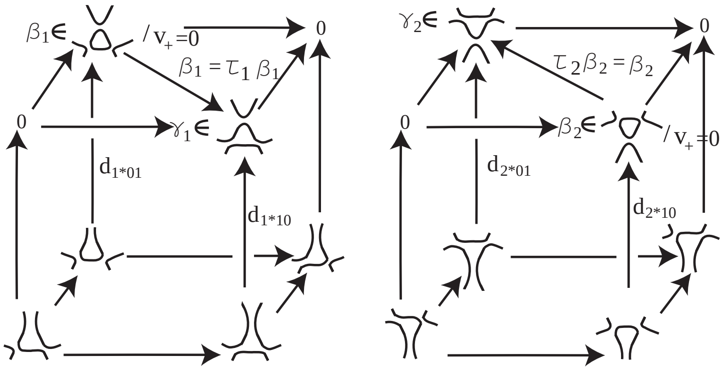


Fig. 5.3 The invariance under Ω_3.

The two obtained complexes are actually isomorphic to each other (an isomorphism can be defined by the map, which preserves low levels shown in Fig. 5.3 and maps the pair (β_1, γ_1) to the pair (β_2, γ_2) in the upper level). The fact that the map is actually an isomorphism of linear spaces is evident. To prove that this map is also an isomorphism in the level of complexes, we have to prove that their maps commute with the maps on the vertical edges. It is sufficient to note that $\tau_1 \circ d_{1*01} = d_{2*01}$ and $d_{1*10} = \tau_2 \circ d_{2*10}$.

Theorem 5.1 is proved.

Remark 5.4. The same local arguments can be applied to any cubes (e.g. to those which have differentials of any form related to bifurcations of type $1 \to 1$).

Definition 5.3. Let us call by the *height* $h(\mathrm{Kh}(K))$ of the Khovanov homology of a virtual link K the difference between the leading and lowest non-zero quantum gradings of non-zero Khovanov homology of K.

The height of the Khovanov homology justifies the estimates coming from the span of the Kauffman bracket polynomial. The latter is responsible for non-cancellability of the leading and lowest terms in the decomposition (4.1); in the same time chains of the Khovanov complex are in natural one-to-one correspondence with monomials of the bracket multiplied by $(-a^2 - a^{-2})$.

By construction it is clear that

$$h(\mathrm{Kh}(K)) - 2 \geqslant \frac{\mathrm{span}\langle K\rangle}{2}.$$

Note that the complex $\mathcal{C}(K)$ splits into the direct sum of two complexes: the complex with an even grading and the complex with an odd grading (recall that the differential preserves the grading).

We get two types of the Khovanov homology: the *even* one Kh^e and *odd* one Kh^o.

They correspond to monomials of the Jones polynomial, having degrees congruent to two modulo four (Kh^o), and monomials the degrees of which are divided by four (Kh^e). A classical link has only one of these two types, more precisely, the following theorem holds.

Theorem 5.2. *For a classical link with even number of components the isomorphism* $\mathrm{Kh}^o \cong 0$ *holds. For a classical link with odd number of components the isomorphism* $\mathrm{Kh}^e \cong 0$ *holds.*

This theorem is completely analogous to Proposition 4.1 about degrees of monomials occurring in the Jones polynomial.

Moreover, it is easy to check that this theorem is true not only for classical links but also for virtual links having a diagram with orientable atom.

Example 5.1. Let us consider the diagram K depicted in Fig. 5.4 (left).

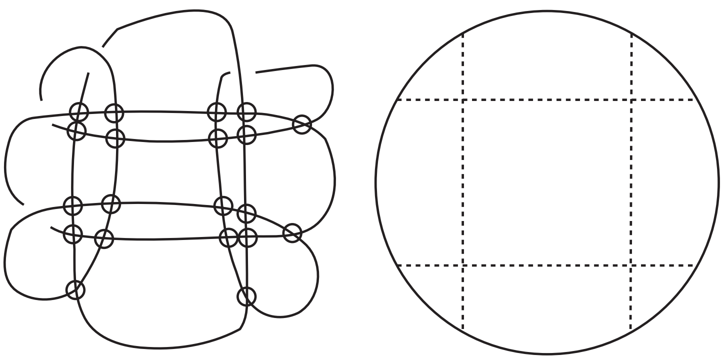


Fig. 5.4 A virtual knot with orientable atom with genus 2.

The chord diagram corresponding to the leading state of the Kauffman bracket polynomial is depicted in the picture on the right. In this state there exists one circle, and in any of four crossings this circle can be transformed into one circle by using the corresponding dashed chord (with framing 1).

We assert that this link has no diagrams with orientable atoms. Indeed, for the given diagram both complexes Kh^o and Kh^e (with coefficients in $\mathbb{Z}_2$) have non-trivial homology. Actually, the A-state of the diagram with one circle with a label 1 gives non-trivial cycle (since all differentials coming from the A-state to neighboring states are zero). Further, in states where one crossing is B-smoothed and the other three crossings are A-smoothed there exists exactly one circle. Let us consider the chain equal to the sum of chains having label 1 at each of these four states. It is easy to check that this chain is a cycle. Further, it cannot be a boundary, since all chains in the A-state are cycles.

There are two homology groups, whose quantum gradings differ by 1, therefore, the link has no diagram with orientable atoms.

In particular, we have shown that the atom genus (the Turaev genus) of the link is equal to one.

Note that this fact cannot be revealed by using the Kauffman bracket polynomial. Indeed, in the A-state (as well as in the B-state) there exists exactly one circle, at each state with one (or three) crossing A-smoothed we have one circle, and if we have two A-smoothed crossings and two B-smoothed crossings, then in two cases we shall have one circle and in the remaining four cases we shall have two circles. Therefore, the Kauffman bracket polynomial of K looks like:

$$\langle K \rangle = a^4 + 4a^2 + 2 + 4(-a^2 - a^{-2}) + 4a^{-2} + a^{-4} = a^4 + 2 + a^{-4}.$$

All terms of this Kauffman bracket polynomial have degrees congruent to each other modulo four. Therefore, in the given case the Khovanov homology is more sensitive to non-orientability of atoms than the Kauffman bracket polynomial.

Note that the constructed Khovanov complex with coefficients in the field $\mathbb{Z}_2$ is completely defined by the structure of the bifurcation cube and the numbers n_+, n_-. Therefore, the Khovanov $\mathbb{Z}_2$-homology does not change under the virtualization of the given link.

In the next section, we shall give another approach to the construction of the Khovanov complex (for framed links) which is sensitive to the virtualization. The Khovanov complex given here coincides with the general Khovanov complex with coefficients in $\mathbb{Z}_2$ in the classical case; in this case it is easy to overcome the difficulty with bifurcations of type $1 \to 1$. In the next two sections we shall construct the Khovanov complex for not all diagrams of virtual links but only for "right" virtual diagrams, which have no partial differentials of type $1 \to 1$ on the cube. As we shall see later, "right" virtual diagrams are those diagrams which orientable atoms correspond to. Later on, we shall construct a "right" virtual diagram for each virtual diagram by some invariant way and see how the Khovanov homology of the corresponding "right" virtual diagram changes under the generalized Reidemeister moves applied to the initial diagram (not necessarily "right"). In the next section we shall construct the Khovanov complex for framed links where the double diagram plays the role of a "right" diagram.

Example 5.2. Let us consider the virtual knot diagram shown in Fig. 5.5. This knot was first considered by Kauffman.

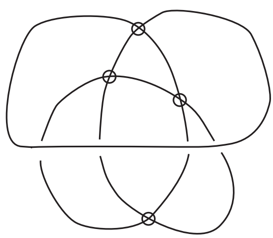


Fig. 5.5 A virtual knot reduced to the unknot by the virtualization and the generalized Reidemeister moves.

This knot can be reduced to the unknot with virtualizations and generalized Reidemeister moves, see Fig. 5.6.

In Fig. 5.6 by the transformation B' we mean a move applied to one classical and one virtual crossing; it represents a composition of the virtualization and the second Reidemeister move, see Fig. 5.7. For each of the transformations shown in Fig. 5.6, we pick out a domain which this transformation is applied to.

Thus, the Khovanov $\mathbb{Z}_2$-homology of the knot depicted in Fig. 5.5 coincides with the Khovanov $\mathbb{Z}_2$-homology of the unknot. One can show (e.g. using the techniques of virtual quandles) that this virtual knot is not trivial.

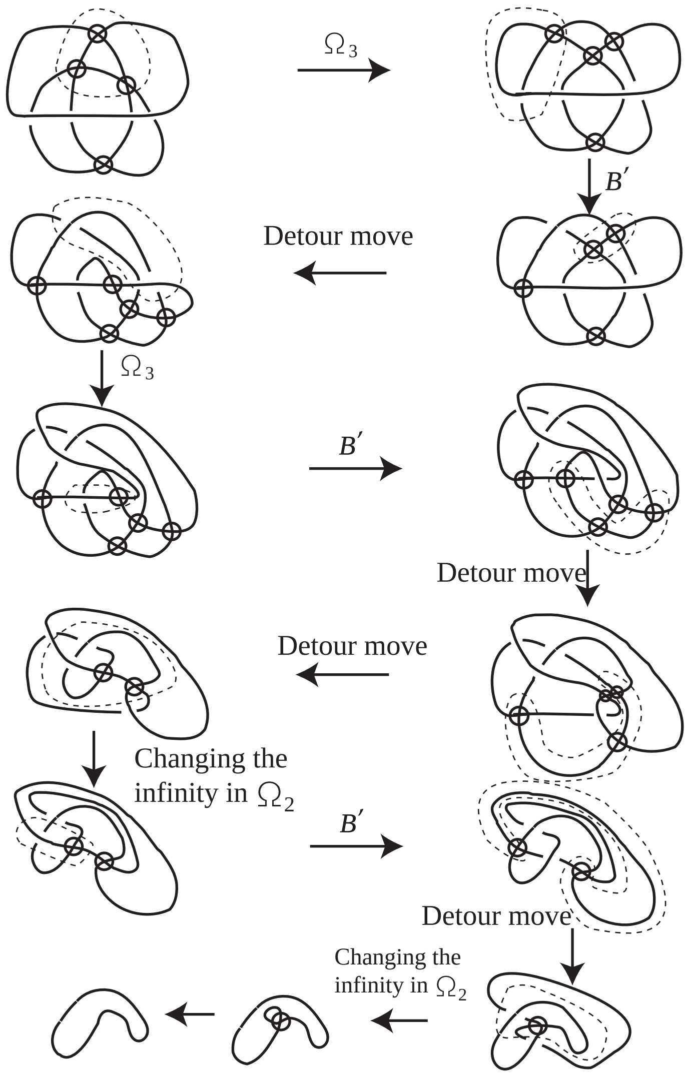


Fig. 5.6 Reducing to the unknot by virtualizations and generalized Reidemeister moves.

5.4 Khovanov homology of double knots

In the next three sections, we shall use the construction connecting atoms with virtual knots. Recall this construction given in Chap. 4 which assigns to a height atom a classical link. This construction is as follows. We embed

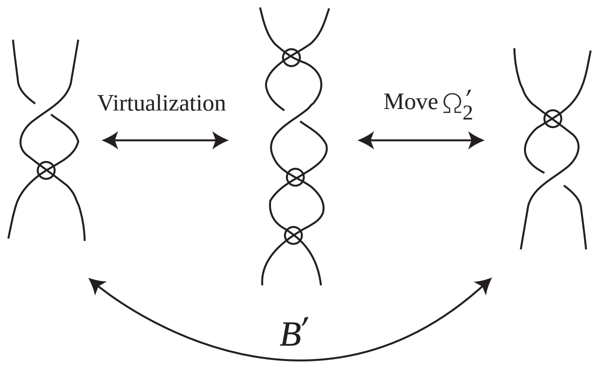


Fig. 5.7 The move B' is expressed in terms of the virtualization.

the frame of an atom in the plane with its A-structure preserved, and each crossing is equipped with the over/undercrossing structure according to the B-structure of the atom.

Let an arbitrary atom be given. Let us immerse its frame in the plane with the A-structure preserved, construct a virtual diagram K from the atom in the way given in Chap. 4.

As it was shown in Theorem 4.2 the equivalence class of K is well defined up to virtualizations.

Definition 5.4. Let us call a diagram of a virtual link *orientable*, if the corresponding atom is orientable, and *non-orientable* otherwise. Call a virtual link *orientable*, if it can be represented by an orientable diagram.

The notion of an orientable diagram under another name was in Kamada's papers [149, 150, 152]; it said about a possibility to swap some classical crossings (overcrossing to undercrossing and vice versa) such that we got an alternating diagram. It is not difficult to check that the class of such diagrams coincides with the class of orientable diagrams.

Remark 5.5. It is not worth to confuse the notion of an *orientable* diagram (in the sense of orientability of the corresponding atom) with the *orientability* (in the sense of existence of an orientation for each component of a link).

Let K be a virtual diagram with an orientable atom.

Define the complex $\mathcal{C}(K)$ as follows. Fix a ring R of coefficients and two-dimensional free module V over this ring such that $\operatorname{qdim} V = q + q^{-1}$.

The chain space of our complex is the same as in the case of coefficients from $\mathbb{Z}_2$. After that a differential is defined as the sum of partial differentials with signs, and partial differentials are defined with the maps m and Δ.

Definition 5.5. The cube with partial differentials ∂' going along edges in the coordinate increasing direction is called *commutative*, if each two-dimensional face of this cube is a commutative diagram and *anticommutative*, if each two-dimensional face is an anticommutative diagram.

In the case of coefficients from the field $\mathbb{Z}_2$ the commutativity of each face is equivalent to its anticommutativity. In the case of coefficients from $\mathbb{Z}$ one can make an anticommutative cube from a commutative cube in the following way.

Assign to all edges of the cube $\{0,1\}^n$ sequences consisting of elements from $\{0,1,*\}$ and having the length n and one element $*$. Each such edge connects two vertices obtained by replacing $*$ by one and zero.

Definition 5.6. The *height* $|\xi|$ of an edge ξ is the smallest height among two heights of its ends.

Thus, if we denote the map corresponding to an edge ξ by ∂'_ξ, then the differential looks like:

$$\partial^r = \sum_{\{|\xi|=r\}} (-1)^\xi \partial'_\xi.$$

Now we have to explain what the sign $(-1)^\xi$ means and define the map ∂_ξ. To well define the operator ∂ such that the property $\partial \circ \partial = 0$ holds, it is sufficient to show that partial differentials ∂'_ξ on two-dimensional faces of the cube are anticommutative diagrams.

A commutative cube can be transformed to an anticommutative cube as follows. First, we have to construct maps on edges such that each two-dimensional face is a commutative diagram, and then we shall equip partial differentials ∂'_ξ with signs. A sign is defined by the following rule. Vertices of the cube are ordered (the homology will not depend on an order). To each vertex of the cube we assign the numbers of all its unit coordinates in the increasing order: $i_1, i_2, \ldots, i_k$ and the formal exterior product $x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_k}$. For example, for $n = 3$ we assign to the vertex $\{1,0,1\}$ the exterior product $x_1 \wedge x_3$. Each edge of the cube, increasing some jth coordinate, can be treated as the exterior multiplication in the right by x_j. If as a result of application of this exterior multiplication to a "lower" vertex we get an exterior product assigned to an "upper" vertex, we put the sign "plus" on the edge, and the sign "minus" otherwise. For example, for the edge $\{1, *, 1\}$ we have the sign minus since $(x_1 \wedge x_3) \wedge x_2 = -x_1 \wedge x_2 \wedge x_3$.

Thus, we got a collection of chain groups $[[K]]$ with the differential ∂.

Let an atom be given. Assume that for the A-structure of the atom there exists an orientation of all edges of the atom such that at each vertex two opposite edges are emanating and two other opposite edges are coming.

Definition 5.7. We call this structure the *source–sink* structure.

Remark 5.6. The same structure was investigated in the theory of virtual knots by Kamada, see, e.g. [150]. This structure was called an *alternating orientation* for a graph (in the present work we call this graph a frame of an atom).

Proposition 5.1. *The frame of an atom admits a source–sink structure if and only if the atom is orientable.*

Proof. Assume that the A-structure of an atom admits a source–sink structure. Let us consider black (supercritical) cells of the atom. We define the orientation of all black cells in such a way that at each vertex two local orientations defined by the black cells incident to this vertex agree with each other (i.e. at a vertex incident to two black cells two orientations defined by consecutive edges are directed either clockwise manner or counterclockwise manner). It is easy to see that in this case the atom is orientable.

The boundary of each black cell of the atom is a cycle consisting of consecutive edges, and each edge in this sequence is not opposite to its predecessor accordingly to the A-structure. Since at each vertex the source–sink condition holds, the cycle is oriented. The orientation of the cell boundary gives the orientation in the interior of the cell: It suffices to choose a basis, the first vector of the basis is oriented along an edge and the second one is directed to the interior of the cell. Perform the same procedure for each cell. The existence of a source–sink structure at each vertex means that the orientation of cells incident to the same vertex agree with each other. The orientation can be easily extended to all white cells.

Let us suppose that the atom is orientable. Fix an arbitrary orientation of the atom and define the orientation of edges of the atom in such a way that for each cell C and each edge e incident to C the orientation of the pair $\vec{e}$, ν is positive, where ν is a normal vector pointing from a point on the edge e inward the cell C, and $\vec{e}$ is a vector directed along the edge accordingly to the selected orientation.

It is evident that the obtained orientation of edges gives a source–sink structure. □

Remark 5.7. It follows from Proposition 5.1 that if an atom (M, Γ) with a frame Γ is orientable, then each atom (M', Γ) with the same frame and A-structure is orientable, too.

Remark 5.8. The source–sink structure given on the whole atom defines an orientation for circles at *all* states of the Kauffman bracket polynomial of the corresponding link. Thus, if one constructs a diagram obtained by smoothing of some crossings and deleting unlinked circles not being incident to chosen crossings, then the frame of the atom corresponding to the new diagram will inherit the source–sink structure from the initial one. Therefore, the obtained atom will be orientable.

From Proposition 5.1, it follows that if the atom corresponding to a virtual diagram is orientable, then there is no bifurcation of type $1 \to 1$ in the bifurcation cube corresponding to the diagram. Indeed, let us consider the frame Γ of the corresponding atom. Each state of the diagram is an atom having the frame Γ. Circles of the state serve for pasting black cells to the frame Γ. According to Proposition 5.1, the new atom is also orientable. Therefore, this atom cannot have a black cell approaching to itself in the non-orientable way (the way the smoothing at the crossing where this cell touches itself, does not change the number of circles).

Thus, bifurcation cubes are well defined for virtual diagrams with orientable atoms, namely, all bifurcations have the following types $1 \to 2$ and $2 \to 1$; partial differentials are defined by the maps m and Δ; the differential is defined as the sum of partial differentials with signs, and the statement that $\partial^2 = 0$ is checked analogously to the classical case.

Note the following two important lemmas.

Lemma 5.2. *Let K be a virtual diagram with an orientable atom. Then the collection of the groups $[[K]]$ together with the differential ∂ gives a complex, i.e. $\partial^2 = 0$.*

Proof. We have to check that each two-dimensional face of the cube $[[K]]$ is anticommutative. This is equivalent to the verification of the commutativity of two-dimensional faces before putting the signs ± 1.

Each two-dimensional face of the cube $[[K]]$ represents the atom with two vertices. Each two-dimensional face of the cube corresponds to a smoothing of some $(n-2)$ classical crossings of the diagram K, see Fig. 5.23.

The remaining two crossings can be smoothed arbitrarily; four possibilities of such a smoothing correspond to vertices of the two-dimensional face.

In these four states there are some number of common circles not being incident to the two crossings under consideration. After deleting these circles, we get an atom with two vertices.

Thus, we have to check that each two-dimensional face which can correspond to some atom with two vertices represents an anticommutative diagram.

Since the atom corresponding to K is orientable, then the atom corresponding to any two-dimensional face of the corresponding complex is also orientable (according to Remark 5.8).

Let us now use the theorem from [205] which tells us that all orientable atoms with two vertices are height atoms.

This means that each atom corresponding to a two-dimensional face of the bifurcation cube corresponding to an orientable atom occurs in the classical case. All such two-dimensional faces are sorted out in [17] and for them the commutativity of the corresponding diagrams is proved (before placing sings in differentials).

After that the proof follows line-by-line the proof in the classical case (see, e.g. [17]) and from the verification of properties of the maps m and Δ. □

Thus, we have shown that the collection of chains $[[K]]$ with the differential ∂ represents an anticommutative cube. Therefore, the complex $\mathcal{C}(K)$ is well defined.

Denote the homology of the complex by $\mathrm{Kh}(K)$.

Lemma 5.3. *Let K, K' be two virtual diagrams with orientable atoms, herewith K' differs from K by applying a detour move or one of the three classical Reidemeister moves. Then there exist an isomorphism of the Khovanov homology* $\mathrm{Kh}(K) \cong \mathrm{Kh}(K')$.

Proof. By applying the detour move, the structure of classical crossings does not change. Thus, the state cube does not change either, and, therefore, the complex does not change.

In the case of the *classical Reidemeister moves* we use the same proof (following Bar-Natan) based on the cancellation principle which was earlier used for the Khovanov homology with coefficients from $\mathbb{Z}_2$. It was local, i.e. it used only the local structure of Reidemeister moves (not depending on the fixed part of the link under the move). Note that in that proof we did not use the fact that the ring of coefficients was the field $\mathbb{Z}_2$ (we only used the injectivity of the map Δ and the surjectivity of the map m). Therefore, the proof passes word-for-word for virtual knots under the condition that all complexes are well defined. □

Proposition 5.2. *Let K be a diagram of a virtual link. Then the atom corresponding to the double diagram $D_2(K)$ is orientable.*

Proof. The proof follows from Proposition 5.1 in the following way.

Let K be a virtual diagram. Let us orient the diagram $D_2(K)$ such that for each point A on the link diagram the basis $(\partial K_A, \tau_A)$ gives the positive orientation of the plane, see Fig. 5.8.

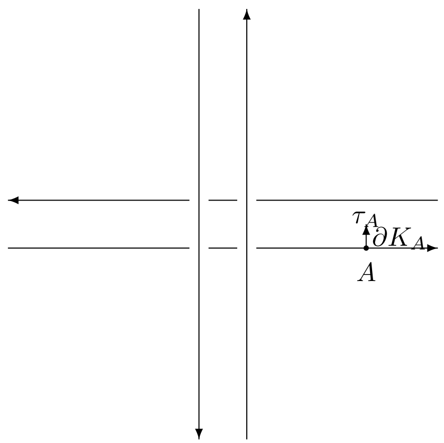


Fig. 5.8 Local orientation for the double diagram.

Here, ∂K_A is a tangent vector to the link, and τ_A is a vector from the point A to a point on the neighboring component and normal to the vector ∂L_A.

The desired source–sink structure can be obtained in the following manner: All edges (i.e. the images of edges of the corresponding atom) of the shadow of the double diagram $D_2(K)$ are partitioned into "long" edges (which originate from edges of the atom corresponding to K) and "short" ones (four short edges correspond to each crossing of K, see Fig. 5.8). Let us change the orientation of all short edges and leave the orientation of long edges. It is evident that the new orientation gives the source–sink structure. □

Taking into account Proposition 5.2 and Lemma 5.2 we conclude that the Khovanov complex for cables $D_{2n}(K)$ is well defined for any ring of coefficients. The map $K \mapsto D_{2n}(K)$ is almost invariant (it is invariant under all combinations of Reidemeister moves which do not change the writhe number). Therefore, it is natural to expect that the homology of the Khovanov complex for double diagrams (the double, triple diagrams and more general the operation of taking l-cable of a diagram $K \mapsto D_l(K)$ were defined in Sec. 4.3) of a knot is an invariant of framed links. Namely, the following lemma is true.

Lemma 5.4. *Let K, K' be two diagrams of equivalent framed virtual links. Then there exists a collection of diagrams $D_2(K) = K_0, K_1, \ldots, K_n = D_2(K')$ such that*:

(1) *all atoms corresponding to the diagrams K_i are orientable*;
(2) *for each $i = 0, \ldots, n-1$ the diagram K_{i+1} is obtained from the diagram K_i by applying one of the generalized Reidemeister moves.*

Proof. The detour move applied to the diagram K induces the composition of detour moves for the diagram $D_m(K)$.

Let us pass to the classical Reidemeister moves.

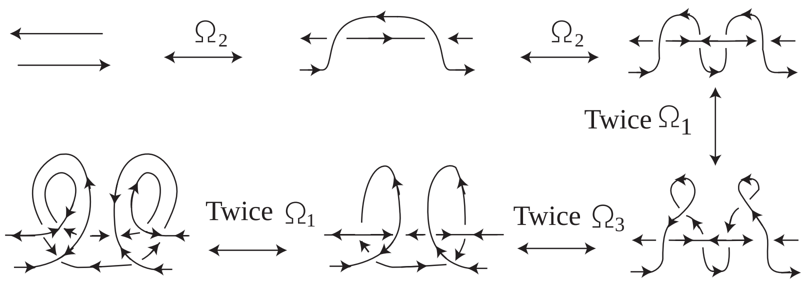


Fig. 5.9 Doubling the move Ω_1.

Let us use Proposition 5.2. For each double Reidemeister move from the chain from $D_2(K)$ to $D_2(K')$ we shall point out how the source–sink structure changes. It was done in Figs. 5.9–5.11.

In some evident cases, arrows in Figs. 5.9 and 5.11 mean not one but two "symmetric" classical Reidemeister moves or several Reidemeister moves,

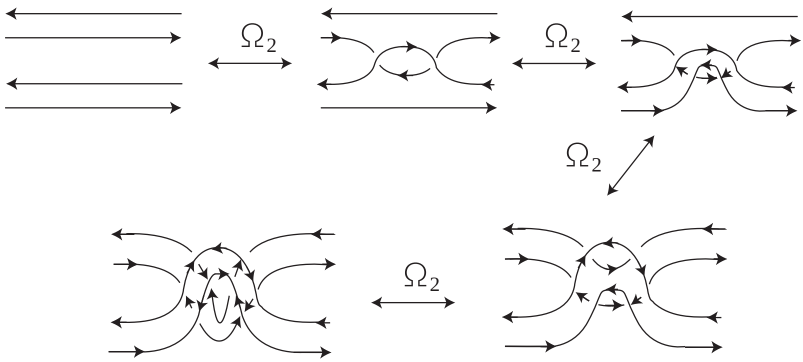


Fig. 5.10 Doubling the move Ω_2.

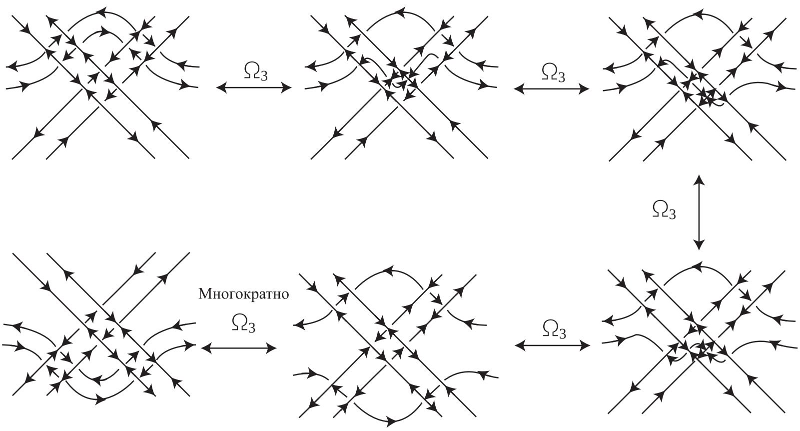


Fig. 5.11 Doubling the move Ω_3.

where the sequence of their applications is evident. They can be applied one by one and in the case of any move depicted in Figs. 5.9–5.11 it is easy to see that this successive transformation leads to an unambiguous transformation of the source–sink structure. □

Note the following. Under the construction of the complex $\mathcal{C}$ for virtual diagrams, it is worth to consider only classical Reidemeister moves, since the detour move does not change an atom.

So, let us consider all classical Reidemeister moves.

If diagrams K and K' are differed by applying the first or third Reidemeister move, then the local source–sink structure for the diagram K is in one-to-one correspondence with the local source–sink structure for K' such that outside the domain of the application of the move these diagrams coincide. Here the source–sink structure of lines depicted by dashed lines is defined as opposite to "thick" lines joining to them, see Fig. 5.12.

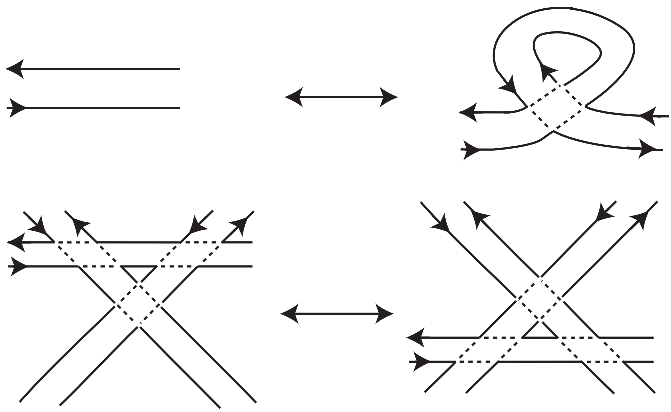


Fig. 5.12 Labeling for the doubling moves Ω_1 and Ω_3.

The second Reidemeister move has two principal different cases, depicted in Fig. 5.13.

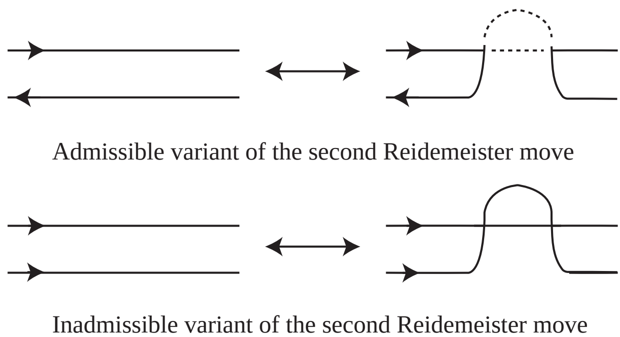


Fig. 5.13 Labeling for the doubling move Ω_2.

In the first case (the upper picture), we have two opposite directed arcs (according to the orientation of the source–sink structure), and in the second case we have two codirectional arcs.

In the first case, it is mentioned how the local labeling and the source–sink structure change.

The second case is not possible, i.e. it can lead to the fact that after applying the second Reidemeister move the atom becomes non-orientable.

Thus, the (increasing) Reidemeister move is the only move from the classical Reidemeister moves which can violate the orientability of the atom. There is no such move among moves described in the proof of Lemma 5.4.

Remark 5.9. One can consider the set of diagrams of virtual knots with orientable atoms and the set of moves on it consisting of all Reidemeister moves not violating the property of orientability (i.e. the detour move, the first and third classical Reidemeister moves and the "orientable" version of the second classical Reidemeister move).

This set was investigated by Kamada under the name *alternating virtual links*.

In particular, from the arguments given above (Lemma 5.3), it follows that the Khovanov complex is well-defined over any ring of coefficients and invariant in the category of orientable virtual links.

Theorem 5.3. *Let n be a natural number. Then* $\mathrm{Kh}(D_{2n}(K))$ *is an invariant of framed virtual links.*

Proof. According to Proposition 5.2, $\mathcal{C}(D_{2n}(K))$ is a well-defined complex. Let K, K' be two diagrams of equivalent framed virtual links. Then by virtue of Lemma 5.4, there exists a collection of virtual diagrams $D_{2n}(K) = K_0, \ldots, K_m = D_{2n}(K')$ corresponding to orientable atoms such that the diagram K_{i+1} is obtained from the diagram K_i by applying generalized Reidemeister moves. By Lemma 5.2 for each of the diagrams K_j the homology $\mathrm{Kh}(D_{2n}(K_j))$ is well defined. The invariance of the homology Kh under the detour move is obvious by construction. Thus, by virtue of Lemma 5.3 (which asserts the invariance under the classical Reidemeister moves), we get $\mathrm{Kh}(D_{2n}(K)) = \mathrm{Kh}(K_1) = \cdots = \mathrm{Kh}(D_{2n}(K'))$. □

Note that the double diagram of K and the double diagram of K' obtained from K by virtualizing one crossing, have different state cubes. Thus the complex constructed in the section can *a priori* distinguish framed virtual diagrams obtained from each other by virtualization.

However, the "double" Khovanov complex constructed in this section essentially differs from the "general" Khovanov complex for classical knots.

In the classical case as well as in the virtual case we have to double and after that we have to calculate the Khovanov homology.

It is natural to raise the question whether the "general" Khovanov homology $\mathrm{Kh}(K)$ is invariant in the case of diagrams with orientable atoms. The positive answer to this question will be given (with some restrictions) in the next section and (completely) in the sections devoted to the Khovanov homology for virtual links.

5.5 Khovanov homology of two-sheeted coverings and atoms

The main goal of this section is the construction of the Khovanov homology with the help of two-sheeted coverings. From this construction we have the following statement.

Statement 5.1. *Let* F *be a field, and let* K, K' *be two equivalent virtual diagrams with orientable atoms. Then there exists an isomorphism of graded homology* $\mathrm{Kh}_{\mathrm{F}}(K) \cong \mathrm{Kh}_{\mathrm{F}}(K')$.

Note that the general assertion about the invariance of this homology with arbitrary coefficients follows from the parity arguments (see below) and from the explicit construction of the Khovanov homology for virtual knots.

The main construction is as follows. For each virtual diagram K one can consider the atom $\mathrm{At}(K)$ corresponding to it. Later on, we shall use the techniques of *orientable covering*. Namely, if the atom $\mathrm{At}(K)$ is orientable, we consider two copies of $\mathrm{At}(K)$, if it is not, then we consider the atom $\widetilde{\mathrm{At}}(K)$ which is the orientable two-sheeted covering above the atom $\mathrm{At}(K)$. It is defined as the two-sheeted covering above the corresponding surface; here, the preimage of the frame is a graph which we consider as the frame, the preimage of a black cell is a pair of black cells, and the preimage of a white cell is a pair of white cells. The atom obtained in such a way can be either two-component or one-component, and it depends on the orientation of the initial atom.

Denote the virtual diagram corresponding to the atom $\widetilde{\mathrm{At}}(K)$ by $\widetilde{K}$.

If we apply a classical Reidemeister move Ω_i to the initial diagram K, then the move Ω_i will be applied to the diagram $\widetilde{K}$ in two places; here in the case of the move Ω_2, the *admissible* variant of the second Reidemeister move will be applied to $\widetilde{K}$ twice.

This construction can be treated as follows: We consider two sets of vertices of the atom with the A-structure at them and connect vertices by edges.

Thus, for each virtual knot we can consider its "covered version":

$$K \to \mathrm{At}(K) \mapsto \widetilde{\mathrm{At}}(K) \mapsto \mathrm{Kh}_{\mathrm{F}}(\widetilde{K}).$$

In terms of a knot diagram, this construction is described as follows. Let a virtual diagram K be given, this diagram has n classical crossings $v_1, \ldots, v_n$. These crossings are connected with each other in some way. Thus, we have a graph Γ immersed in the plane. Each crossing v_i has four (adjacent) ends v_{i1}, v_{i2}, v_{i3}, v_{i4} enumerated, for example, in clockwise manner, with crossings connected by branches of the diagram which edges of the atom correspond to. Let an edge e_j connect the ends $v_{j_1 j_2}$ and $v_{j_3 j_4}$, where j_2, $j_4 \in \{1, 2, 3, 4\}$.

The diagram $\widetilde{K}$ is constructed as follows. It contains $2n$ crossings $v'_1, \ldots, v'_n$, $v''_1, \ldots, v''_n$, which are connected by branches. Each edge e_j of the initial diagram has two preimages: e_j^1 and e_j^2. Each of two edges e_j^i connects the end $v'_{j_1 j_2}$ or $v''_{j_1 j_2}$ with the end $v'_{j_3 j_4}$ or $v''_{j_3 j_4}$. For each edge e_j^1 we have to choose which ends are connected (v' or v''). Here we have an arbitrariness in the description. The matter is that before describing edges we have not had a natural ordering of vertices: Which vertex of the vertices v'_i or v''_i is the "first" and which one is the "second". To overcome this difficulty let us choose some maximal tree T in Γ and say that all edges e_j^1 corresponding to edges of this graph connect ends $v'_{j_1 j_2}$ with $v'_{j_3 j_4}$ (thereby edges e_j^2 connect ends $v''_{j_1 j_2}$ and $v''_{j_3 j_4}$).

The choice of another tree will correspond to changing notations: v'_j and v''_j swap places in some pairs. After that the rule for connecting the remaining ends by edges e_i^1 and e_i^2 is the following. Pointing out which pairs of ends are connected by an edge e_i^α, we shall either connect them by the edge e_i^1 or e_i^2: the "symmetric" pair of ends corresponding to it obtained by swapping $v' \longleftrightarrow v''$ will also be connected by an edge. Henceforth, for constructing a virtual diagram it is not important for us to remember the notation of these edges.

We shall not pay attention to how we place edges e_i^α on the plane. The resulting class of the virtual link will not depend on it (by construction, diagrams will differ from each other by applying a finite sequence of detour moves).

So, we have fixed a maximal tree $T \subset \Gamma$. Each edge e_j not belonging to this tree represents the minimal cycle on the subgraph $T \cup e_j \subset \Gamma$. In the case when this cycle is *right* (see below), we connect the ends $v'_{j_1j_2}$ and $v'_{j_3j_4}$ by the edge e^1_j, and the ends $v''_{j_1j_2}$ and $v''_{j_3j_4}$ by the edge e^2_j. In the case of a *bad* cycle we connect the ends $v'_{j_1j_2}$ and $v''_{j_3j_4}$ by the edge e^1_j, and the ends $v''_{j_1j_2}$ and $v'_{j_3j_4}$ by the edge e^2_j. The notion of *right and bad* edges goes back to orientable and non-orientable cycles on the corresponding atom. An edge is called *right* if the corresponding cycle is oriented. Under the covering of the atom, orientable cycles are sent to cycles, and non-orientable cycles are sent to unclosed paths (with some ends in v'_k, v''_k). Let us define the notion of a *right edge* (for edges not belonging to T), and the notion of a *right* (*orientable*) cycle in terms of a diagram of the virtual link. For this we consider all edges of the given cycle e_{j_1}, $e_{j_2}, \ldots, e_{j_k}$, $e_{j_{k+1}} = e_{j_1}$, where edges e_{j_i}, $e_{j_{i+1}}$ meet at a vertex (the indices i are taken modulo k) and let us try to define locally the source–sink structure along them. Let us orient the edge e_{j_1} in some way. Further, if the edge e_{j_2} is opposite to the edge e_{j_1} at a vertex, then we orient e_{j_2} such that either both edges e_{j_1} and e_{j_2} come into the vertex, or both edges emanate from it; in the case when the edges are not opposite, we shall make one of them come into the vertex and the other emanate from it. Further, we shall make the same thing at the orientation of the edges e_{j_3}, $e_{j_4}, \ldots$. If the process succeeds, i.e. we have defined on the edge $e_{j_{i+1}} = e_{j_1}$ an orientation coinciding with the initial one, we call the cycle *right*, and *bad* otherwise. Namely, a cycle is called *right* (*oriented*) if the number of its transversal passing through classical crossings, vertices of the atom, is even.

Remark 5.10. For a plane diagram the parity of the number of transversal passing through classical crossings coincides with the parity of passing through virtual crossings (all these passings are transversal).

It is easy to check that this definition of a right cycle coincides with the definition of an orientable cycle on the atom defined by the A-structure. Setting successively orientations of edges according to the source–sink structure we define orientations of black cells approaching (locally) to these edges. The first vector of the basis is directed along the orientation of the edge, and the second one is directed inward the black cell. If we return to the initial edge with the same orientation, then this means that we have traveled along an orientable cycle, and a non-orientable cycle otherwise. Indeed, if we pass through a classical crossing, then orientations of neighboring cells defined in such a way, are opposite to each other. Thus, getting a compatible orientation means precisely that our path goes transversely evenly many times.

So, we have defined the notion of a *right* (*orientable*) cycle and a *right* edge (for edges not belonging to the tree T). Therefore, we have completely constructed the virtual diagram $\widetilde{K}$. Note that the definition of a right cycle does not depend (up to detour moves) on the choice of the tree T.

Moreover, by the atom $\widetilde{\mathrm{At}}(K)$ the knot corresponding to the two-sheeted covering is restored up to virtualizations (it does not change the Khovanov homology by the way); we have already mentioned the explicit way of constructing the diagram $\widetilde{K}$ with the diagram K; it corresponds to some immersion of the frame of the atom $\widetilde{\mathrm{At}}(K)$ (with preserving the A-structure).

It is easy to see that the detour move in the initial diagram K of the link induces some combinations of the detour moves on the diagram $\widetilde{K}$. Moreover, the following lemma takes place.

Lemma 5.5. *By applying one of the classical Reidemeister moves to a diagram K the diagram $\widetilde{K}$ will change in the following way: The same Reidemeister move is applied to it in two places. Herewith, the atom corresponding to the "middle" diagram obtained from $\widetilde{K}$ by applying the second Reidemeister move in one place (any of two places) is orientable.*

Proof. We shall denote diagrams before and after applying Reidemeister moves by K and K', respectively, the framings of the corresponding atoms by Γ, Γ', and the corresponding diagrams of the coverings by $\widetilde{K}$, $\widetilde{K}'$.

Each Reidemeister move represents a transformation of a diagram inside some domain; inside this domain the diagram K has some form P, and the diagram K' has some form Q. Here, we have some collection of tails $t_1, \dots, t_m$ connecting the subdiagram P (of the diagram K) or the subdiagram Q (of the diagram K') with the remaining (fixed) part of the diagram. In the case of the first Reidemeister move m is equal to two, in the case of the second Reidemeister move this number equals four, and it equals six in the case of the third Reidemeister move. On the diagrams $\widetilde{K}$ and $\widetilde{K}'$, each tail t_i is lifted to two tails t_i', t_i''. When the subdiagram P or the subdiagram Q does not contain non-orientable cycles, on $\widetilde{K}$ we get two copies of the subdiagram P, and on $\widetilde{K}'$ we get two copies of the subdiagram Q. The claim of the theorem is that these copies are connected by tails in a compatible way. An example of two connections is given in Fig. 5.14.

In the first case, all vertices t_i', t_i'' are connected in the same manner when we lift the diagrams K and K', namely, all t_j' are connected with each other, and t_l'' are connected with each other. In the second case in Fig. 5.14 the connection is the first one when we lift the diagram K, and the connection when we lift the diagram K', is another. For example, one

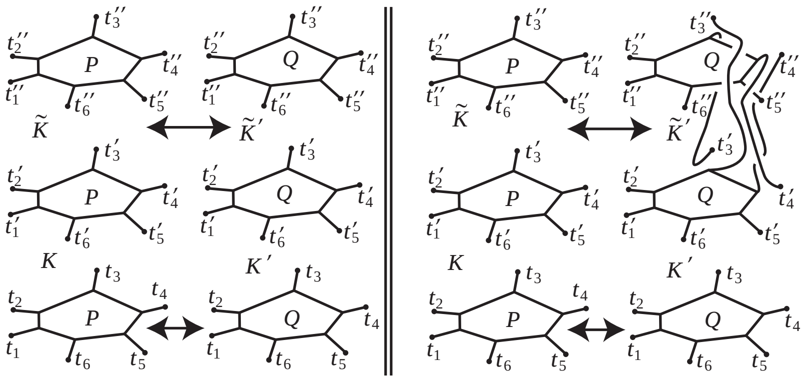


Fig. 5.14 Two different connections.

copy is connected with vertices t_1', t_2', t_3'', t_4'', t_5', t_6', and the second one is connected with t_1'', t_2'', t_3', t_4', t_5'', t_6''. We have to show that the second case cannot occur.

When we apply the increasing first Reidemeister move, a loop appears which consists of one edge e_j, the initial point (it is also the final point) of which splits some edge e_l. It is obvious that the edge e_j considered as cycle is right. Moreover, the edge e_l is split into two edges e_{l_1} and e_{l_2} which are not opposite at the crossing of splitting. Let us consider the cycle C on the graph Γ' containing the edge e_l. The cycle C' containing the edges e_{l_1} and e_{l_2} on the graph Γ' corresponds to it in a natural way. If the cycle C is orientable, then the cycle C' is also orientable, and vice versa. This follows from the fact that if the edges e_{l_1} and e_{l_2} are not opposite, then under the definition of a cycle one can define the orientation successively originating from the orientation of the edge e_l on these edges. Therefore, if on the diagram $\widetilde{K}$ the edge e_l^1 connects, say, ends v'_{pq} with v''_{rs}, then the pair of the edges $e_{l_1}^1$, $e_{l_2}^1$ will connect the ends corresponding to them (with the same notations) v'_{pq} and v''_{rs}. This means that the connection has the first type.

We have the same situation with the remaining two Reidemeister moves. Each of these moves, Ω_2 or Ω_3, represents a transformation of some domain in the plane; from this domain several tails come out (in the case of the second Reidemeister move we have four tails and in the case of the third Reidemeister move we have six tails). Let us consider the collection of ends $\{v_{ij}\}$ of these tails.

Let us consider the second Reidemeister move. The diagram K' has the bigon cd and four emanating edges a, b, e, f, see Fig. 5.15.

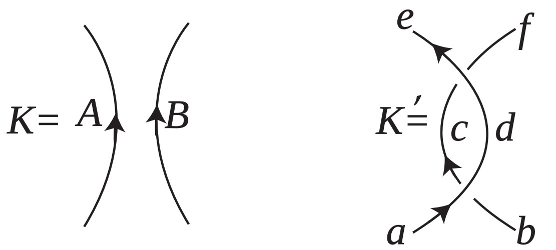


Fig. 5.15 The case of the second Reidemeister move.

This bigon, considered like a cycle, is right, since it has no transversal passing through crossings. Thus, the whole collection of edges is uniquely lifted on the diagram $\widetilde{K}'$. As a result, we get two collections of branches a', b', c', d', e', f' and a'', b'', c'', d'', e'', f''. The branches c', d' form a bigon to which we can apply the decreasing second Reidemeister move. We can also make the bigon with branches $c''d''$. By applying the decreasing second Reidemeister moves to them, we get the diagram $\widetilde{K}$. For this it is sufficient to note that after this decreasing Reidemeister move, the edge a' is connected with the edge e', and the edge b' with the edge f' (herewith a'' is connected with e'', and b'' with f''). The latter follows from the fact that any cycle on K' passing successfully through a, c, e has as many transversal passings through vertices as the cycle on K corresponding to it and passing through the edge A has.

From this the second claim of the lemma follows.

In the case of the third Reidemeister move on the diagram K as well as on the diagram K' we have a triangle (h, i, f) and (k, m, n) and six exterior branches (a, b, g, d, c, e) and (a, b, l, d, c, j).

Both triangles represent right cycles, since they do not contain transversal passings through classical crossings. Therefore, the corresponding domains P and Q are lifted to two copies of the domains P and Q. It remains to check that their lifts have the first type.

For this we have to show that any two paths $\gamma \subset K$ and $\gamma' \subset K'$ connecting ends t_i, t_j are "equally" lifted on K and $\widetilde{K}$. For example, if one of two preimages $\widetilde{\gamma}$ of the path γ connects points t_i', t_j' and passes inside the domain P, then for each path γ' having the same ends as the path γ

has one of the preimages $\widetilde{\gamma}'$ connects the same points t'_i, t'_j (but not t'_i, t''_j). In Fig. 5.16 an example of such paths is shown.

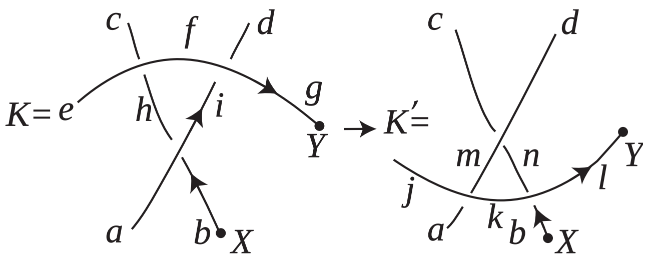


Fig. 5.16 The case of the third Reidemeister move.

Namely, let us consider two paths between points X and Y on the diagrams K and K'. We assert that they are lifted in the same manner on $\widetilde{K}$ and $\widetilde{K}'$, respectively. This follows from the fact that the number of points passed transversally is the same for these paths (it equals zero). Analogously, one can prove the sameness of the lifting of any two paths in Fig. 5.16 with the same ends. □

According to Lemma 5.2, the homology $\mathrm{Kh}(\widetilde{K})$ is well defined.

Therefore, by Lemma 5.3 the Khovanov homology of a "covered" knot does not change under applying Reidemeister move to the initial knot. From here we get the theorem.

Theorem 5.4. *The map* $K \mapsto \mathrm{Kh}(\widetilde{K})$ *gives a well-defined invariant of virtual links.*

Remark 5.11. Note that only the second Reidemeister move Ω_2 can change the type of the corresponding atom (i.e. it can convert a non-orientable atom to an orientable one and vice versa). If, for example, we have an orientable atom $\mathrm{At}(K)$ and two components of the atom $\widetilde{\mathrm{At}}(K)$, then the application of the second Reidemeister move (inadmissible version) to K can "connect" these components into one (this corresponds to the fact that after applying the second Reidemeister move, the atom may be non-orientable).

Herewith the moves Ω_1, Ω_3 preserve the orientability of the atom.

Now let the atom corresponding to a diagram K be orientable. Then $\widetilde{K}$ consists of two copies of the atom corresponding to K. Since F is a field, we have $\mathrm{Kh}_{\mathrm{F}}(\widetilde{K}) = \mathrm{Kh}_{\mathrm{F}}(K)^{\otimes 2}$.

Therefore, the homology $\mathrm{Kh}(K)$ is obtained from the invariant homology $\mathrm{Kh}(\widetilde{K})$ by "extracting of the tensor square root". In the case when the ring of coefficients is a field, we have the Poincaré polynomial $\mathfrak{P}$ in two variables with all integer non-negative coefficients. From this polynomial we have to extract the "square root", i.e. to find the Laurent polynomial $\mathfrak{Q}$ in the same two variables with integer non-negative coefficients (coefficients are non-negative since they are the ranks of Khovanov homology groups) such that the equality $\mathfrak{Q}^2 = \mathfrak{P}$ holds. It is obvious that if we can do this, then it can be done uniquely. Since this operation is unique, if it exists, we get the claim of Theorem 5.1.

Moreover, from these discussions we get the following theorem.

Theorem 5.5. *Let* F *be a field, and let for a virtual diagram K the graded homology* $\mathrm{Kh}_{\mathrm{F}}(\widetilde{K})$ *cannot be represented as the tensor square. Then K has no diagram with an orientable atom. In particular, the virtual link generated by K is not classical.*

It is natural that the Khovanov complex constructed in this section cannot detect non-triviality of the virtual knot depicted in Fig. 5.5, since this knot is obtained from the unknot by generalized Reidemeister moves and virtualizations.

The question about whether two non-isotopic classical links can be obtained from each other by a finite sequence of generalized Reidemeister moves and virtualizations is an important and interesting conjecture. The Khovanov complex gives a partial answer to this question.

From Theorem 5.1 and the invariance of the Khovanov homology under virtualization, we have the following theorem.

Theorem 5.6. *If a classical link is obtained from a classical link by applying generalized Reidemeister moves and virtualizations, then these links have the same Khovanov homology with coefficients from any preassigned field.*

Later in this chapter we shall show that this theorem is true for arbitrary coefficients (e.g. from the ring $\mathbb{Z}$), see Theorems 5.11 and 5.12.

5.6 Khovanov homology and parity

Let us have a map $\widetilde{f}$ sending the set of diagrams of virtual knots into itself and having the following properties:

(1) for each virtual diagram K the diagram $\widetilde{f}(K)$ is a virtual diagram with an orientable atom;
(2) if a diagram K has an orientable atom, then $\widetilde{f}(K) = K$;
(3) if two diagrams K and K' are equivalent by means of Reidemeister moves, then $\widetilde{f}(K)$ and $\widetilde{f}(K')$ are equivalent by means of Reidemeister moves, where all middle diagrams connecting the diagrams $\widetilde{f}(K)$ and $\widetilde{f}(K')$ have orientable atoms.

Theorem 5.7. *The map* $K \mapsto \mathrm{Kh}(\widetilde{f}(K))$ *is an invariant of virtual links.*

The map $\widetilde{f}$ is constructed by means of consecutive application of the following operation: For a virtual diagram we replace all odd classical crossings with virtual crossings, and continue this operation until all crossings are even, for more details see Chap. 8.

5.7 Khovanov homology for virtual links

5.7.1 *Atoms and twisted virtual knots*

Bifurcations of types $2 \to 1$ and $1 \to 2$ in the Khovanov complex will (see Sec. 5.7.2) correspond to *partial differentials* ∂' which the differential ∂ consists of (see below); the bifurcation of type $2 \to 1$ corresponds to the multiplication m, and the bifurcation of type $1 \to 2$ corresponds to the comultiplication Δ.

In Sec. 5.2 by the following variable change $a = \sqrt{-q^{-1}}$ we got instead of the Jones polynomial X its modified version J, and also the polynomial $\widehat{J} = J \cdot (q + q^{-1})$.

In this chapter we deal with the polynomial $\widehat{J}$ and call it the *Jones polynomial.*

As it was shown earlier, all necessary information for calculating the Jones polynomial is contained in the atom corresponding to a virtual diagram.

The whole information about the number of circles in states of the diagram can be extracted from the corresponding atom. In other words, the state cube can be completely restored from the atom.

An actual problem is the problem of finding the *genus of a virtual link*: the minimal genus of atoms corresponding to diagrams of the virtual link. From the definition, it follows that diagrams of classical links having the genus zero represent connected sums of alternating diagrams.

This genus is also called the *Turaev genus* due to [294]. It turned out [199] that this genus had an important significance in studying Heegaard–Floer homology of classical knots.

We shall construct the Khovanov complex starting with a given virtual link diagram. As we shall see, the homology of the complex constructed in this way really depends only on the corresponding atom. Thus the homology will be invariant under virtualization. This supports the *virtualization conjecture* (see Conjecture 4.1) saying that if two classical diagrams are equivalent by a chain of generalized Reidemeister moves and virtualizations, then the corresponding links are isotopic in the usual (classical) sense.[1] From arguments above it follows that the corresponding links have isomorphic Khovanov homologies. Note that the virtualization conjecture is true for the unknot (i.e. if a classical diagram of a knot is obtained from a diagram of the unknot by applying a finite sequence of the generalized Reidemeister moves and the virtualization, then the classical diagram represents the unknot), since the Khovanov homology detects the unknot, see [187].

Twisted virtual knots [42, 310] are close relatives of virtual knots. They are represented by knots in oriented thickenings of not necessarily orientable surfaces modulo stabilization/destabilization.

A particular case of the theory of twisted virtual knots is the theory of knots in $\mathbb{R}P^3$.

Definition 5.8. An *orientable thickening* of two-dimensional surface M is an orientable three-dimensional manifold I-bundled over M, where I is a segment.

Let us consider a non-orientable surface S and construct the canonical oriented I-bundled over it. It represents a three-dimensional manifold $S\widetilde{\times}I$ with boundary.

A nice example of such a thickened surface is $\mathbb{R}P^2\widetilde{\times}I$, which is homeomorphic to $\mathbb{R}P^3\backslash\{*\}$. Thus, by constructing the Khovanov homology for such knots, we shall get the Khovanov homology theory for knots in $\mathbb{R}P^3$.

[1] Note that if two classical diagrams are virtual equivalent, then they are equivalent in the ordinary case; this follows, for example, from Kuperberg's theorem [189].

Given a surface M and its thickening $M\tilde{\times}I$. Then links in $M\tilde{\times}I$ can be considered by means of their diagrams: projections on M.

There are two types of stabilization/destabilization of such thickening surfaces: along orienting cycles and along non-orienting cycles. In the second case, we add/remove a thickened Möbius band.

In general position, one gets a framed 4-graph. In order to restore the link, one should indicate for each crossing how the two branches behave in a neighborhood of this crossing. In the orientable case, one just indicates which branch should be over, and which branch should be under. However, in the non-orientable case this indication is relative. While walking along a non-orienting circuit, the direction upwards changes to the direction downwards. So, for example, knots in $\mathbb{R}P^3 \backslash * = \mathbb{R}P^2\tilde{\times}I$ can be represented by diagrams in $\mathbb{R}P^2$ such that all crossings lie inside the disc $D^2 \subset \mathbb{R}P^2$; by passing the boundary of the disc the direction changes, see Fig. 5.17. To handle this, we choose an affine chart such that the complement to this chart in S is one-dimensional. For this chart we have a well-defined notion of an over/undercrossing.

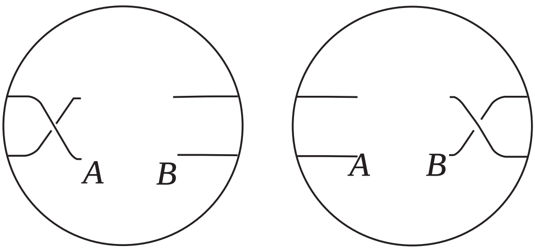


Fig. 5.17 A branch AB forms overcrossing in the left picture and undercrossing in the right picture.

Note that links in such surfaces are well described by atoms. Indeed, fix (once for all) an orientation on $M\tilde{\times}I$. Now, for a link diagram in M, we already have the frame of the atom: a framed 4-graph.

Now, the way for attaching black cells is the following (see Fig. 5.18). For a vertex v, we take two emanating non-opposite half-edges a and b. The corresponding virtual link contains two points projected in the vertex v, one of which is incident to the edge corresponding to a, and the other one is incident to the edge corresponding to b. In a neighborhood of v, denote by c the small vector going from a point on the edge a to a point on b. If the basis (a, b, c) is positively oriented in our three-dimensional manifold, then the angle between half-edges a and b is decreed to be white, as well as the opposite angle. Otherwise they are both black.

Note that this choice does not depend on the ordering of the pair (a, b), nor on their directions.

In the case of general virtual links which are a particular case of twisted virtual links, the way of pasting black cells described above is agreed with the way described in Chap. 4.

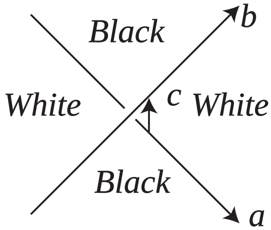


Fig. 5.18 Constructing the atom from a diagram.

This leads to the following theorem.

Theorem 5.8. *There is a well-defined map from the set of twisted virtual knots to the set of virtual knots modulo virtualization.*

Knots in such surfaces were considered by Asaeda, Przytycki and Sikora in [13], and Viro [310] (Bourgoin first considered stabilizations that led to twisted virtual knots). In [13] a Khovanov homology theory for such surfaces was constructed by using an additional topological information coming from surfaces.

From Theorem 5.8 and the invariance of the Khovanov homology under virtualization (Lemma 5.6, see below), it follows immediately that the Khovanov homology constructed below can be generalized for twisted virtual knots.

5.7.2 *Khovanov complex for virtual knots*

Our aim is to define a homology theory for virtual knots (with arbitrary atoms) over an arbitrary ring in such a way that:

(1) the homology we are defining is invariant under the (generalized) Reidemeister moves;

(2) for the case of virtual knots with orientable atoms (also known as alternatible virtual knots) this homology theory coincides with the one constructed in the previous sections;
(3) the tensor product of the complex with $\mathbb{Z}_2$ coincides with the theory constructed in Sec. 5.3;
(4) the graded Euler characteristic of the complex which will be constructed coincides with the Jones polynomial.

Remark 5.12. The coefficient ring might be an arbitrary abelian group with unit, for example, $\mathbb{Z}$.

For the sake of simplicity we shall sometimes call modules over rings "linear spaces", not depending on whether the ring is a field or not.

We have constructed the Khovanov homology theory for orientable virtual knots with arbitrary coefficients. The main obstruction to extend this theory over non-orientable atoms is the possibility of $1 \to 1$-bifurcations on edges of the cube. Here, we used a two-dimensional graded module V with the graded dimension $(q + q^{-1})$ over the main coefficient ring.

If no $1 \to 1$-bifurcations occur, we may construct the Khovanov cube by using the standard differentials, the multiplication m (for $2 \to 1$-bifurcations) and the comultiplication Δ (for $1 \to 2$-bifurcations).

The situation with the $1 \to 1$-bifurcation (the essential phenomenon of the theory of virtual knots appearing because of the existence of non-orientable atoms) makes the problem more complicated. Indeed, if we wish to construct a grading-preserving theory without introducing any new grading, this partial differential should be identically equal to zero because of the grading reasons (there should be a map from V to V that lowers the grading by one). In the space V, the basis consists of two elements with gradings $+1$ and -1. If we set this partial differential to be equal to zero with all other differentials (m and Δ) defined in the standard way, we get a straightforward generalization for the $\mathbb{Z}_2$ case.

In this section, we involve two additional structures: The basis change in the space V corresponding to a circle and generated by $\{1, X\}$ (the homology group of the unknot) while passing from one crossing to another and the exterior product of "circles" instead of their usual tensor products.

Notational agreement. Given an unordered set of vector spaces. Enumerate them arbitrarily: $V_1, \ldots, V_n$. We shall define a new space not depending on the ordering of the spaces, which will be denoted[2] by

[2] In the case of coincidence of the linear spaces $V = V_1 = \cdots = V_n$ we shall use also the notation $V^{\wedge n}$.

$V_1 \wedge V_2 \wedge \cdots \wedge V_n$ as follows. Consider all possible tensor products of these spaces and identify them according to the following rule. Let $x_i \in V_i$, $i = 1, \ldots, n$. We set $x_{\sigma_1} \otimes \cdots \otimes x_{\sigma_n} = \text{sign}(\sigma)\, x_1 \otimes \cdots \otimes x_n$.

We shall denote such tensor product $x_1 \otimes \cdots \otimes x_n$ of elements $x_i \in V_i$ by $x_1 \wedge x_2 \wedge \cdots \wedge x_n$. We call this space the *ordered tensor product.*

Remark 5.13. To avoid confusion, note that, in writing $X \wedge X$, we always assume that the first X and the second X belong to different (but possibly isomorphic) spaces; thus $X \wedge X$ is not zero (unlike the wedge product of 1-forms in the commutative case).

Let us consider a virtual diagram K.

To handle it and to make the whole cube anticommutative we have to add two ingredients, sensitive to orientability of the atom.

(1) With each circle C in each state we associate a vector space of graded dimension[3] equal to $q+q^{-1}$. Namely, given an orientation o of the circle C; we associate with this circle the graded vector space generated by elements 1 and $X_{C,o}$ of gradings 1 and -1, respectively. The orientation change of the circle (passing to $-o$) leads to $X_{C,-o} = -X_{C,o}$.

(2) Given a state s of a virtual link diagram K having l circles $C_1, \ldots, C_l$. With this state, we associate an ordered tensor product $V^{\wedge l}$; as a basis of this product we take the product $(p^1)_{C_{a_1}} \wedge (p^2)_{C_{a_2}} \wedge \cdots \wedge (p^l)_{C_{a_l}}$, where $(p^i)_{C_{a_i}}$ represents an element from $V_{C_{a_i}}$.

Thus, we have defined the chain space of the complex corresponding to the virtual diagram K. We denote it by $[[K]]$. All the basis elements of this space correspond to some states of K with an additional choice of the elements ± 1 or $\pm X$. Let s be a state of K with the set of circles $C_1, \ldots, C_l$, whence for these circles we have chosen elements $\gamma_1, \ldots, \gamma_l$, each of them being ± 1 or $\pm X$. Then these elements form a chain of the complex $[[K]]$ having the height h, where h is the number of B-smoothings of s, and the grading which is equal to $h + \#1 - \#X$, where $\#1$ is the number of elements of type ± 1 among $\gamma_1, \ldots, \gamma_l$, and $\#X$ is the number of elements $\pm X$ among $\gamma_1, \ldots, \gamma_l$.

Our next goal is the description of the differential ∂ in this complex, which increases the height by one and does not change the grading.

[3]From now on, we have passes from the notation v_+ and v_- to the notation 1 and X (before v_+ play the role of unity). This leads to the same homology theory up to a grading shift and a normalization. In the sequel we should not pay attention to these normalizations and shifts, this agrees with [178] in verbatim.

Set $n_+ =$ the number of crossings , $n_- =$ the number of crossings .

Denote by $\mathcal{C}(K)$ the complex obtained from $[[K]]$ by the height shift and the grading shift: $\mathcal{C}(K) = [[K]]\{n_+ - 2n_-\}[-n_-]$, i.e. the height of each chain decreases by n_-, and the grading increases by $(n_+ - 2n_-)$; all differentials remain coordinately. Here we assume that $[[K]]$ is a complex, this fact will be proved below.

Whatever the differential ∂ is, from the construction of chains of the complex $\mathcal{C}(K)$ follows Theorem 5.9.

Theorem 5.9. *For any virtual diagram K we have $\chi(\mathcal{C}(K)) = \widehat{J}(K)$.*

We shall think of all classical crossings as oriented upwards.

Consider a state s of a diagram of an oriented virtual link. Choose a classical crossing and consider all circles of the state s incident to this crossing. There are one or two such circles. Fix orientations on these circles according to the orientation of the edge emanating upwards to the right (and opposite to the orientation of the edge incoming to the crossing from the bottom left, see Fig. 5.19, upper part). As we shall see further, in the case of one circle, these two orientations defined locally can be uncoordinated, but this case will not be under our consideration.

Thus, the orientations of these circles of the state s locally agree with the orientation of the edge emanating upwards to the right (as well as with the edge incoming from the bottom-right) and disagree with the orientation on the left side. We orient the half-edges as shown in the lower-left part of Fig. 5.19. Thus, we have fixed a choice of the generator X for any circle incident to a given crossing. Note that for another crossing for the same circle the choice of X may differ from this one by a sign.

Differentials will be defined according to the orientations of circles at classical crossings and local orderings of components with the following rule.

The orientations described above are well defined unless the case when the edge corresponding to the crossing of the diagram bifurcates one circle to one circle. *In such cases, we set the partial differential to be zero.*

Assume we have a $1 \to 2$ or $2 \to 1$-bifurcation at a crossing.

If we deal with two circles incident to the crossing from the opposite sides, we order them in such a way that the upper (respectively, left) circle is locally first; the lower (respectively, right) one is thus, the second. In the sequel, when defining partial differentials we assume that all circles are

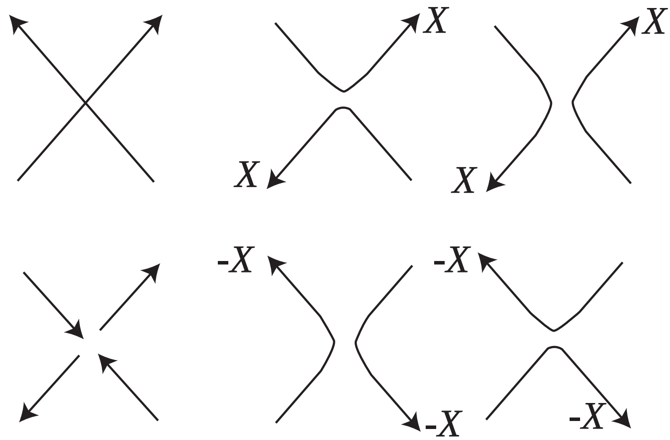


Fig. 5.19 Definition of a basis at a crossing.

ordered in such a way that the circles we deal with are in the very first position in our tensor product; this can always be obtained by means of a permutation, which might lead to a sign change. The map on the other circles is identical.

Let there be given an edge of the bifurcation cube where the number of circles is changed by one. This bifurcation corresponds to a certain crossing; we have two options $2 \to 1$ or $1 \to 2$. In those states when we have two circles incident to the crossing, the circles are ordered. Moreover, all three circles are oriented, thus, we have chosen a basis for the space corresponding to each of these circles.

Now we define the maps $\Delta\colon V \to V \wedge V$ and $m\colon V \wedge V \to V$ locally according to the prescribed choice of generators at the crossing and local ordering:

$\Delta(1) = 1_1 \wedge X_2 + X_1 \wedge 1_2$; $\Delta(X) = X_1 \wedge X_2$ and

$m(1_1 \wedge 1_2) = 1$; $m(X_1 \wedge 1_2) = m(1_1 \wedge X_2) = X$; $m(X_1 \wedge X_2) = 0$, see Fig. 5.20.

Note that the map m is surjective and the map Δ is injective.

If we have some circles $C_1, \ldots, C_l$ not incident to the crossing in question, and elements $\gamma_1, \ldots, \gamma_l$ on them, the formulae for the partial differen-

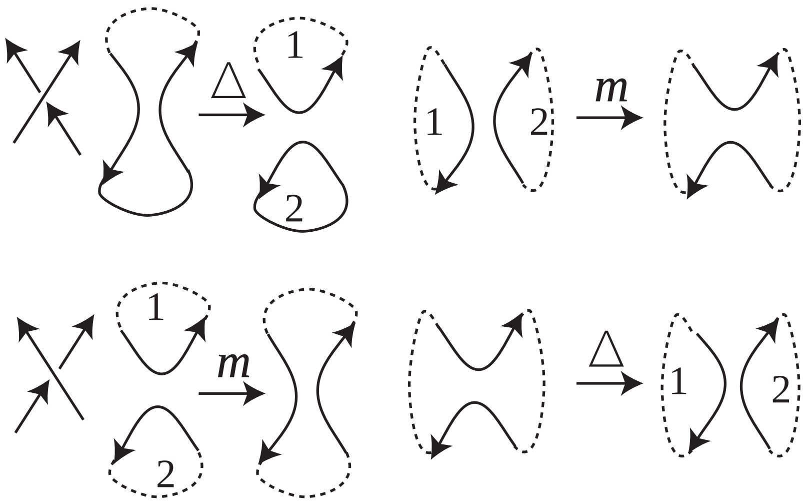


Fig. 5.20 Defining operations m and Δ.

tials ∂' are written as:

$$\begin{aligned}\partial'(1 \wedge \gamma_1 \wedge \cdots \wedge \gamma_l) &= \Delta(1) \wedge \gamma_1 \wedge \cdots \wedge \gamma_l \\ &= 1_1 \wedge X_2 \wedge \gamma_1 \wedge \cdots \wedge \gamma_l + X_1 \wedge 1_2 \wedge \gamma_1 \wedge \cdots \wedge \gamma_l, \\ \partial'(X \wedge \gamma_1 \wedge \cdots \wedge \gamma_l) &= \Delta(X) \wedge \gamma_1 \wedge \cdots \wedge \gamma_l = X_1 \wedge X_2 \wedge \gamma_1 \wedge \cdots \wedge \gamma_l\end{aligned}$$

(in the case of a $1 \to 2$-bifurcation) and

$$\begin{aligned}\partial'(1_1 \wedge 1_2 \wedge \gamma_1 \wedge \cdots \wedge \gamma_l) &= m(1_1 \wedge 1_2) \wedge \gamma_1 \wedge \cdots \wedge \gamma_l \\ &= 1 \wedge \gamma_1 \wedge \cdots \wedge \gamma_l, \\ \partial'(X_1 \wedge 1_2 \wedge \gamma_1 \wedge \cdots \wedge \gamma_l) &= \partial'(1_1 \wedge X_2 \wedge \gamma_1 \wedge \cdots \wedge \gamma_l) \\ &= m(X_1 \wedge 1_2) \wedge \gamma_1 \wedge \cdots \wedge \gamma_l \\ &= m(1_1 \wedge X_2) \wedge \gamma_1 \wedge \cdots \wedge \gamma_l \\ &= X \wedge \gamma_1 \wedge \cdots \wedge \gamma_l, \\ \partial'(X_1 \wedge X_2 \wedge \gamma_1 \wedge \cdots \wedge \gamma_l) &= m(X_1 \wedge X_2) \wedge \gamma_1 \wedge \cdots \wedge \gamma_l = 0\end{aligned}$$

(in the case of a $2 \to 1$-bifurcation).

After that we define the differential ∂ on the chain space corresponding to the state s as the sum of partial differentials acting on the state s.

Example 5.3. Thus, if we wish to comultiply the second factor X_2 in $X_1 \wedge X_2$, we get $X_1 \wedge X_2 = -X_2 \wedge X_1 \to -X_2 \wedge X_3 \wedge X_1 = -X_1 \wedge X_2 \wedge X_3$, where X_3 belongs to the newborn third component (under the condition that at the crossing of splitting the circle X_2 is locally first (i.e. upper and left), and the circle X_3 is locally second).

Given an oriented diagram K of a virtual link, we have constructed a set of bigraded groups with differential ∂. Denote the set of groups by $[[K]]$. The differential increases the height and does not change the grading.

In this section, we shall prove the main theorem.

Theorem 5.10. *The set of groups $[[K]]$ together with the differential ∂ is a well-defined bigraded complex, i.e. $\partial^2 = 0$. Herewith the differential preserves the grading and increases the height by one.*

The complex $\mathcal{C}(K)$ is obtained from $[[K]]$ by the height shift and grading shift. From the constructions it will follow that the homology of the complex $\mathcal{C}(K)$ coincides with the homology constructed for the case of virtual knots with orientable atoms.

Further, from the proof of Theorem 5.10 by construction the claim of Theorem 5.9 follows.

The complex with coefficients in the field $\mathbb{Z}_2$ coincides with the complex over $\mathbb{Z}_2$ described in Sec. 5.3.

Theorem 5.11. *The homology of the bigraded complex $\mathcal{C}(K)$ is an invariant of the virtual link K under the generalized Reidemeister moves.*

We first prove Theorem 5.10. After that, we shall prove Theorem 5.11; its proof will be more technical and it will follow the standard scheme of [17], however, some additional sign checks for partial differentials, appearing while ordering and orienting the circles, will be needed. We shall also show that the homology of $\mathcal{C}(K)$ coincides with the homology constructed for the case of virtual knots with orientable atoms.

Let us pass to the proof of Theorem 5.10.

We first prove two lemmas that establish some properties of our complex $\mathcal{C}(K)$ and simplify further arguments.

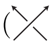
Let K be a virtual diagram. Consider a classical crossing v of it. Let the diagram K' be the diagram obtained from K by the virtualization of v. Then there exists a one-to-one correspondence between the sets of classical crossings of the diagrams K and K'. It generates a one-to-one correspondence ϕ between the states (for the corresponding vertices we have either A-smoothings or B-smoothings). Note that such a bijection does not change the number of circles in the states; it follows from the fact that all states can be restored from the atom, and the atom does not change under virtualizations. Let us orient circles of corresponding states *equally* outside the crossing v. This identification defines the map $g\colon [[K]] \to [[K']]$ of the chain spaces according to the following rule. For any state s and the corresponding state $\phi(s)$, the diagrams K and K' look identical outside a neighborhood of v. Thus, we can establish the bijection between oriented circles of s and oriented circles of $\phi(s)$, that leads to the definition of g. We shall use the same notation g for maps of vector spaces (modules) corresponding to the circles in states s and $\phi(s)$.

With each state s of the diagram K we associate the subspace of the space $[[K]]$; we denote it by C_s. Denote the corresponding space for K' by $C_{s'}$.

Lemma 5.6. *Let* K, K' *be two diagrams obtained one from another by the virtualization. Then there is a grading-preserving chain map* $f\colon [[K]] \to [[K']]$ *that maps* C_s *isomorphically to* $C_{s'}$ *and commutes with the local differentials.*

In particular, if $[[K]]$ *is a well-defined complex, then so is* $[[K']]$; *herewith their homology groups are isomorphic.*

Proof. Suppose the diagram K' is obtained from the diagram K by the virtualization at a crossing v.

The map f is constructed according to the crossing type of v ( or ). By construction, partial differentials of the complex $[[K']]$ coincide with the images of partial differentials of $[[K]]$ under g, except, maybe, those partial differentials corresponding to the crossing v. Furthermore, differentials corresponding to v split our cube to the "lower subcube" and the "upper subcube", as shown in Fig. 5.21.

Now, the remaining partial differentials differ possibly by signs on edges corresponding to the crossing v. Our goal is to show that they either all agree or all differ by -1 sign, as shown in Fig. 5.21.

Indeed, the bases at all crossings but v agree for K and K'. This leads to the identification of chains of the corresponding complexes. For this isomorphism for every circle C incident to v and the circle $g(C)$ corresponding to it in the corresponding state of the diagram K' we have $g(X_{C,o_K}) = -X_{g(C),o_{K'}}$, where o_K and $o_{K'}$ are the orientations of the circles C and C' at the crossing v of the diagrams K and K' chosen according to the rule depicted in Fig. 5.19. The latter identity holds because in any state s the circle C that tends from the upper-right to the crossing v of K, corresponds to the circle $\phi_*(C)$ in the state $\phi(s)$ that tends to v from the

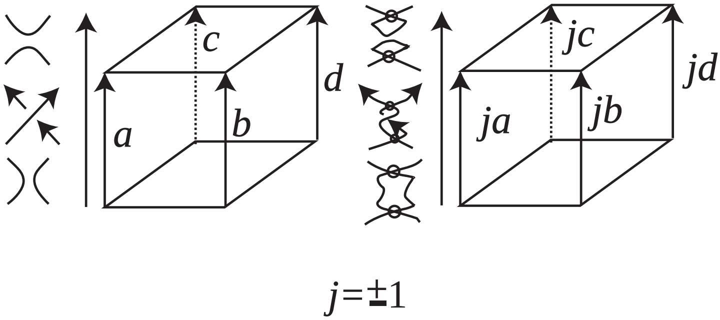


Fig. 5.21 The behavior of the cube under the virtualization.

upper-left, this corresponds to the change X to $-X$ in the local basis of spaces V corresponding to circles of the state incident to the given crossing, see Fig. 5.19. If we dealt with the usual tensor product case regardless of the circle ordering, the transformation $X \to -X$ would leave m invariant and change Δ to $-\Delta$.

Assume now that the crossing v is positive (⤫). All maps of type m corresponding to v, represent bifurcations of two circles (a left one and a right one) into one circle. After the virtualization, the circles interchange their roles, see Fig. 5.22.

Globally we get a sign change for all m-type partial differentials. For partial differentials of type Δ we have one circle that bifurcates into two ones, the upper one, and the lower one; the "up-down" position remains unchanged under virtualization, that preserves all Δ-type partial differentials. The first component is shown locally by solid line, whence the second component is shown by a dashed line.

Summing up (and reminding about the sign change of the partial differential Δ because of passing $X \to -X$), we see that the virtualization of a positive crossing changes the signs of all partial differentials corresponding to this crossing.

Now divide $[[K]]$ and $[[K']]$ into two parts each, according to the smoothing of v; we call one part of the cube "upper", the remaining part being lower. Now set $f\colon [[K]] \to [[K']]$ as g for all elements from the lower subcube and as $-g$ for the upper subcube.

Evidently, this map commutes with partial differentials. Indeed, the commutativity of the map f with partial differentials inside one of the

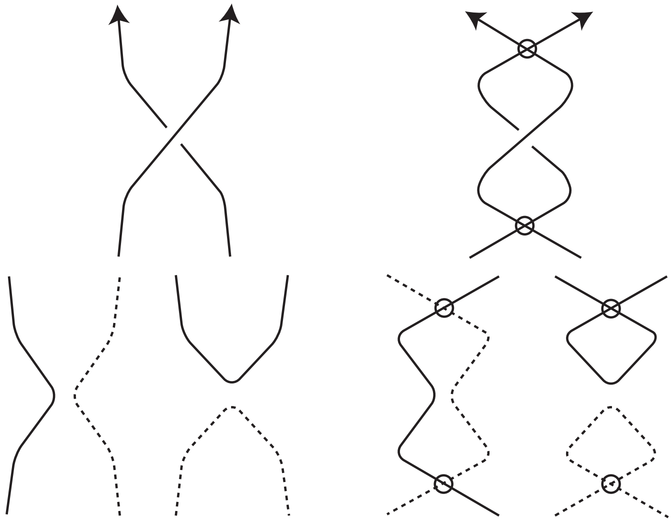


Fig. 5.22 Virtualization.

subcubes follows from the fact that the map g is anticommutative, therefore, the map f commutes.

Thus if the initial cube were anticommutative, then the constructed map would be an isomorphism in homology.

Analogous reasonings show that the virtualization of a negative crossing does not change the cube at all. The minus sign that appears on edges corresponding to Δ is canceled by the minus sign caused by the permutation of circles (the right one and the left one). This completes the proof of the lemma. □

This lemma means that the homology of a virtual diagram with two classical crossings (if well defined) can be restored from an atom endowed with an orientation of the link components.

Thus, to prove that the cube $[[K]]$ anticommutes, we can make some preliminary virtualizations for classical crossings of K and consider the analogous question for the obtained diagram K'.

To check the anticommutativity of the cube $[[K]]$ we have to consider all 2-faces of it. Each 2-face is represented by fixing a way of smoothing some $(n-2)$ classical crossings of K, see Fig. 5.23. The remaining two crossings can be smoothed arbitrarily; the four possibilities correspond to the vertices of the 2-face.

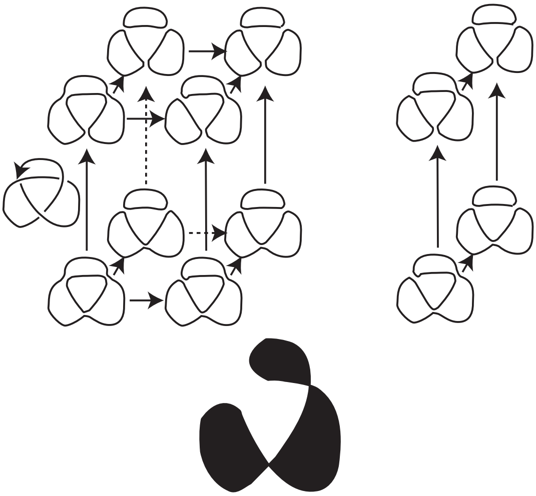


Fig. 5.23 A 2-face generates an atom.

In Fig. 5.23 the bifurcation cube is shown in the left part and the 2-face and the corresponding atom are shown in the right part. The atom can be restored from a knot diagram, as described above in Sec. 4.2.3.

Now, for these four states, there are some "common" circles which do not touch any of the two vertices in question (in the case depicted in Fig. 5.23 there are no such circles). After removing these circles, we get an atom with two vertices.

What we actually have to check is that any face corresponding to any possible atom with two vertices anticommutes.

For the two vertices of such an atom, we have some local orientations of the link at each of these vertices; they are needed to fix the local ordering of components (see Fig. 5.19) while defining the differentials.

Note that globally these orientations might not agree on the circles; namely, an edge of the atom with two vertices consists of several edges of the diagram which might have opposite orientations, see Fig. 5.24.

It turns out, however, that these local orientations can be chosen arbitrarily without losing the anticommutativity property and without changing the homology.

Namely, fix an atom with two vertices. All possible occurrences of this atom in the cube correspond to local orientations of edges at these vertices. Fix an orientation for one crossing v_1 and choose two distinct orientations

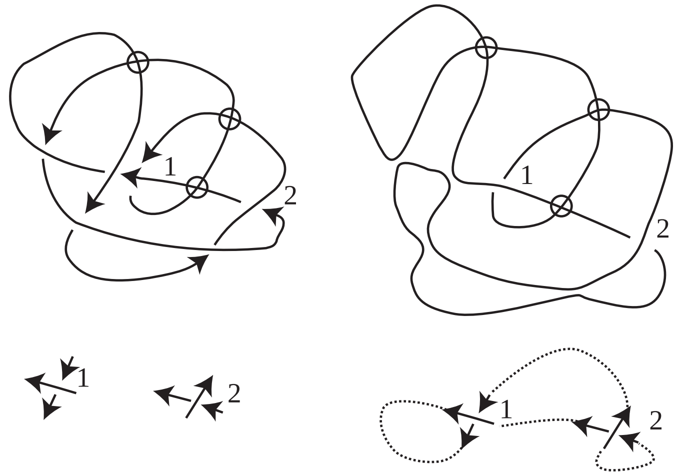


Fig. 5.24 Orientation for atom crossings.

for the second crossing v_2 that differ from each other by the clockwise $\frac{\pi}{2}$-turn of the arrows, see Fig. 5.25. Thus, we get two pictures and two two-dimensional discrete cubes, Q_1 and Q_2. These cubes coincide as sets of linear spaces. Let V_s and $V_{s'}$ be linear spaces of Q_1 and Q_2 corresponding to some fixed state s and the state s' corresponding to it.

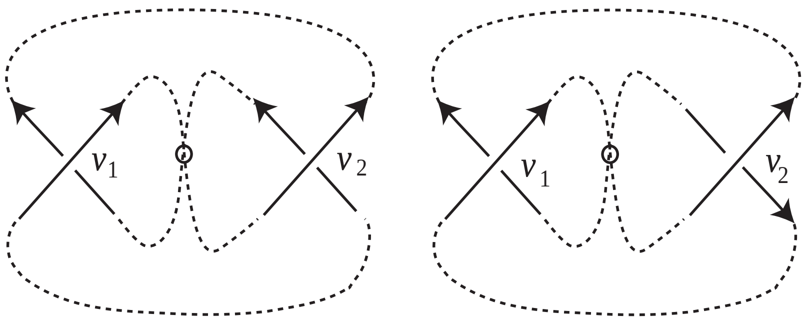


Fig. 5.25 Q_1 and Q_2.

Lemma 5.7. *If Q_1 is anticommutative, then so is Q_2. Moreover, there exists a grading preserving chain map $f\colon Q_1 \to Q_2$ that takes V_s isomorphically to $V_{s'}$ and commutes with partial differentials.*

Proof. The proof of Lemma 5.7 is very much similar to that of Lemma 5.6.

A sketch of the proof goes as follows. After rotating all arrows at v_2 in the clockwise direction, we get the local sign change of X for all circles incident to this crossing. Analogously to Lemma 5.6, we consider two complexes and identify their chain spaces by means of the map g (analogous to the map g from Lemma 5.6) in such a way that the differentials corresponding to any other crossing coincide.

After that we shall correct g, as in Lemma 5.6, to get a map f that commutes with all partial differentials, which would yield the statement of the lemma. If we dealt with the usual unordered tensor product, this would lead to the sign change of all partial differentials of type Δ corresponding to v_2.

Furthermore, in the case of a positive crossing, all differentials of type m corresponding to this crossing, change their sign, too.

In the case of negative crossings, partial differentials of type $2 \to 1$ do not change, and $1 \to 2$-bifurcations change the sign again. Thus, we have the same situation as in Lemma 5.6, which completes the proof of Lemma 5.7. □

Let us continue the proof of Theorem 5.10.

Lemma 5.7 means that in order to check the anticommutativity of all possible faces, it is sufficient to enumerate all atoms with two vertices and check the anticommutativity for each of them. We first fix a representation of such an atom in $\mathbb{R}^2$ (i.e. an immersion of its frame preserving the A-structure); such immersions differ by a possible virtualization which does not change the complex (up to isomorphism) by Lemma 5.6; then we choose a local orientation, which does not matter either by Lemma 5.7.

Note that among atoms with two vertices there are disconnected atoms, i.e. those for which each edge connects some vertex with itself. For such atoms in the case of ordinary tensor product we get by evident reasons commutative 2-faces. In the case of ordered tensor products the corresponding faces will obviously anticommute.

Some (connected) atoms with two vertices are inessential in the following sense. We have set the $1 \to 1$ differential to be zero. By parity reasons, in the 2-face of any atom there might be 0, 2 or 4 such edges. The case when we have no such edges is orientable. When we have four edges representing differentials of type $1 \to 1$, then the proof follows from the identity $0 = 0$. The same takes place in the case when in the diagram, the anticommutativity of which we prove, we have two compositions of maps and one of the maps at each composition is zero.

There are some inessential atoms, where two vertices are not connected to each other. For any of them, anticommutativity is obvious. There are six essential connected atoms with two vertices, as shown in Fig. 5.26. All these atoms except the first one are orientable.

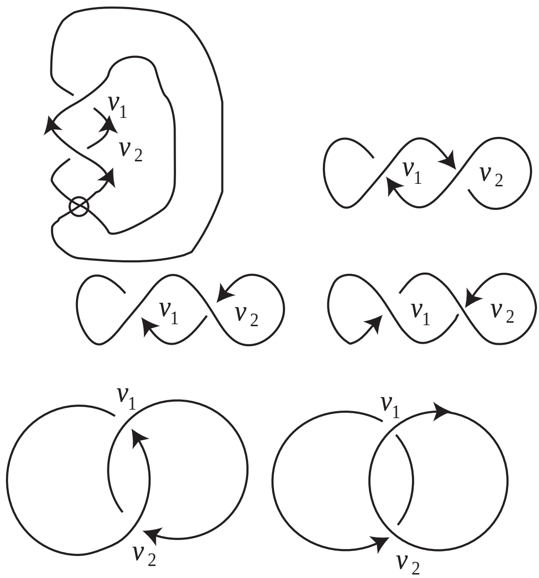


Fig. 5.26 Essential atoms with two vertices.

For the first one, an accurate calculation corresponding to Fig. 5.27 shows that both compositions give zero.

Indeed, the lower composition is zero. Substituting X into the upper composition, we get $\pm X \wedge X$ at the first step and zero at the second step. If we start with 1, we get $1_{1,o_{v_1}} \wedge X_{2,o_{v_1}} + X_{1,o_{v_1}} \wedge 1_{2,o_{v_1}}$ at the first step; here the first index is the local number of the circle (the first circle is big and the second one is small), and the second index is the name of the vertex. When passing to the second vertex v_2, the first and second circles change their roles: The circle number 1 becomes the lower one and number 2 becomes the upper one. Also, for the big circle, X changes to $-X$. Thus we get $-X \wedge 1 + 1 \wedge X$ which is transformed by m to zero.

Let us now check orientable atoms. For any of them, we fix an orientation as shown in Fig. 5.26. Such an orientation gives a coordinated

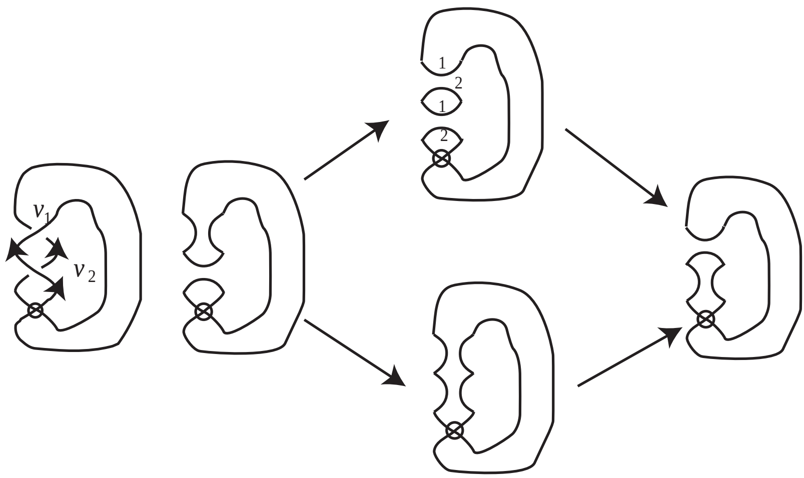


Fig. 5.27 The non-orientable atom.

orientation of circles at two crossings which are under consideration in the sense of Fig. 5.19. After that, we can fix the bases $\{1, X\}$ for all circles at vertices according to the rule shown in Fig. 5.19.

Now, the anticommutativity is checked as follows. If we dealt with the usual (unordered) tensor product case, everything would commute. Now, the enumeration of circles might cause minus signs on some edges. We have to check that for any of these five atoms the total sign would be minus.

For instance, in Fig. 5.28 we have an oriented atom with two vertices. The analogous check of the unordered tensor product case means the usual associativity $m \circ (m \otimes 1) = m \circ (1 \otimes m)$, where the circles are enumerated from the left to the right. In the left part of the figure, one pair of numbers of the circles 1 and 2 is drawn upside down to underline which circle is assumed to be locally the first (left); the other one is the second (right).

Here we have to take into account the global ordering of the components. Note that for three components, we always have to apply $m \wedge \mathrm{Id}$ first, taking those components to be multiplied with the first and second positions.

Thus, $m \circ (m \wedge \mathrm{Id})$ applied to $A_1 \wedge A_2 \wedge A_3$ gives us $m(m(A_1, A_2), A_3) = -(A_1 \cdot A_2 \cdot A_3)$; here $\cdot$ means the usual multiplication in Khovanov's sense: $X \cdot X = 0$; $X \cdot 1 = 1 \cdot X = X$; $1 \cdot 1 = 1$. Here the minus sign appears at the second crossing, we have two branches oriented downwards, thus, the rightmost circle occurs to be locally the left one.

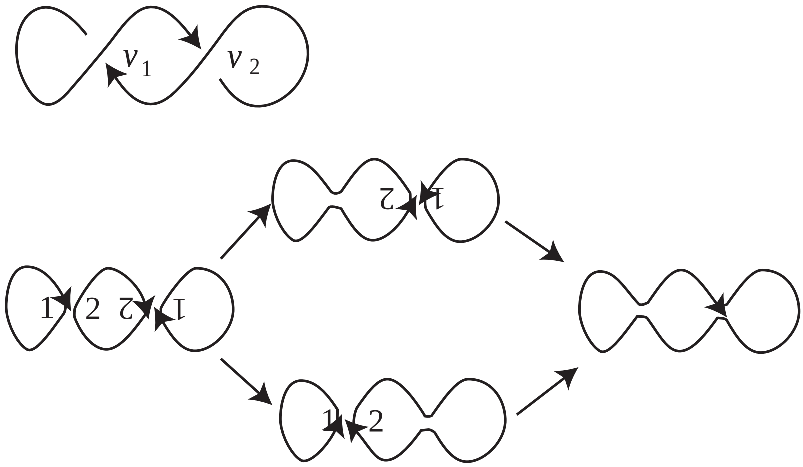


Fig. 5.28 An orientable two-vertex atom.

On the other hand, if we consider the second crossing first, we get $A_1 \wedge A_2 \wedge A_3 = (A_2 \wedge A_3) \wedge A_1 = -(A_3 \wedge A_2) \wedge A_1 \to -(A_2 \cdot A_3) \wedge A_1 = A_1 \wedge (A_2 \cdot A_3)$. Applying m to that, we get $A_1 \cdot A_2 \cdot A_3$.

All other atoms are checked analogously. Note that our setup gives directly an anticommutative cube, unlike the Khovanov original setup, where we got an anticommutative cube from a commutative one by adding some minus signs on edges. Thus, Theorem 5.10 is proven. Therefore, Theorem 5.9 is also proven.

Let us prove Theorem 5.11.

Remark 5.14. Throughout the rest of the proof of Theorem 5.11, we shall not care about height and degree shifts. The proof of their coincidence for diagrams differed by Reidemeister moves repeats verbatim that in the classical case, see, e.g. [17].

***Proof of Theorem* 5.11.** First, note that the complex $\mathcal{C}(K)$ itself does not change at all if we perform the detour move. Therefore, the homology does not change.

In the case of classical Reidemeister moves, the proof goes along the line of the proof from the previous section.

Let us be more specific. The case of the first Reidemeister move is evident. The complex corresponding to a diagram with a curl looks like:

$$[[\text{⤫}]] = \left([[\text{⤫}_0]] \xrightarrow{m} [[\text{⤫}_1]]\{1\} \right).$$

Thus, recalling that m is surjective, we see that the left subcomplex of the complex [[ ] generated by ± 1 on the small circle is mapped onto the right subcomplex [[ ]]. Thus, we see that the whole complex has the same homology as its quotient by its acyclic part. The latter one has the same homology as [[ ]].

Analogously, one treats the other curl; here one should take into account the injectivity of Δ.

As in the case of the first Reidemeister move, the invariance under the second Reidemeister move repeats that by Bar-Natan in the classical case. We give it here because we shall need this proof for proving the invariance under the third Reidemeister move.

As opposed to the case in the first part of this chapter, we should pay attention to orientations of circles when we prove the invariance under the second Reidemeister move.

For the second Reidemeister move we note that we can choose orientations of all circles incident to a given crossing locally agreed (such that under passing along one circle from one crossing to the other one the variable X does not change the sign), see Fig. 5.29.

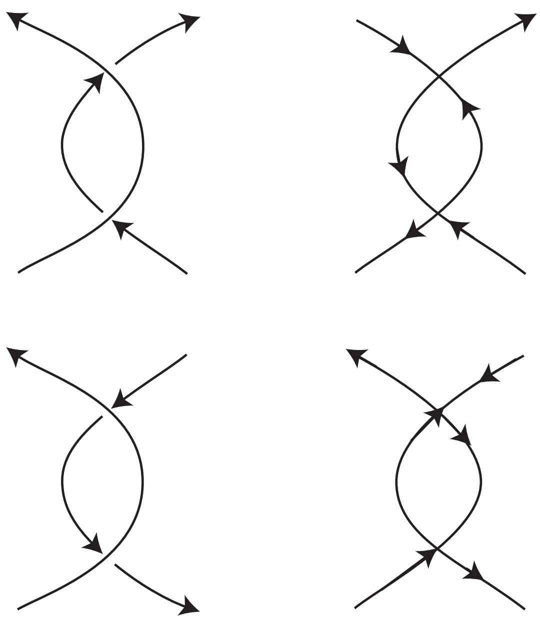
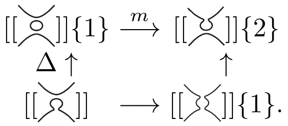


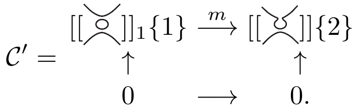
Fig. 5.29 Orientations of upper-right agrees for Ω_2.

Now, we proceed our discussion by taking into account that the sign in X does not change under passing from one vertex to the second one under moving along a circle of any state. The initial complex $\mathcal{C}$ looks like:



(5.1)

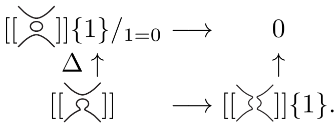
This complex contains the following subcomplex $\mathcal{C}'$:



Here and further the subindex 1 in the upper-left angle means the label on the small circle.

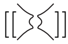
The acyclicity of $\mathcal{C}'$ is evident.

Factorizing $\mathcal{C}$ by $\mathcal{C}'$, we get



(5.2)

In the upper-left angle $1 = 0$ means that we have factorized the space $\{1, X\}$ corresponding to the small circle by the subspace spanned by 1, i.e. in the corresponding (ordered) tensor product instead of 2-space we have one-dimensional space generated by X.

In the last complex, the arrow Δ directed upwards, is an isomorphism. Thus, this complex (after some normalization) has the same homology as . This proves the invariance under Ω_2 (up to height and grading shifts).

This argument will be used later for proving the invariance of the Khovanov homology under the third Reidemeister move.

Returning to the initial (5.1) complex $\mathcal{C}$ we see that its homology groups are in one-to-one correspondence with those from the lower-right corner of (5.2). Also note that in the initial complex all non-trivial cycles have "local" height corresponding to the lower-right and upper-left angles. Thus, in the initial complex, every element α staying in the upper left corner of the initial complex, is homologous to precisely one element $-\tau(\alpha)$ staying in the lower-right corner. The map τ is obtained by composing (Δ^{-1}) downwards and the map directed to the right. For the case of the third Reidemeister moves, we shall simplify subcomplexes, corresponding to the second Reidemeister moves. This simplification will be performed twice, and it will lead to two maps analogous to τ, denoted by τ_1 and τ_2.

Let us now consider the third Reidemeister move shown in Fig. 5.30.

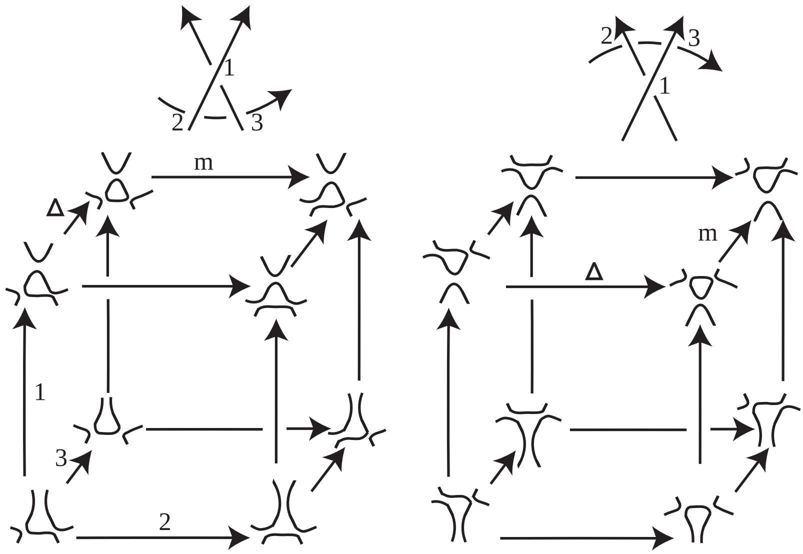


Fig. 5.30 The third Reidemeister move.

It is well known (see, e.g. [259]) that any variant of the third Reidemeister move can be obtained as a composition of Ω_1, Ω_2 and one prefixed version of the third Reidemeister moves, in which a choice for over/undercrossing and orientations of edges is chosen. Consider only one case, shown in Fig. 5.32, with crossing smoothings as in Fig. 5.30.

At any crossing in Fig. 5.32 there is a local rule for orientations for all edges incident to it, according to the rule shown in Fig. 5.19. If two crossings are adjacent, the orientation might or might not be coordinated. We see that the orientation (defined according to Fig. 5.19) in the third crossing (left picture) does not agree with the orientations in the first and second crossings analogously, for the right picture, the second crossing disagrees with the first one and with the third one. Note that the rule in Fig. 5.19 does not depend of types of crossings, but does depend on the orientations of branches.

Apply virtualizations to crossings 1, 2 of the first diagram and to the second crossing of the second diagram; after that, all local orientations (in the sense of variable X) will be coordinated, see Fig. 5.31.

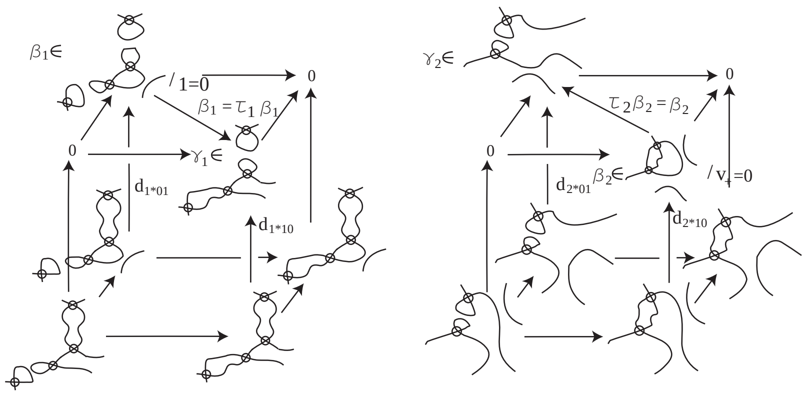


Fig. 5.31 The diagrams after the virtualization.

The positive smoothings at crossing 1 are the same (up to virtualizations) for both diagrams. The negative smoothing of them gives two pictures obtained one from another by a sequence of (virtualizations and) two classical Reidemeister moves.

Thus, the complexes of the two diagrams in question can be rearranged to have coinciding bottom levels, and top levels have the same homology (in both cases we applied Ω_2).

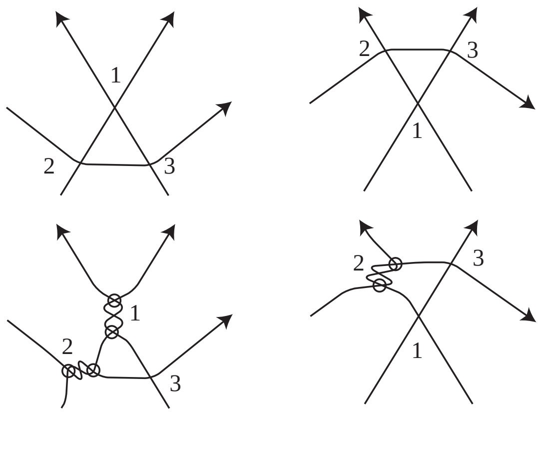


Fig. 5.32 Virtualizing crossings under Ω_3 to make all bases agreed.

The main thing to check is that the differentials going upwards agree for these complexes, i.e. the "upwards" maps in both cases either coincide or differ by a sign. These complexes are shown in Fig. 5.3.

In our situation, the only difference from the classical case which may occur is that they differ by a minus sign (because of ordered tensor products taken instead of the usual tensor products).

In the classical case the final complexes (after factorizing) have the form shown in Fig. 5.3.

In Fig. 5.3 the virtualization applied by us in Fig. 5.32 is not designated. The picture shows only what circles are transformed, but does not show what circle is the first at a crossing, and what circle is the second (for this it is necessary to take into consideration the virtualization in Fig. 5.32.

Here $v_+ = 0$ (in our case $1 = 0$) in the left upper corner of Fig. 5.3 means that the space corresponding to the given state is factorized by the subspace where the small circle is marked by 1. Here τ_1 and τ_2 are not differentials; they are chain maps taking an element to the element which is minus homologous to the initial one.

To establish the isomorphism in homology, it is sufficient to show that $\tau_1 \circ d_{1*01} = d_{2*01}$ and $d_{1*10} = \tau_2 \circ d_{2*10}$. In this case we shall show that all the maps "upwards" in both complexes differ by a sign (since in both cases τ_i is minus the identity in homology). After that the homotopy equivalence of the two complexes corresponding to the third Reidemeister move is proved as in Lemma 5.6: By means of a natural map that identifies lower subcubes and minus that map that corresponds to the complex which the upper subcube is reduced to.

The ordered tensor product case differs from the usual one, possibly, by signs on edges.

Let us check that the signs agree in our setup. We shall show that $\tau_1 \circ d_{1*01} = d_{2*01}$ (the remaining case $d_{1*10} = \tau_2 \circ d_{2*10}$ is completely analogous).

Let us view Fig. 5.3 and take into account the virtualization of the right and left diagrams at crossings. The required identity will look like $p = q \circ \Delta^{-1} \circ \Delta$, see Fig. 5.33.

Here d_{1*01} is a $1 \to 2$-bifurcation (we denoted it by Δ); $\tau_1 = \nu \circ \Delta^{-1}$, where ν is a partial differential and Δ^{-1} is assumed as an operation inverse to Δ (note that the space in the upper-left corner in which the element β_1 stays is factorized by $1 = 0$; i.e. the space associated with the small circle C, is one-dimensional with generator X). Then, the comultiplication map for which C is a resulting circle becomes an isomorphism.

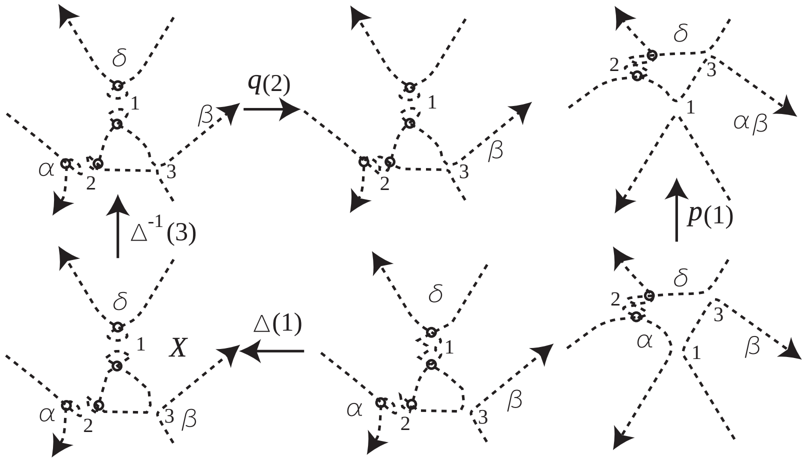


Fig. 5.33 Checking the invariance under Ω_3.

Consider Fig. 5.33. For each of the maps in the brackets the number of a crossing is indicated which this map is applied to.

The maps p and q are just the usual local differentials, either both multiplications, or both comultiplications, or both zeros.

If $p = q = 0$, there is nothing to prove.

Consider the remaining cases. We have three fragments of circles α, β, δ. In the very initial state (which the map p in the right picture and Δ in the left picture are applied to) they may belong to one, two or three different circles. We shall first consider the case when all fragments containing α, β, δ belong to three different circles.

For simplicity we denote the elements of the algebra V (of type 1 or $\pm X$) related to these circles, by the same letters as fragments α, β, δ.

In our case, both operations p and q are multiplications.

Starting with $\alpha \wedge \beta \wedge \delta$, we get on the right picture the map d_{2*01}:

$$p\colon \alpha \wedge \beta \wedge \delta \to (\alpha \cdot \beta) \wedge \delta,$$

where $(\alpha \cdot \beta)$ means an ordinary product in the Frobenius algebra.

On the left picture we have:

$$\alpha \wedge \beta \wedge \delta = \delta \wedge \alpha \wedge \beta \xrightarrow{\Delta} \delta \wedge X \wedge \alpha \wedge \beta.$$

Here we applied the comultiplication to δ to get two circles at the crossing number 1; the two resulting circles are denoted by δ (the upper one) and X (the lower one).

Now, $\delta \wedge X \wedge \alpha \wedge \beta = -\beta \wedge X \wedge \alpha \wedge \delta$. We then perform Δ^{-1} at the crossing 3. This map joins the two circles marked by β and X.

At this crossing the generator X is related to the left circle, and β is related to the right circle. Thus, we have

$$-\beta \wedge X \wedge \alpha \wedge \delta = X \wedge \beta \wedge \alpha \wedge \delta \stackrel{\Delta^{-1}}{\rightarrow} \beta \wedge \alpha \wedge \delta.$$

Now, the operation q is the comultiplication at the crossing 2, where the circle marked by β is the first (upper), and the one marked by α is the second one (lower). Thus, we get: $(\alpha \cdot \beta) \wedge \delta$.

Now assume that α and β form one circle (in the initial state), and δ forms a separate circle. Denote the mark (an element from V) corresponding to the first circle by A, and the mark corresponding to the second circle by δ.

The map p looks like:

$$A \wedge \delta \stackrel{\Delta}{\rightarrow} \sum_i A_{i,1} \wedge A_{i,2} \wedge \delta,$$

where $\sum_i A_{i,1} \otimes A_{i,2}$ is the result of application of the comultiplication to A in the ordinary sense (in the case of unordered tensor product), see Fig. 5.34.

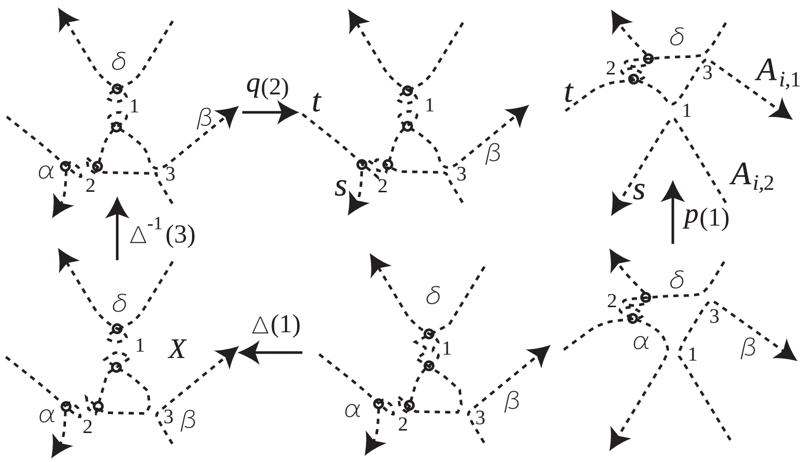


Fig. 5.34 Checking the invariance under Ω_3.

In the further proof for simplicity of writing we shall not use the sum sign $\sum_i$.

In the left picture we have

$$A \wedge \delta = -\delta \wedge A \to -\delta \wedge X \wedge A$$

(at the first crossing the marking δ corresponds to the upper circle and X corresponds to the lower circle).

Then for the map Δ^{-1} at the crossing 3 we have

$$-\delta \wedge X \wedge A = -X \wedge A \wedge \delta \to -A \wedge \delta$$

(here X was on the left side, and A was on the right side).

Finally, the map q at the crossing 2 gives us

$$-A \wedge \delta \to -A_{i,1} \wedge A_{i,2} \wedge \delta.$$

Here $A_{i,1}$ corresponds to the locally upper component s at the crossing 2, and $A_{i,2}$ is locally lower component t. But, in the right picture they have opposite ordering. More precisely, we have

$$-A_{i,1,s} \wedge A_{i,2,t} \wedge \delta.$$

In the first case (the map p) we had

$$A_{i,1,t} \wedge A_{i,2,s} \wedge \delta = -A_{i,2,s} \wedge A_{i,1,t} \wedge \delta.$$

These two results coincide because of cocommutativity of Δ in the ordinary case.

One can consider the remaining cases analogously.

Suppose that α and δ belong to one circle (the corresponding element being denoted by α), and β belongs to another circle. Then we have the following maps.

In the simplest case (the map p) we have

$$\alpha \wedge \beta \to (\alpha \cdot \beta).$$

On the left picture we have

$$\alpha \wedge \beta \to \alpha \wedge X \wedge \beta = X \wedge \beta \wedge \alpha \to \beta \wedge \alpha \to (\beta \cdot \alpha).$$

Consider the case of multiplication when β and δ form one circle (the corresponding element being denoted by β). We get:

$$\alpha \wedge \beta \to (\alpha \cdot \beta)$$

on the right picture (the map p) and

$$\alpha \wedge \beta = -\beta \wedge \alpha \to -\beta \wedge X \wedge \alpha = X \wedge \beta \wedge \alpha \to \beta \wedge \alpha \to (\beta \cdot \alpha)$$

on the left picture.

Finally, consider the case when at the beginning we had exactly one diagram, we get two comultiplications:

$$A \to A_{i,1,t} \wedge A_{i,2,s}$$

in the simplest case (the map p) and

$$A \to A \wedge X = -X \wedge A \to -A \to -A_{i,1,s} \wedge A_{i,2,t} = A_{i,2,t} \wedge A_{i,1,s}.$$

Thus, we have proved the equality $\tau_1 \circ d_{1*01} = d_{2*01}$. The proof of the equality $d_{1*10} = \tau_2 \circ d_{2*10}$ is completely analogous. □

Theorem 5.12. *Let K be a virtual diagram for which the corresponding atom is orientable. Then the homology* $\mathrm{Kh}(K)$ *coincides with the Khovanov homology constructed in the previous chapter.*

During the proof of this theorem, we denote our complex and our homology by $\mathcal{C}(K)$ and $\mathrm{Kh}(K)$, respectively, and the ones constructed in the previous sections by $\mathcal{C}'(K)$ and $\mathrm{Kh}'(K)$ respectively.

***Proof of Theorem* 5.12.** First we note that the normalizations for $\mathcal{C}$ and $\mathcal{C}'$ are performed in the same manner. Thus, we can forget about additional normalizations of type $[-n_-]\{n_+ - 2n_-\}$.

First, we assume the diagram of K is chosen in such a way that all X's for all crossing and circles agree (that is, for a given state circle, while passing from one classical crossing P to another one Q, we get $X_{C,o_P} = X_{C,o_Q}$, not $X_{C,o_P} = -X_{C,o_Q}$). This is possible since the atom corresponding to K is orientable. Indeed, since the atom corresponding to K is orientable, we can globally define the orientation of all edges to be compatible with the orientation of the circles in each state. At each crossing of K this orientation may agree or disagree with the local orientation of edges determined by Fig. 5.19 (the orientation originates from the source–sink structure). Let us apply the virtualization to all crossings of K where these orientations disagree. By Lemma 5.6, the homology of the complex $\mathcal{C}(K)$ remains the same, and the orientations of circles given locally at crossings according to the rule in Fig. 5.19 become compatible.

After that, we should just care about signs of local differential and enumeration of circles for any crossing.

We construct a homology-preserving map between two cubes. Fix an enumeration of the classical crossings of K. Let us associate a maximal spanning tree for the cubes $\mathcal{C}(K)$ and $\mathcal{C}'(K)$ as follows. This tree consists of all edges of the form $(\alpha_1, \ldots, \alpha_l, *, 0, \ldots, 0), \alpha_j \in \{0, 1\}$, i.e. an edge in

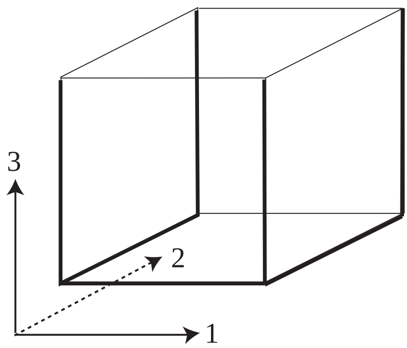


Fig. 5.35 Choosing a spanning tree.

the direction x_{l+1} belongs to this tree if all the coordinates of $x_{l+2}, \dots, x_n$ vanish, see Fig. 5.35.

With each state s of the complex $\mathcal{C}(K)$ we associate the ordered tensor power $V^{\wedge l}$, and with the corresponding state for the complex $\mathcal{C}'(K)$ we associate $V^{\otimes l}$, where l is the number of circles in the state s. Enumerate the circles in the A-state in some way. Then the ordering determines a map between the space corresponding to the A-state s in $\mathcal{C}(K)$ and the space corresponding to some state $g(s)$ of the complex $\mathcal{C}'(K)$. After that we can successively renumber the circles at all vertices of the tree in order that the identification of the chains in the corresponding states of the complexes $\mathcal{C}(K)$ and $\mathcal{C}'(K)$ commute with the partial differentials acting along the edges of the spanning tree. Thus we have constructed a map between the whole chain space of $\mathcal{C}(K)$ and the chain space of $\mathcal{C}'(K)$.

This map g commutes with all the partial differentials for the following reasons. Let ∂', ∂'' be the partial differentials corresponding to the same edge of the complexes $\mathcal{C}$ and $\mathcal{C}'$. Then we have $g \circ \partial' = \pm \partial'' \circ g$.

If the compatibility holds for three of four edges of some two-dimensional face, then it also holds for the fourth edge, since both complexes are anticommutative and no one of the partial differentials is the identical zero.

To complete the proof, we note that all the edges of the cube can be exhausted if we start from the maximal tree and successively add the missing edges of the two-dimensional faces (add the fourth edge provided that we have three). $\square$

As it was done in Definition 5.3 we call by the *height* $h(\mathrm{Kh}(K))$ of the Khovanov homology of a virtual link K the difference between the leading and lowest non-zero quantum gradings of non-zero Khovanov homology groups of the virtual link K. From Theorem 5.12 it follows that the definition given in Sec. 5.3 (using Khovanov homology for orientable atoms) is agreed with the definition for the ordinary case based on the construction of the present section.

5.8 Spanning tree for Khovanov complex

In this section we shall show how the model of the spanning tree for the Kauffman bracket polynomial given in Sec. 4.2.6 is categorified. We shall describe slightly different approach to calculating (more precisely, to estimation) of the Khovanov homology for classical and virtual links, thanks to which some properties of the Khovanov homology became clearer.

Let us formulate the lemma from the theory of algebraic complexes, we shall follow Wehrli [315].

Lemma 5.8. *Let C_0 and C_1 be graded complexes and $C_i = A_i \oplus B_i$, where the complexes B_i have zero homology. Let $w\colon C_0 \to C_1$ be a map of chains preserving the grading, and let $w_{AA}\colon A_0 \to A_1$ be a "part" of the map w, i.e. the composition of the map w with the evident projection and embedding. Let A be a cone of the map w_{AA}, C be a cone of the map w, and B be (contractible) complex of type $B_0 \oplus B_1[1]$. Then the complexes C and $A \oplus B$ have the same homology.*

The proof of this theorem is pure algebraic, it does not concern the "internal" structure of differentials in the complexes A_i and B_i. The lemma is a key point in the proof of Theorem 5.13 about the spanning tree for the Khovanov complex of classical links.

The main idea of constructing the spanning tree leading to the proof of the theorem is the same as the Thistlethwaite idea which was used for constructing the spanning tree of the Kauffman bracket polynomial: It is necessary to take the bifurcation cube and split it into small subcubes corresponding to states from $\mathcal{V}_1$ (see Sec. 4.2.6). After that we have to consider the Khovanov homology for each of these subcubes, i.e. the copies of the homology groups of the unknot and apply Lemma 5.8 to them repeatedly. We should apply this lemma at each splitting of the cube into two parts.

We asserts that this proof is fit for all models of the Khovanov complex of virtual knots in those cases when this complex is well defined.

Let us describe this construction in more details. We shall consider a non-normalized Khovanov complex of virtual knots. In what follows we should take the "common normalizing factor" out, i.e. shift the height and the grading.

Let K be a virtual diagram. Let us consider its non-normalized bifurcation cube $[[K]]$ with the differential ∂. Enumerate all crossings of K and we shall split the cube $[[K]]$ successively into cubes according to Thistlethwaite's scheme (see Sec. 4.2.6). Namely, in the first step we investigate whether the first crossing is splitting (we call a crossing *splitting*, if under deleting the corresponding vertex from the frame of the atom, the new frame is not connected) and, if it is not splitting, we pass to considering two cubes obtained from $[[K]]$ by fixing the first coordinate. These two cubes represent non-normalized Khovanov complexes for the diagrams K_0 and K_1 obtained from K by smoothings of type A and B. The Khovanov complex (non-normalized) for K_i has some set of homologies; if we consider K_0 and K_1 as non-separated complexes but compound parts of the Khovanov complex corresponding to K, we get some new differentials corresponding to passing from K_0 to K_1. Lemma 5.8 asserts that the initial (non-normalized) Khovanov complex for the diagram K has the same homology as the complex made only from homology of the complexes K_0 and K_1 (and as well as some acyclic part).

Further, we apply the second step: We consider the complexes K_0 and K_1 (as compound parts of the new complex the homology of which coincides with the Khovanov homology of the link K) and investigate whether the corresponding diagrams split in the second crossing. If some of them (say, K_0) does not split, then we reconstruct the complex K_0 and get the complex of type $(K_{00} \to K_{01}) \oplus \langle \text{acyclic part} \rangle$.

We continue the process until we reach a diagram from the set $\mathcal{S}_K$ (see Sec. 4.2.6). Each of these diagrams represents the unknot, therefore, we conclude that the Khovanov homology can be calculated with the help of a complex consisting of the Khovanov homology of the unknot. In terms of formula it looks like the following.

Theorem 5.13. *The non-normalized Khovanov complex of a diagram K of a virtual link is isomorphic to some complex whose chain group looks like*

$$\bigoplus_{s \in \mathcal{V}_1} \mathcal{A}[\beta(s) + w(K_s)]\{\beta(s) + 2w(K_s)\},$$

where $\mathcal{A}$ is the homology group of the unknot.

Note that here we have not used the fact that a link is classical. Therefore, everything can be generalized word by word for virtual links in the case of field on which the initial Khovanov complex is well defined.

Later on, we shall use also the phrase *Wehrli's complex*, by bearing in mind the complex which is quasiisotopic to the Khovanov complex, the existence of the latter is given by Theorem 5.13.

5.9 The Khovanov polynomial and Frobenius extensions

The Khovanov theory of virtual knots described earlier in this chapter is not unique what one can get with the help of the Kauffman model and the (anti)commutative state cube. The present section is devoted to a generalization of the Khovanov theory which uses Frobenius extensions for classical and virtual links.

5.9.1 *Frobenius extensions*

Let $\mathcal{R}$, $\mathcal{A}$ be commutative rings, and let $\iota\colon \mathcal{R} \to \mathcal{A}$ be an embedding of the commutative rings such that $\iota(1) = 1$. The restriction functor taking $\mathcal{A}$-modules to $\mathcal{R}$-modules has right and left adjoint functors: the induction functor $\mathrm{Ind}(M) = \mathcal{A} \otimes_{\mathcal{R}} M$ and the coinduction functor $\mathrm{CoInd}(M) = \mathrm{Hom}_{\mathcal{R}}(\mathcal{A}, M)$. One says that ι is a *Frobenius mapping*, if the induction functor coincides with the coinduction functor. Equivalently: the embedding ι is *Frobenius* if the restriction functor has a 3-sided dual functor. In this case one says also that the ring $\mathcal{A}$ is a *Frobenius extension* over $\mathcal{R}$ by means of ι.

The following proposition takes place.

Proposition 5.3 ([145]). *The embedding ι is Frobenius if there exist a mapping $\mathcal{A}$-bimodules $\Delta\colon \mathcal{A} \to \mathcal{A} \otimes_{\mathcal{R}} \mathcal{A}$ and a mapping $\mathcal{R}$-modules $\varepsilon\colon \mathcal{A} \to \mathcal{R}$ such that Δ is a coassociative and commutative multiplication, herewith $(\varepsilon \otimes \mathrm{Id})\Delta = \mathrm{Id}$.*

A Frobenius extension with a choice ε and Δ is denoted by $\mathcal{F} = (\mathcal{R}, \mathcal{A}, \varepsilon, \Delta)$ and called a *Frobenius system*, [145].

Frobenius extensions are convenient for constructing the Khovanov homology theory for the following reasons. In the module $\mathcal{A}$ defined over the ring $\mathcal{R}$ there are two natural operations: multiplication and comultiplication, the operation Δ.

We are going to use these operations for constructing the Khovanov homology theory for virtual links. Meanwhile we (by evident reasons) restrict ourselves only with the case of commutative rings; moreover, we forget the operator ε (this operator is used for defining invariants of cobordisms and proving functoriality). In other aspects we follow the paper [178] by Khovanov.

5.9.2 *Khovanov construction for Frobenius extensions*

As it was described earlier in this chapter the standard Khovanov theory is constructed over some arbitrary ring $\mathcal{R}$ (for example, the ring $\mathbb{Z}$ or the field $\mathbb{Q}$, or the field $\mathbb{Z}_p$), herewith the homology of the unknot is a graded two-dimensional module $\mathcal{A}$ over this ring, generated by vectors v_+ and v_- having gradings $+1$ and -1, respectively. Two maps are defined on these vectors: the multiplication m and comultiplication Δ. If one shifts the gradings of vectors (this requires a slight change (renormalization) in the construction of the homology theory), then one can set deg $v_+ = 0$, deg $v_- = 2$. Then the element v_+ can be considered as unit (let us denote it by 1, and denote v_- by X), and the multiplication and comultiplication defined earlier turn the module $\mathcal{A}$ into a Hopf algebra over $\mathcal{R}$, in which the multiplication is defined by rules $X^2 = 0$, and the comultiplication looks like $\Delta(1) = 1 \otimes X + X \otimes 1$, $\Delta(X) = X \otimes X$.

In [178] Khovanov solved the following problem: How can one find a condition for a couple of linear spaces $(\mathcal{A}, \mathcal{R})$ to get a link homology theory, where $\mathcal{R}$ is the basic coefficient ring and $\mathcal{A}$ (some Hopf algebra over $\mathcal{R}$) is the homology of the unknot (the main building bricks)? That means that we consider the state cube, with each vertex associated with a tensor power of $\mathcal{A}$ (over $\mathcal{R}$), with exponent equal to the number of circles in the given state, and define partial differentials by means of multiplication and comultiplication, and then add signs on edges and normalize the whole construction by grading shifts.

Khovanov showed that the invariance under the first Reidemeister move requires that $\mathcal{A}$ is two-dimensional as an $\mathcal{R}$-module and gave necessary and sufficient conditions for the existence of such a link homology theory.

In the same paper [178], it is shown that any such theory can be obtained by some operations (base change, twisting and dualtiy) from the following solution:

(1) $\mathcal{R} = \mathbb{Z}[h, t]$,

(2) $\mathcal{A} = \mathcal{R}[X]/(X^2 - hX - t)$,
(3) deg $X = 2$, deg $h = 2$, deg $t = 4$,
(4) $\Delta(1) = 1 \otimes X + X \otimes 1 - h1 \otimes 1$,
(5) $\Delta(X) = X \otimes X + t1 \otimes 1$.

As we see, the multiplication in the algebra $\mathcal{A}$ preserves the grading, and the comultiplication raises it by two.

We omit normalizations regulating these gradings.

We call this construction the *universal* $(\mathcal{R}, \mathcal{A})$*-construction*. The corresponding homology of a (classical oriented) link K will be denoted by $\mathrm{Kh}_U(K)$.

Khovanov proved that all other cases followed from the universal $(\mathcal{R}, \mathcal{A})$-construction. First, he investigates Frobenius extensions for the invariance of the obtained homology theory under the first classical Reidemeister move Ω_1. This leads it to two-dimensional $\mathcal{A}$ as an $\mathcal{R}$-module.

Later, Khovanov considers the universal topological construction by Bar-Natan [20], and constructs a functor from the topological category of Bar-Natan to the category of Frobenius extensions of rank two. The constructed functor is neither injective nor surjective, but it enjoys all nice properties needed for the invariance under the Reidemeister moves.

Thus Khovanov shows that any rank two Frobenius extension as above defines an extraordinary link homology theory. He shows also that any such theory without loss of information can be reduced to the universal theory described above by some algebraic operations.

We shall not go into the details of Khovanov's and Bar-Natan's constructions. We shall just take the universal $(\mathcal{R}, \mathcal{A})$-construction together with some structural statements from Khovanov's theory for building a theory for virtual links.

Also, note that Khovanov also studied functoriality of his new homology theory, for example, its "good behavior" under cobordisms (projective functoriality). To this end, besides multiplication and comultiplication operations, he also defined the unit and counit map and their transformations; we shall not touch on this subject. We shall use only the fact that Khovanov's proof is local, i.e. under that and other Reidemeister moves the invariance does not use any assumption about the structure of the link outside a part of the plane where the move is performed (and, per se, it repeats the proof of the invariance given earlier for the "general" Khovanov homology, in which the injective of the map Δ and the surjective of the map m were used).

In this section, we show that Khovanov's universal construction works in the case of orientable atoms straightforwardly, and write down the algebraic equations the partial differentials have to satisfy for the case of arbitrary virtual links.

5.9.3 *Geometrical generalizations by means of atoms*

With each virtual link diagram having an orientable atom, the universal $(\mathcal{R}, \mathcal{A})$-construction associates some bifurcation cube, the bigraded chain space with partial differentials, whose homology leads to an invariant of virtual links (after a normalization).

Here, with the state cube and the bifurcation cube we associate bigraded complexes with tensor powers of the ring $\mathcal{A}$ over the ring $\mathcal{R}$ staying in vertices of the cube; the tensor power corresponds to the number of circles in the given state; partial differentials in these cubes are defined by using m and Δ, and differentials are sums of partial differentials with signs.

From Khovanov's theory [178] it follows that there exists a *local* proof of the invariance for the universal $(\mathcal{R}, \mathcal{A})$-construction, i.e. there is a number of algebraic steps (equivalences, analogous to the cancellation principle and short exact sequences) which leads to the following.

Let us fix a classical Reidemeister move Ω_i. Then for any classical diagrams K and K' which differ locally by a Reidemeister move Ω_i, there exists, see ahead, a consequence of algebraic transformations taking $\mathrm{Kh}_U(K)$ to $\mathrm{Kh}_U(K')$ and not depending explicitly on the behavior of partial differentials of the Khovanov complexes for K and K' except for those whose explicit form (μ or Δ) follows from the structure of our Reidemeister move Ω_i.

This argument leads to the fact that the universal $(\mathcal{R}, \mathcal{A})$-construction can be generalized for virtual diagrams with orientable atoms. Namely, given a diagram K with an orientable atom, we can construct the corresponding bifurcation cube with differentials, corresponding to the multiplication and comultiplication operations (with signs) and calculate its homology. Furthermore, if two diagrams K, K' have orientable atoms and are obtained from each other by some classical Reidemeister move Ω_i, then according to the principle described above, there is an isomorphism between the graded homologies $\mathrm{Kh}_U(K) \cong \mathrm{Kh}_U(L')$. Since the universal $(\mathcal{R}, \mathcal{A})$-construction is tautologically invariant under the detour move (the bifurcation cube does not change), the following analogue of Lemma 5.3 holds.

Lemma 5.9. *Let K, K' be two diagrams with orientable atoms such that K' differs from K by an application of a detour move or one of the three classical Reidemeister moves. Then* $\mathrm{Kh}_U(K) \cong \mathrm{Kh}_U(K')$.

This argument together with Lemmas 5.4, 5.5 yields that the universal $(\mathcal{R}, \mathcal{A})$-construction works for

- the construction of the Khovanov homology theory Kh_U for framed virtual links by taking the $2l$ parallel copies;
- the construction of the Khovanov homology theory Kh_U for virtual knots by taking two-sheeted orienting coverings over the corresponding atoms;
- the construction of the Khovanov homology theory Kh_U for virtual knots obtained by taking parity projections, see Chap. 8 ahead.

More precisely, the following theorem holds.

Theorem 5.14. (1) *Let l be a natural number. Then* $\mathrm{Kh}_U(D_{2l}(K))$ *is an invariant of the framed virtual link K.*
(2) *The map* $K \mapsto \mathrm{Kh}_U(\widetilde{K})$ *gives a well-defined invariant for virtual links.*

5.9.4 *Algebraic generalizations*

As we have shown before, for virtual knots with orientable atoms the Khovanov homology with $\mathbb{Z}_2$-coefficients can be defined straightforwardly if we set all partial differentials of type $1 \to 1$ to be zero.

Let us now consider the universal $(\mathcal{R}, \mathcal{A})$-construction, and let us generalize it for the case of virtual knots.

Note that if with each knot we associate a well-defined complex, then the homology of this complex will be automatically invariant under classical Reidemeister moves (according to the locality of the invariance proof) and the detour move (there is nothing to prove in this case).

Thus, we have reduced the problem of finding the extension for the ring $\mathcal{A}$ in order to construct the Khovanov homology theory for arbitrary virtual link diagrams, to the following problem. Find an operator (a homomorphism of $\mathcal{R}$-modules) $\mathfrak{I}\colon \mathcal{A} \to \mathcal{A}$ corresponding to maps of type $1 \to 1$ in such a way that for every virtual diagram the bifurcation cube with partial differentials obtained from m, Δ, $\mathfrak{I}$, is anticommutative.

Thus, we require the commutativity of the cube in order to turn it into an anticommutative cube (just as it was done in the usual case).

This problem is purely algebraic. In order to solve it, one has to consider all possible 2-faces of the bifurcation cube for a diagram K; there are finitely many such types (with each face, one associates some atom with two vertices). For each face, one has to check some algebraic conditions for the maps $\mathfrak{I}$, Δ and m.

For the space $\mathcal{A}$, let us take the basis $\{1, X\}$, and for the space $\mathcal{A} \otimes \mathcal{A}$ we take the basis $\{1 \otimes 1, 1 \otimes X, X \otimes 1, X \otimes X\}$.

Then in these bases the maps Δ and m are represented by the following matrices:

$$\Delta = \begin{pmatrix} -h & t \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad m = \begin{pmatrix} 1 & 0 & 0 & t \\ 0 & 1 & 1 & h \end{pmatrix}.$$

After that we shall use the sign of matrix multiplication instead of the composition of the operators. So, for example, we write $\mu \cdot \Delta$ instead of $\mu \circ \Delta$. One of the particular cases given here, is considered in detail in [303].

We look for a matrix

$$\mathfrak{I} = \begin{pmatrix} p & q \\ r & s \end{pmatrix},$$

which corresponds to bifurcations of type $1 \to 1$ and give, at the same time, the (anti)commutativity of the bifurcation cube.

Let a coefficient ring $\mathcal{R}$ containing elements h and t with gradings 2 and 4, respectively, be given. Denote the obtained bifurcation cube by $[[K]]_{\mathcal{R}}$. Let us define the differential as the sum of the partial differentials corresponding to edges (of type m, Δ, $\mathfrak{I}$) with signs arranged as it was done on page 197.

Lemma 5.10. *The bifurcation cube* $[[K]]_{\mathcal{R}}$ *is anticommutative if and only if the following properties hold*:

$$m \cdot \Delta = (\mathfrak{I})^2,$$

$$\Delta \cdot \mathfrak{I} = (\mathfrak{I} \otimes 1) \cdot \Delta = (1 \otimes \mathfrak{I}) \cdot \Delta, \tag{5.3}$$

$$\mathfrak{I} \cdot m = m \cdot (\mathfrak{I} \otimes 1) = m \cdot (1 \otimes \mathfrak{I}). \tag{5.4}$$

Proof. For checking the (anti)commutativity of the state cube it is necessary for us to consider every possible sorts of faces of the cube. Later on, we disregard additional signs on edges and prove the commutativity.

In the "simple" case where we have the field $\mathbb{Z}_2$ and null-differentials corresponding to bifurcations of type $1 \to 1$, everything was reduced to the "classical" cases, and as well as to the case depicted in Fig. 5.1.

For the $(\mathcal{R}, \mathcal{A})$-theory we have to check more cases, since maps of type $1 \to 1$ are not assumed to be zero, and the maps m (multiplication) and Δ (comultiplication) are more complicated than in the case of the homology Kh.

Each two-dimensional face of the cube represents a collection consisting of four states, see Sec. 5.4. Under a passage from one state to another, some circles are reconstructed and the others remain. Denote these four states by s_{00}, s_{01}, s_{10} and s_{11} depending on the values of two changing coordinates. Delete "common components" of the states s_{ij}, i.e. those components of the state s_{00} which do not adjoin to the crossings at which the substitution of the smoothing occurs. Then the given two-dimensional face of the cube will represent some virtual knot and, therefore, the atom corresponding to it. This atom will have exactly two vertices. If the atom is height, then the corresponding diagram is realized by a bifurcation of embedded circles into the plane, thus, the (anti)commutativity of the corresponding face belongs to the number of classical cases checked by Khovanov.

For atoms with disconnected frames the check is obvious. Further, each orientable atom with the connected frame having two vertices is height. Thus, the required checking is reduced to sorting out unoriented atoms with two vertices (all of them by definition are not height). Sorting out these atoms, eventually we shall come to relations which are satisfied identically, e.g. $\mathfrak{I} \circ \mu = \mathfrak{I} \circ \mu$, see Fig. 5.36. Three atoms giving non-trivial relations pointed out in the claim of the lemma are given in Figs. 5.37, 5.38 and (the example considered above), Fig. 5.1. $\square$

We met the first equation already in the case of the general Khovanov homology $\mathcal{C}$ (there the composition $m \cdot \Delta$ looks more simple). In the case of the universal $(\mathcal{R}, \mathcal{A})$-theory we have:

$$m \cdot \Delta = \begin{pmatrix} -h & 2t \\ 2 & h \end{pmatrix}.$$

If we want to construct a $\mathbb{Z}$-*graded* theory, then it is necessary for us that the matrix $\mathfrak{I}$ increases the grading of elements of the ring $\mathcal{R}$ by one. This means that all elements p, q, r, $s \in \mathcal{R}$ should be homogeneous. In this case $\deg p = 1$, $\deg q = 2$, $\deg r = 0$, $\deg s = 1$, herewith it is possible that any of the elements p, q, r, s are equal to zero (in this case the grading

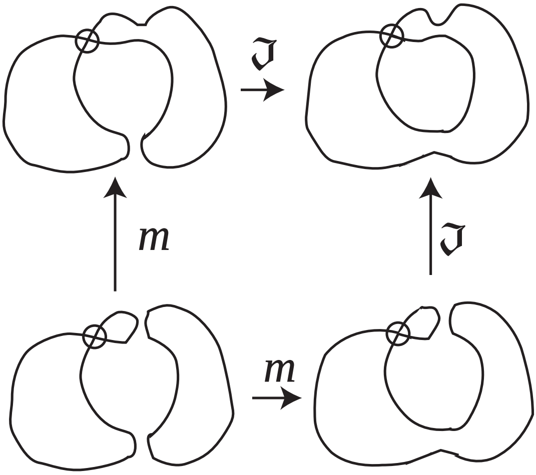


Fig. 5.36 Bifurcation corresponding to tautological relation.

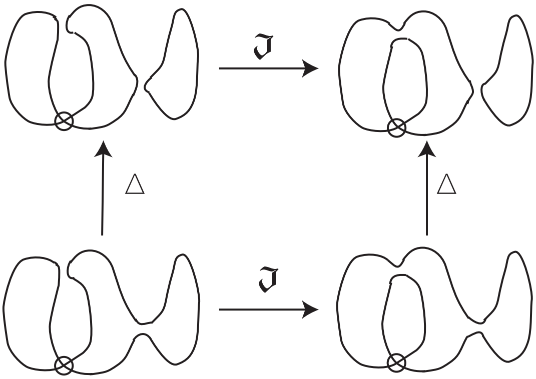


Fig. 5.37 Relations $\Delta \cdot \mathfrak{I} = (\mathfrak{I} \otimes 1) \cdot \Delta = (1 \otimes \mathfrak{I}) \cdot \Delta$.

is not defined). Then from the equality $(\mathfrak{I})^2 = m \cdot \Delta$ it follows deg $(2t) =$ deg $t = 3$, which leads us to a contradiction, if $2 \neq 0$.

Thus (as well as in the case of the general Khovanov homology), under this approach the $\mathbb{Z} \oplus \mathbb{Z}$-bigraded homology theory is possible only in the case of a field with characteristic two.

Let us consider the case of a field with characteristic two. It turns out that in this case we have a simple non-trivial solution. Namely, in the case

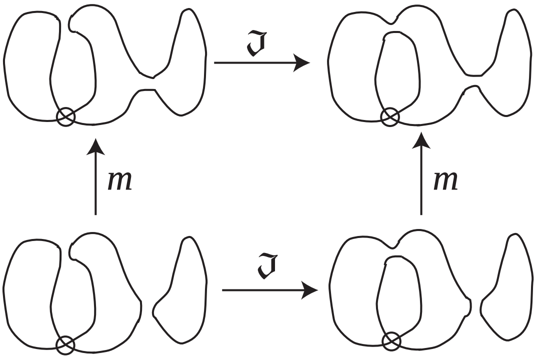


Fig. 5.38 Relations $\mathfrak{I} \cdot m = m \cdot (\mathfrak{I} \otimes 1) = m \cdot (1 \otimes \mathfrak{I})$.

$2 = 0$ the matrix $m \cdot \Delta$ is turned into the diagonal matrix

$$m \cdot \Delta = \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix}.$$

Let us add to the ring $\mathcal{R}$ a new element $u = \sqrt{h}$, deg $u = 1$. Now set $\mathcal{R}' = \mathbb{Z}_2[u, t]$, herewith the algebra $\mathcal{A}$ takes the form $\mathcal{A}' = \mathcal{R}'[x]/(X^2 - u^2X - t)$, where deg $X = 2$, deg $t = 4$, deg $u = 1$.

Set

$$\mathfrak{I} = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}. \tag{5.5}$$

In this case the matrix $\mathfrak{I}$ is scalar, and Eqs. (5.3) and (5.4) are satisfied automatically.

Thus, we conclude with the following theorem.

Theorem 5.15. *Over the field $\mathbb{Z}_2$ the pair of algebras $(\mathcal{R}', \mathcal{A}')$ together with multiplication m, comultiplication Δ defined by $\Delta(1) = 1 \otimes X + X \otimes 1 - u^2 \cdot 1 \otimes 1$, $\Delta(X) = X \otimes X + t1 \otimes 1$ and the scalar map $\mathfrak{I}$ looking like* (5.5), *gives an invariant homology theory for virtual links.*

In the general case, i.e. in the case of the Khovanov homology for virtual links, we have the following theorem.

Theorem 5.16. *The restriction of Khovanov's universal theory for the case $h = 0$ (no restrictions on t) can be extended to virtual links by the method suggested in Sec.* 5.7.

The main idea of the proof of Theorem 5.16 is the following. Δ and m behave nicely under the involution $I\colon 1 \mapsto 1$, $X \mapsto -X$ that takes place while inverting the circle: The multiplication m does not change, and Δ changes the sign. Note that this takes place only for $h = 0$ (for arbitrary t). The case when $h \neq 0$ can be handled by using a more sophisticated twisting.

This generalizes straightforwardly for the case when $h = 0$ (where all differentials of type $1 \mapsto 1$ are assumed to be zero). As a particular case, this leads to an analogue of Lee's theory, see [193, 194].

5.10 Minimal diagrams of links

In the classification and tabulation of (virtual) knots the important step is to describe diagrams having a minimal number of (classical) crossings. One of the main achievements in the development of knot theory is Kauffman–Murasugi–Thistlethwaite theorem (Theorem 4.6) and the classification of alternating links by Menasco and Thistlethwaite [246] following from this theorem. In Chap. 4 we also proved some minimality theorems (Theorems 4.5, 4.7, 4.8).

In this section we shall prove theorems establishing the minimality of virtual and classical diagrams, see also [137, 227]. We refine results about the minimality of diagrams of virtual links described in Sec. 4.3 by using Khovanov complex. The main idea is as follows. Theorem 4.5 (the inequality span $\langle K\rangle \leqslant 4n + 2(\chi - 2)$) for a virtual diagram K with n classical crossings and the atom with the Euler characteristic χ allowed one to prove the minimality in those cases, when the Euler characteristic could not be increased. If the inequality turns into the equality, then to decrease the number of crossings we have to increase the Euler characteristic of the atom or, the same, to decrease its genus. It turns out that by using Khovanov homology one can make estimation on the atom genus, at the same time in some cases one can see that this genus cannot be decreased. In this case the previous arguments together with non-reducibility of the genus lead to the minimality of the diagram.

We shall first mention the spanning tree theorem for Khovanov homology, proved independently by Wehrli [315] and Champanerkar and Kofman [54].

More precisely, in [315] it is shown that the Khovanov homology is isomorphic to the homology of a certain complex. Recall that $\mathcal{V}_1(K)$ (see

Sec. 4.2.6) is the set of states of the virtual diagram K, where the number of circles equals one. From this a generalization of Theorem 5.13 for the case of arbitrary virtual knots follows.

Lemma 5.11. *The non-zero Khovanov homology* $\mathrm{Kh}(K)$ *can have the bigrading only of the form* $(C_1 + \beta - w, C_2 + \beta - 2w \pm 1)$, *where* w *belongs to some finite set of integers,* β *belongs to the set of values* $\beta(s)$ *over all states* $s \in \mathcal{V}_1(K)$, *and* C_1, C_2 *are constants.*

The proof of Lemma 5.11 given in [315] is generalized verbatim for the case of virtual diagrams. An important particular case of this lemma is the statement of the Khovanov homology thickness (thickness was first introduced by Shumakovitch [279]). Let us define this notion accurately by extending it for all virtual diagrams with arbitrary coefficients.

Consider a virtual diagram K and its Khovanov homology over a certain non-graded ring R. Denote by $t_{\max}$ and $t_{\min}$ the maximal and minimal values of $2x - y$ over all pairs x, y such that the homology group of K with the bigrading (x, y) is non-trivial.

Definition 5.9. The *thickness* (*width*) $T_R(K)$ of the Khovanov complex is $(t_{\max} - t_{\min})/2 + 1$.

Remark 5.15. This quantity is integer in the case of orientable atoms, and might be half-integer in the case of non-orientable atoms.

Later on, by a *diagonal* we call the set of pairs of integer numbers (x, y) for which the number $2x - y$ is constant. Among diagonals there are the extreme left and the extreme right, at which the number $2x - y$ is minimal and maximal, respectively. Thus, the thickness measures the number of diagonals between two extreme diagonals.

Definition 5.10. By *thickness* (*width*) $T(K)$ of the virtual diagram K we mean the maximum of all $T_R(K)$ over all R without additional grading.

From Lemma 5.11 and the definition of atom, we get the following lemma.

Lemma 5.12. *For any connected* (*in the sense of atoms*) *diagram* K *of a virtual link we have*: $T(K) \leqslant g(K) + 2$, *where* $g(K)$ *is the genus of the atom corresponding to* K*.*

The notion of the 1-complete virtual diagram was defined in Chap. 4, see Definition 4.10.

Definition 5.11. Let us call a virtual diagram K 2-*complete*, if $T(K) = g(K) + 2$.

Indeed, for an estimation of the number of diagonals of the Wehrli complex (see Theorem 5.13) it is necessary for us to estimate the range of numbers $\beta(s)$ over all states $s \in \mathcal{V}_1(L)$. It is easy to see that in the case of alternating link diagrams all these numbers equal each other (this leads to the presence of two diagonals $t_{\max}$ and $t_{\min}$ such that $t_{\max} = t_{\min} + 2$), in the case of atoms with genus one the numbers $\beta(s)$ can equal x, $x+1$, $x+2$ for some x; in the case of atoms with the Euler characteristic χ they can take values in an interval from some number x to $x + (2 - \chi)$.

From Theorem 4.5 and Lemma 5.12, we have the following theorem.

Theorem 5.17. *Let* $T(K) = g + 2$, $\operatorname{span}\langle K\rangle = s$. *Then the number of classical crossings of a connected diagram of the virtual link generated by* K *cannot be smaller than* $s/4 + g$.

In particular, if a diagram with n *crossings and the atom with genus* g *is* 1-*complete and* 2-*complete, then it is minimal.*

The last assertion means that all diagrams for which two properties of "natural non-reducibility" hold (in the decomposition of the Kauffman bracket polynomial the leading and lowest terms are not equal to zero and in the Wehrli complex each of the two extreme diagonals has at least one non-trivial element of the Khovanov homology) are minimal.

Theorem 5.17 holds in any category in which the Khovanov complex is well defined and invariant. So, if we are interested in the invariance of a classical diagram in the category of classical diagrams, we can consider the thickness in the classical category.

Chapter 6

Virtual Braids

6.1 Introduction

This chapter is devoted to virtual braids and its connection to the theory of virtual knots. The main result of the present chapter is the construction of an invariant for virtual braids, whose restriction to classical braids is a complete invariant. This invariant was first constructed by the first-named author and it is a generalization of a complete invariant of classical braids for the virtual case. The statement that this invariant is complete in the case of classical knots follows from the faithfulness of the Hurwitz action [126] on free groups; the invariance was first proved by Artin [12]. The question whether the invariant is complete for the case of virtual braids with the number of strands more than two, is still an open problem.

From the existence of an invariant of virtual braids generalizing a complete invariant of classical braids it follows that the natural mapping from the group of classical braids to the group of virtual braids is an inclusion. This result was first proved by Fenn, Rimanyi and Rourke in [87].

The present chapter is organized as follows. First we give all necessary definitions in the theory of classical and virtual braids and formulate known results about virtual braids. The remaining part of the present chapter is devoted to the invariant of virtual braids. We also show that this invariant is complete for virtual braids with two strands.

6.2 Definitions of virtual braids

Classical braids have many different definitions: the geometrical definition uses diagrams on the plane, considered up to Reidemeister moves; the algebraic definition uses a concrete presentation; the algebro-geometric definition says that the braid group is the fundamental group of the space of polynomials without multiple roots; and the topological definition says that the braid group is the fundamental group of the configuration space for a set of distinct points on the plane.

Virtual braids, a generalization of classical braids, were first mentioned by Kauffman in his first talk about virtual knots [157]. The first papers about virtual braids belong to Kamada [153] and Vershinin [307]. In the papers [166, 167] by Kauffman and Lambropoulou Markov's theorem is proved for the case of virtual braids.

Virtual braids admit a combinatorial definition. They are defined as equivalence classes of virtual braid diagrams by virtual Reidemeister moves (all Reidemeister moves except the first classical and virtual moves, are admitted). The fact that we forbid the first virtual Reidemeister move leads to a remarkably simple construction generalizing *all* quantum invariants of classical knots for the case of "virtual knots" with the following clause. Here one should consider the theory where virtual knot diagrams are factored by all generalized moves except the first virtual one. In this case, one talks about *rigid virtual knots.* While considering *virtual braids* there exists a canonical way for generalizing all quantum invariants (here we add the operator of transposition of tensor summands, which corresponds to a virtual crossing, to the given operator which is a solution of the Yang–Baxter equation). However, this generalization cannot be canonically extended to the case of virtual knots: In the classical case there exists some normalization which turns an invariant of braid into an invariant of knots, closures of braids, and the new invariant is obtained by adding some tensor which regulates the invariance under the first classical Reidemeister move. In the case of virtual knots this construction fails since we have to require the invariance under the first virtual Reidemeister move under the same conditions. However, in the case of rigid virtual knots the theory of quantum invariants works. For more details concerning this theory see [158].

Definition 6.1. A *virtual braid diagram* on n strands is a union of n smooth curves in general position on the plane connecting points $(i, 1)$ with points $(a_i, 0)$, these curves are monotonic with respect to ordinate (here $(a_1, \ldots, a_n)$ is some permutation of the numbers $(1, \ldots, n)$), herewith some crossings are marked as virtual crossings, and at the other (classical) crossings the under/overcrossing structure is specified, i.e. it is indicated which branch forms an *overcrossing* and which one forms an *undercrossing*, see Fig. 6.1. In this case it is said that the braid realizes the permutation

$a\colon 1 \mapsto a_1, \ldots, n \mapsto a_n$.

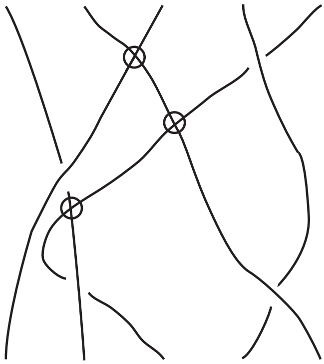


Fig. 6.1 A virtual braid diagram.

Definition 6.2. A *virtual braid* is an equivalence class of virtual braid diagrams by planar isotopies and all virtual Reidemeister moves except the first classical move and the first virtual move.

Remark 6.1. The first classical and virtual Reidemeister moves are impossible because a strand of a braid cannot be ascending.

Definition 6.3. Let us call a virtual braid *pure* (*even*), if the permutation corresponding to it is trivial (even).

Like classical braids, virtual braids form a group (with respect to juxtaposition and rescaling the vertical coordinate). The unit element e of this group is the braid represented by all vertical parallel strands. The reverse element for a given braid is just its mirror image with respect to a horizontal line. The generators of the group of virtual braids with n strands are: $\sigma_1, \ldots, \sigma_{n-1}$ (classical crossings) and $\zeta_1, \ldots, \zeta_{n-1}$ (virtual crossings), see Fig. 6.2.

It is evident that for each $i = 1, \ldots, n-1$ the following equality $\zeta_i^2 = e$ holds (by virtue of the second virtual Reidemeister move).

One can show that the following set of relations [307] generates the group of virtual braids with n strands:

(1) the relations of the (classical) braid group:
$\sigma_i \sigma_j = \sigma_j \sigma_i$ for $i, j = 1, \ldots, n-1, |i-j| \geqslant 2$;

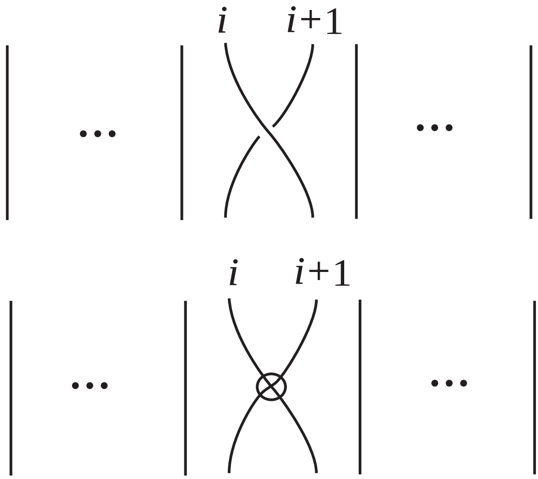


Fig. 6.2 The generators of the group of virtual braids: σ_i (up) and ζ_i (below).

$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $i = 1, \ldots, n-2$;

(2) the relations of the symmetric group:
$\zeta_i\zeta_j = \zeta_j\zeta_i$ for $i, j = 1, \ldots, n-1$, $|i-j| \geqslant 2$;
$\zeta_i\zeta_{i+1}\zeta_i = \zeta_{i+1}\zeta_i\zeta_{i+1}$ for $i = 1, \ldots, n-2$;
$\zeta_i^2 = e$ for $i = 1, \ldots, n-1$;

(3) the mixed relations:
$\sigma_i\zeta_{i+1}\zeta_i = \zeta_{i+1}\zeta_i\sigma_{i+1}$ for $i = 1, \ldots, n-2$;
$\sigma_i\zeta_j = \zeta_j\sigma_i$ for $i, j = 1, \ldots, n-1$, $|i-j| \geqslant 2$.

It is not difficult to show that the group of virtual braids is generated by one classical generator, for example, σ_1 and all virtual generators. Then all the other generators σ_i will be conjugate with the generator σ_1. For example, $\sigma_2 = \zeta_1\zeta_2\sigma_1\zeta_2\zeta_1$.

Other questions related to presentations of the group of virtual braids and the presentation of virtual knots by closures of virtual braids for arbitrary number of strands were described in works by Kauffman and Lambropoulou [166, 167, 169, 170] (see also [105]).

The group of virtual braids with n strands is denoted by $\mathrm{VB}(n)$. If we consider the subgroup of $\mathrm{VB}(n)$ consisting of those braids with no occurrence of the generators σ_n and ζ_n, then we shall get the subgroup isomorphic to $\mathrm{VB}(n-1)$. Therefore, we have the inclusions $\mathrm{VB}(1) \subset \mathrm{VB}(2) \subset \cdots \subset \mathrm{VB}(n) \subset \cdots$.

Thus, we can say about the *stably group of virtual braids* VB(∞) as the direct limit of the groups of virtual braids with respect to the inclusion.

6.3 Virtual braids and virtual knots

Before formulating main results about braids and virtual braids we recall some structural moments.

6.3.1 *Closure of virtual braids*

Analogous to the case of classical braids virtual braids have the *closure*, see Fig. 6.3.

Definition 6.4. We say that a (smooth) virtual diagram K on the plane is *braided* with respect to some point A if $A \notin K$ and at each point of K the tangent vector (oriented along the orientation of K) is directed counterclockwise manner if we look from the point A.

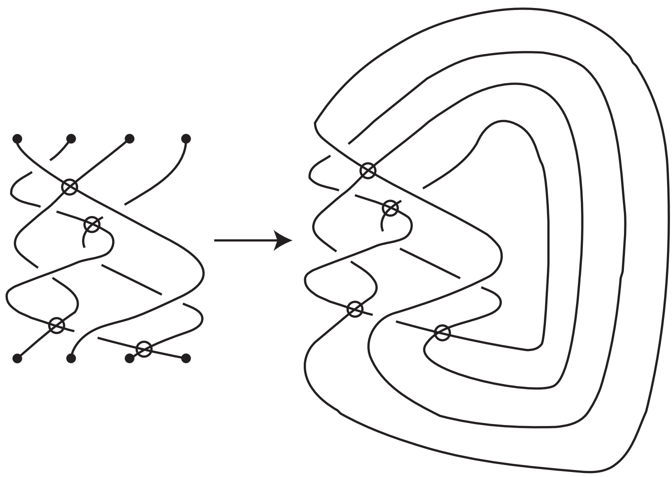


Fig. 6.3 Closure of a virtual braid.

A braided (with respect to some point) diagram can be turned into a diagram of the closure of a virtual braid: For this we have to "cut" the diagram along neighboring radii, the sector between them does not contain crossings, and straighten the diagram, see Fig. 6.4.

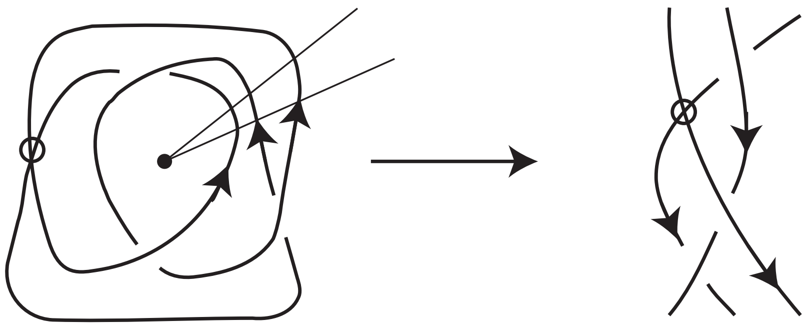


Fig. 6.4 Construction of a virtual braid by a braid diagram.

It is evident that equivalent virtual braids give isotopic virtual links. Moreover, all equivalence classes of virtual links can be represented as closures of virtual braids.

Alexander's theorem [6] asserts that any link can be obtained as the closure of a braid. The virtual analogue of this theorem (see, e.g. [166]) asserts that any virtual link is obtained as the closure of a virtual braid.

Markov's theorem [241] gives the set of moves for braids (virtual braids) such that two braids β_1 and β_2 represent isotopic links, i.e. $\mathrm{Cl}(\beta_1) \equiv \mathrm{Cl}(\beta_2)$, where $\mathrm{Cl}(\beta_i)$ is the closure of β_i, if and only if the braid β_1 can be transformed into the braid β_2 by a finite sequence of these moves. In the classical case there are only two transformations (moves); they are called *Markov's transformations* (*moves*).

Herewith in the classical case we may assume that all braids in any chain from β_1 to β_2 are classical.

In the classical case Markov's theorem holds.

Theorem 6.1. *The closures of two braids β_1 and β_2 are isotopic if and only if β_1 can be obtained from β_2 by a finite sequence of transformations shown in Fig.* 6.5 (*in the right picture the crossing added below can be either*  *or* ).

A proof of this theorem was first announced by Markov [241], but this proof contained some disadvantages. The first unexceptionable proof of (classical) Markov's theorem is due to Birman [26], see also [252, 288].

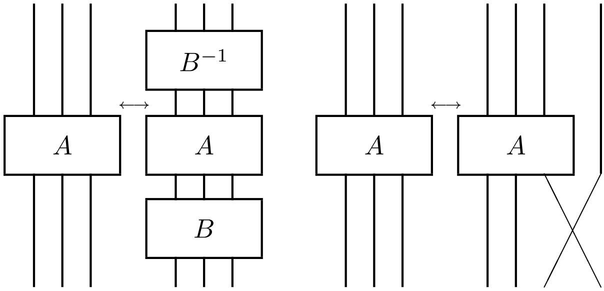


Fig. 6.5 Markov's moves.

In [153] Kamada proved an analogue of Markov's theorem for the case of virtual braids. Namely, he proved the following.

Theorem 6.2. *Two virtual braid diagrams have isotopic closures as virtual links if and only if they are related by a finite sequence of the following moves* (VM0)–(VM3):

- *braid equivalence*;
- *conjugation* (*in the virtual braid group*);
- *right stabilization* (*adding a strand with an extra positive*, *negative or virtual crossing*), *and destabilization* (*the operation being inverse to a stabilization*);
- *right/left virtual exchange move*, *see Fig.* 6.6.

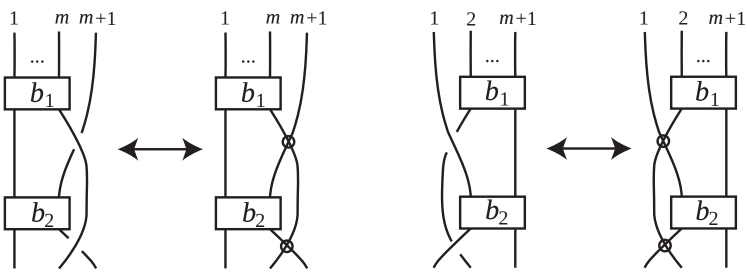


Fig. 6.6 Virtual moves.

The moves (VM0)–(VM2) are analogous to those in the classical case with the only difference that a conjugacy element (the move (VM1)) can be any virtual braid with the same number of strands. The "new" move

(VM3) has two variants: right and left. Each of the virtual exchange moves replaces two classical crossings between which a strand without classical crossings inside passes, with two virtual crossings. These moves do not change the equivalence class of braid closures, since the corresponding closures are obtained from each other by applying the second classical and virtual Reidemeister moves.

The necessity of the moves listed above is obvious; it is left for the reader as a simple exercise. For sufficiency, we refer the reader to the original work [153].

The set of moves introduced by Kauffman and Lambropoulou in [167, 168] is more convenient.

Namely, it is proved.

Theorem 6.3. *Two virtual braid diagrams β_1 and β_2 have isotopic closures as oriented virtual links if and only if β_2 is obtained from β_1 by applying a finite sequence of the following moves:*

(1) *an isotopy of braids;*
(2) *a conjugacy by means of classical braids;*
(3) *the right virtual L_v-move;*
(4) *the right classical L_v-move;*
(5) *the right and left under-threaded L_v-moves.*

These moves are schematically depicted in Figs. 6.7–6.9.

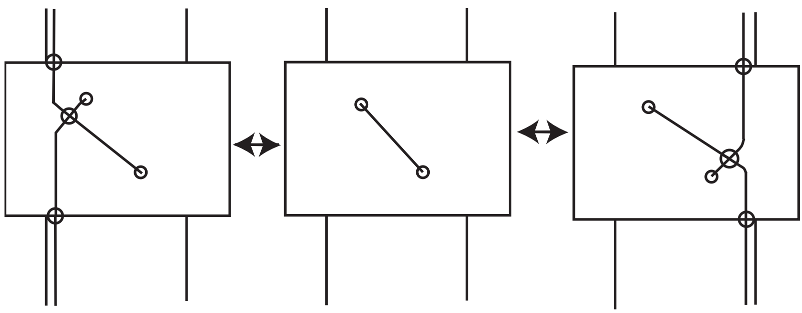


Fig. 6.7 Virtual L_v-moves: right and left.

Note that the proof of Markov's theorem in the formulation of [167] is easier and more convenient than the proof of [153]. It is based on the so-called L-move which is used both under constructing the closure of a virtual

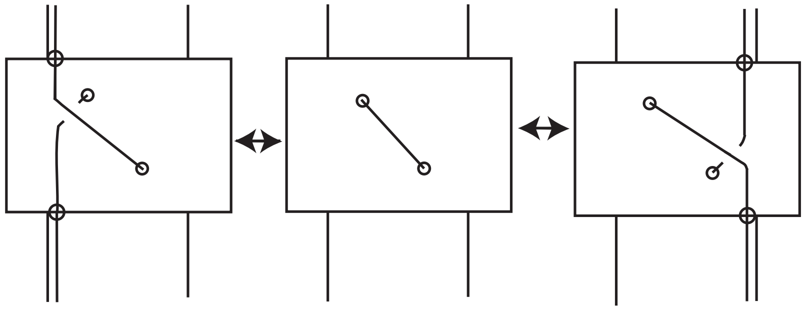


Fig. 6.8 Classical L_v-moves: right and left.

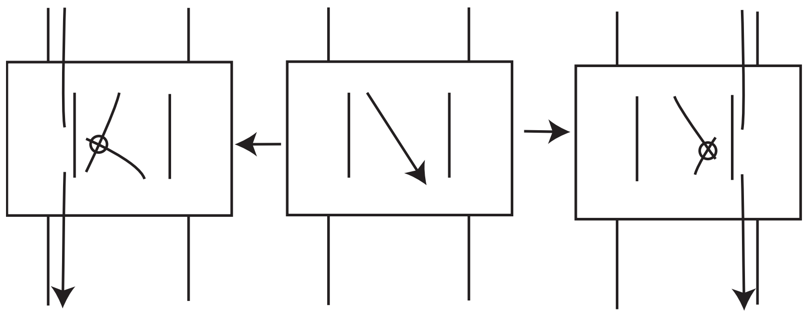


Fig. 6.9 Right and left under-threaded L_v-moves.

braid and describing simplest virtual equivalences for links translated in the language of braids.

Note also that in the list of moves by Kauffman and Lambropoulou there are no conjugacy by means of virtual braids; with intelligent trick the authors showed that a conjugacy with the help of virtual braids could be expressed by using the detour move and L-moves.

Moreover, in [167] the following *algebraic* reformulation of Markov's theorem which allows one to replace the moves from the list by *local moves* is given. Namely, the list of elementary algebraic equivalences for virtual braids (besides an isotopy of braids) is the following:

(1) The classical and virtual conjugacies: $\zeta_i \alpha \zeta_i = \alpha = \sigma_i^{-1} \alpha \sigma_i$.
(2) The right stabilization (classical and virtual): $\alpha \zeta_n \sim \alpha \sim \alpha \sigma_n^{\pm 1}$.
(3) The algebraic under-threaded move (top and bottom): $\alpha \sim$

$\alpha\sigma_n^{\pm 1}\zeta_{n-1}\sigma_n^{\mp 1}$.

(4) The algebraic left under-threaded move (top and bottom): $\alpha \sim \alpha\zeta_n\zeta_{n-1}\sigma_{n-1}^{\mp 1}\zeta_n\sigma_{n-1}^{\pm 1}\zeta_{n-1}\zeta_n$.

In the second, third and fourth cases the initial braid α has n strands; the resulting braid has $n+1$ strands.

6.3.2 *Burau representation and its generalizations*

The (classical) braid group has a natural representation. This representation is called the *Burau representation* [45]. It is closely related to the Alexander polynomial of the closure.

The more natural way to find representations of the braid group is the following. One can consider the braid group $\mathrm{Br}(n)$ (the group of classical braids with n strands) and try to represent braids by $n \times n$ matrices. More precisely, one can associate with the element σ_i a block-diagonal matrix with (2×2)-blocks situated in two rows $(i, i+1)$ and in two columns $(i, i+1)$, and the remaining blocks have the size (1×1) and equal 1 and are situated on the principal diagonal. It is evident that for such a matrix we have the commutative relation between the images σ_i, σ_j, where $|i - j| \geqslant 2$. If we take two matrices corresponding to σ_i, $i = 1, 2$, with equal diagonal (2×2)-blocks (but in different places), then we only have to check the relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ for matrices of the size 3×3. We get easily the representation in which a (2×2)-block looks like:

$$\begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix}. \tag{6.1}$$

This representation is called the *Burau representation* of the braid group. It was first proposed by Burau [45].

The Alexander polynomial of a classical link is restored from the Burau representation of a braid whose closure gives the given link.

The faithfulness of this representation was an open problem for a long time. In [26] Birman proved the faithfulness of this representation for the group of braids with three strands.

In [250] Moody constructed the first example of a non-trivial element from the kernel of the Burau representation (the group of braids with more than three strands).

For the time being the problem of the faithfulness of the Burau representation is solved: positive for $n \leqslant 3$ and negative for $n \geqslant 5$; see, e.g. [24]. The case $n = 4$ is still open. In [25] Bigelow mentioned a connection between the problem of recognition of the unknot by the Jones polynomial with one variable and the problem of faithfulness of the Burau representation for braids with four strands.

It is known that the Burau representation is reducible. Namely, it has the eigenvector $(1, \ldots, 1)$.

Vershinin [307] suggested the following generalization $\mathfrak{B}$ of the Burau representation [45] for virtual braids. The virtual braid group $\mathrm{VB}(n)$ is represented by $n \times n$ matrices where the generators σ_i, ζ_i are represented by block-diagonal matrices with the unit on the main diagonal and the only non-trivial (2×2)-block on lines and columns $(n, n-1)$. The block for σ_i is shown in (6.1). For ζ_i we use permutations, namely, the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The proof that it really gives a representation is left to the reader as an exercise.

The trivial generalization of the Burau representation for virtual braids under which a transposition is associated with a virtual crossing leads to a generalization of *all* quantum invariants for virtual braids, see [158]. Therefore, we obtained a representation of the virtual braid group which is denoted by $\mathfrak{B}$.

However, this representation is rather weak, it has a non-trivial kernel in the case of braids with two strands. It is easy to check that for the non-trivial virtual two-strand braid represented by the word $\beta = (\sigma_1^2 \zeta_1 \sigma_1^{-1} \zeta_1 \sigma_1^{-1} \zeta_1)^2$ we have

$$\mathfrak{B}(\beta) = \mathfrak{B}(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where e is the unit element. Indeed, the matrix $\mathfrak{B}(\sigma_1)$ has the following eigenvalues: 1 and $1-t$. More precisely,

$$C\mathfrak{B}(\sigma_1)C^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1-t \end{pmatrix},$$

where

$$C = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}.$$

In this case we have

$$C\mathfrak{B}(\zeta_1)C^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

We shall write just ζ instead of $C\mathfrak{B}(\zeta_1)C^{-1}$ and σ instead of $C\mathfrak{B}(\sigma_1)C^{-1}$.

Set $H(k,l,m) = \sigma^k\zeta\sigma^l\zeta\sigma^m\zeta$. Then H is an upper triangular matrix with 1 and -1 on the main diagonal if $k+l+m=0$. Set $k=2, l=-1, m=-1$. Then $\mathfrak{B}(H(2,-1,-1)^2) = e$.

Further, we shall show that the virtual braid $\beta = (\sigma_1^2\zeta_1\sigma_1^{-1}\zeta_1\sigma_1^{-1}\zeta_1)^2$ is non-trivial.

One has the following generalization of the Burau representation [209]: We take polynomial matrices in two variables, t and q (and their inverse), the same image of elements σ_i as before and the matrix

$$\begin{pmatrix} 0 & q \\ q^{-1} & 0 \end{pmatrix}$$

to be the block for expressing images of elements ζ_i. Denote the map, defined above on generators of the braid group, by R.

Theorem 6.4. *The map R can be generated as a representation of the braid group.*

Proof. Obviously, the matrix $R(\sigma_i)$ is invertible, and for the matrix $R(\zeta_i)$ we have $(R(\zeta_i))^2 = e$.

Furthermore, the relations of the braid group for the σ's can be easily checked as in the case of the "weaker" Burau representation.

So, we only have to check the relations $R(\zeta_i\zeta_{i+1}\zeta_i) = R(\zeta_{i+1}\zeta_i\zeta_{i+1})$ and $R(\zeta_i\zeta_{i+1}\sigma_i) = R(\sigma_{i+1}\zeta_i\zeta_{i+1})$.

They can be checked straightforwardly by direct calculation with 3×3 matrices. □

Now, we can prove the following theorem.

Theorem 6.5. *The group* Br(3) *is naturally embedded in the virtual braid group* VB(3).

Proof. Let β_1, β_2 be some braid-words written in σ_1, σ_2, σ_1^{-1}, σ_2^{-1}. Suppose they represent the same braid in $\mathrm{VB}(n)$. Then their Burau matrices coincide. Hence the Burau representation of the classical braid group is faithful for the case of braids with three strands (see [25, 26]), and we conclude that β_1 and β_2 represent the same word in Br(3). □

The question about the existence of a faithful representation of the group of virtual braids with arbitrary number of strands is still open. Recently, Bigelow and Krammer independently constructed a faithful representation of the classical braid group for any number of strands [24, 186].

6.4 The Kauffman bracket polynomial for braids

Analogous to the Kauffman bracket polynomial for classical and virtual knots, one can define the Kauffman bracket polynomial for classical and virtual braids.

Namely, let D be a diagram of a virtual braid with n strands. Let us consider the set of its classical crossings. At each classical crossing we have two smoothings: A:  $\to$)(or B:  $\to$  (the same situation we had for knot diagrams). After a smoothing a braid diagram can be transformed to a diagram which is not a braid diagram. After a smoothing of all crossings we get some state s of the diagram. In this state we have (possibly, empty) the set of closed circles and the set of n segments connecting endpoints of the given braid. We have $2n$ endpoints, therefore, there are $(2n-1)!!$ possibilities for their pairwise connection (possible, with virtual crossings). Let us denote such diagrams by α_i.

In contrast to virtual links, where after smoothings of any diagram we get a set of circles, and the Kauffman bracket polynomial is a Laurent polynomial; in the case of braids the final results of smoothings are diagrams consisting of α_i and a finite set of free-standing circles.

Let us define now the Kauffman bracket polynomial for D in a state s as the corresponding diagram α_i taken with the coefficient $(-a^2 - a^{-2})^{\gamma(s)}$, where $\gamma(s)$ is the number of free-standing circles in the state s.

After that for D we set

$$\langle D \rangle = \sum_s a^{\alpha(s)-\beta(s)} \langle D|s \rangle,$$

where $\langle D \,|\, s \rangle$ is the diagram obtained from D by smoothing according to the state s.

The obtained Kauffman bracket polynomial is an invariant of braids, since we do not have the first Reidemeister move Ω_1; the invariance under the second and third Reidemeister moves Ω_2, Ω_3 can be checked straightforwardly.

One can also consider the Kauffman bracket polynomial for closures of braids; in this case one has to normalize it by the standard way, multiplying by $(-a)^{-3w}$, where w is the writhe number of the braid (the number of  crossings minus the number of  crossings).

6.5 Invariants of virtual braids

In this section, we are going to present an invariant $\mathcal{F}$ of virtual braids. Its restriction on the class of classical braids is a complete invariant; this follows from the fact that this restriction corresponds to a faithful action of the group of classical braids on a free infinite-generated group. From the completeness of the restriction of the invariant $\mathcal{F}$ on the case of classical braids, it follows that the classical braid group is a subgroup of the virtual one with the same number of strands. More precisely, since we have an invariant of virtual braids which is a complete invariant for classical braids, then for two classical braids which are equivalent as elements of the virtual braid group, the values of this invariant coincide; by virtue of the completeness, we conclude that these classical braids represent the same element in the corresponding classical braid group. The question of whether the invariant is complete remains open so far.

Definition 6.5. A virtual braid diagram is called *regular* if any two different intersection points have different ordinates.

Let us start with basic definitions and introduce the notation.

Remark 6.2. In the sequel, the number of strands for a virtual braid diagram is denoted by n, unless otherwise specified.

Remark 6.3. In the sequel, regular (virtual) braid diagrams and corresponding braid words will be denoted by Greek letters (possibly, with indices). Virtual braids will be denoted by Latin letters (with indices, maybe).

Remark 6.4. We shall also treat *braid words* and *braids* familiarly, saying, for example, "a strand of a braid word" and meaning "a strand of the corresponding braid".

Let us describe the construction of the word for a given regular virtual braid diagram. Let us walk along the axis Oy from the level $\{y = 1\}$ down to the level $\{y = 0\}$ and watch all those levels $y = t \in [0, 1]$ having crossings. Each such crossing permutes strands with local numbers i and $(i + 1)$ for some $i = 1, \ldots, n - 1$. If the crossing is virtual, we write the letter ζ_i, if not, we write σ_i if the overpass is the "northwest–southeast" strand, and σ_i^{-1} otherwise.

Thus, we have got the braid word for a given regular virtual braid diagram, see Fig. 6.10.

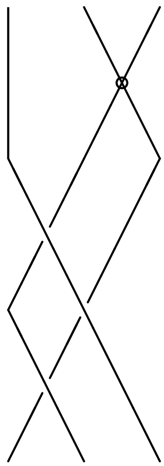


Fig. 6.10 A virtual braid and the corresponding word.

Thus the main question is the word problem for the virtual braid group: How can one recognize whether two different (regular) virtual braid diagrams β_1 and β_2 represent the same braid or not?

One can apply the virtual braid group relations to one diagram without getting the other, and one does not know whether he has to stop and say that they are not isomorphic or he has to continue.

In a natural way the problem of constructing *complete* invariants appears. A partial answer to this question is the construction of a virtual braid group invariant, i.e. a function on virtual braid diagrams (or braid words) that is invariant under all virtual braid group relations. In this case, if for an invariant $\mathcal{F}$ we have $\mathcal{F}(\beta_1) \neq \mathcal{F}(\beta_2)$, then β_1 and β_2 represent two different braids.

6.5.1 *The construction of the main invariant*

Let G be the free group with generators $a_1 \dots, a_n, t$. Let E_i be the quotient set of right residue classes $\{a_i\}\backslash G$ for $i = 1, \dots, n$.

Definition 6.6. A *virtual n-system* is a set of elements $\{x_1 \in E_1, x_2 \in E_2, \dots, x_n \in E_n\}$.

The aim of this subsection is to construct an invariant map $\mathcal{F}$ (non-homomorphic) from the set of all virtual n-strand braids to the set of virtual

n-systems.

Let β be a braid word. Let us construct the corresponding virtual n-system $\mathcal{F}(\beta)$ step-by-step. Namely, we shall reconstruct the function $\mathcal{F}(\beta\psi)$ from the function $\mathcal{F}(\beta)$, where ψ is σ_i or σ_i^{-1} or ζ_i.

First, let us take n residue classes of the unit element of G: $\{e, e, \ldots, e\}$. Let ε be the unit braid word, set

$$\mathcal{F}(\varepsilon) = \{e, e, \ldots, e\}.$$

Now, let us read the word β. If the first letter is ζ_i, then all words but e_i, e_{i+1} in the n-system stay the same, e_i becomes equal to t and e_{i+1} becomes t^{-1} (here and in the sequel, we mean, of course, residue classes, for example, $[t]$ and $[t^{-1}]$, but we write just t and t^{-1} for the sake of simplicity).

Now, if the first letter of our braid word is σ_i, then all classes but e_{i+1} stay the same, and e_{i+1} becomes a_i^{-1}. Finally, if the first letter is σ_i^{-1}, then the only changing element is e_i: it becomes a_{i+1}.

The procedure for each next letter (generator) is the following. Denote the index of this letter (the generator or its inverse) by i. Assume that the left strand of this crossing originates from the point $(p, 1)$, and the right one from the point $(q, 1)$. Let $e_p = P$, $e_q = Q$, where P, Q are some words representing the corresponding residue classes from E_p, E_q. After that all residue classes but e_p, e_q should stay the same.

If the letter is ζ_i, then e_p becomes $P \cdot t$, and e_q becomes $Q \cdot t^{-1}$. If the letter is σ_i, then e_p stays the same, and e_q becomes $QP^{-1}a_p^{-1}P$. Finally, if the letter is σ_i^{-1}, then e_q stays the same, e_p becomes $PQ^{-1}a_qQ$. Note that these operations are well defined, i.e. they do not depend on the choice of representatives of the corresponding residue classes, it can be checked straightforwardly.

Actually, if we take the words a_p^lP, a_q^mQ instead of the words P, Q, we get: $a_p^lPt \sim Pt, a_q^mQt^{-1} \sim Qt^{-1}$ in the first case and $a_p^lP \sim P, a_q^mQP^{-1}a_p^{-l}a_p^{-1}a_p^lP = a_q^mQP^{-1}a_p^{-1}P \sim QP^{-1}a_p^{-1}P$ in the second case. In the third case we obtain $a_p^lPQ^{-1}a_q^{-m}a_q^{-1}a_q^mP = a_p^lPQ^{-1}a_q^{-1}Q \sim PQ^{-1}a_q^{-1}Q, a_q^mQ \sim Q$.

Thus, we have defined the map $\mathcal{F}$ from the set of all virtual braid diagrams to the set of virtual n-systems.

Theorem 6.6. *The function $\mathcal{F}$ defined above is a braid invariant. Namely, if β_1 and β_2 represent the same braid, then $\mathcal{F}(\beta_1) = \mathcal{F}(\beta_2)$.*

Proof. We have to demonstrate that the function $\mathcal{F}$ defined on virtual braid diagrams is invariant under all virtual braid group relations. It suffices to prove that, for the words $\beta_1 = \beta\gamma_1$ and $\beta_2 = \beta\gamma_2$, where $\gamma_1 = \gamma_2$ is a relation we have proved, we can also prove $\mathcal{F}(\beta_1) = \mathcal{F}(\beta_2)$. During the proof of the theorem, we shall call it the *A-statement*.

Indeed, having proved this claim, we also have $\mathcal{F}(\beta_1\delta) = \mathcal{F}(\beta_1\delta)$ for arbitrary δ because the invariant $\mathcal{F}(\beta_1\delta)$ (as well as $\mathcal{F}(\beta_2\delta)$) is constructed step-by-step, i.e. knowing the value $\mathcal{F}(\beta_1)$ and the braid word δ, we easily obtain the value of $\mathcal{F}(\beta_1\delta)$. Hence, for braid words β, δ and for each braid group relation $\gamma_1 = \gamma_2$ we have $\mathcal{F}(\beta\gamma_1\delta) = \mathcal{F}(\beta\gamma_2\delta)$. This completes the proof of the theorem.

Now, let us return to the A-statement.

To prove the A-statement, we must consider all virtual braid group relations. The commutation relation $\sigma_i\sigma_j = \sigma_j\sigma_i$, $\sigma_i\zeta_j = \zeta_j\sigma_i$, and as well $\zeta_i\zeta_j = \zeta_j\zeta_i$ for $|i-j| \geqslant 2$, is obvious: All four strands involved in this relation are different, so the order of applying the operation is immaterial. The same can be said about the other commutation relations, involving one σ and one ζ or two ζ's.

Now let us consider the relation $\zeta_i^2 = \varepsilon$ which is pretty simple too. Actually, let us consider a braid word β, and let the word β_1 be defined as $\beta\zeta_i^2$ for some i. Let $\mathcal{F}(\beta) = \{P_1, \ldots, P_n\}$, $\mathcal{F}(\beta_1) = \{P_1', \ldots, P_n'\}$. Let p and q be the numbers of strands coming to the crossing from the left side and from the right side, respectively. Obviously, for $j \neq p, q$ we have $P_j = P_j'$. Besides, $P_p' = (P_p \cdot t) \cdot t^{-1} = P_p, P_q' = (P_q \cdot t^{-1}) \cdot t = P_q$.

Now let us consider the case $\beta_1 = \beta \cdot \sigma_i \cdot \sigma_i^{-1}$ (obviously, the case $\beta_1 = \beta\sigma_i^{-1}\sigma_i$ is quite analogous to this one).

As before, denote $\mathcal{F}(\beta)$ by $\{\ldots P_i \ldots\}$, and $\mathcal{F}(\beta_1)$ by $\{\ldots P_i' \ldots\}$, and the corresponding numbers of strands by p and q. Again, we have $P_j' = P_j$ for $j \neq p, q$. Moreover, $P_p = P_p'$ by definition of $\mathcal{F}$ (since the pth strand makes an overcrossing twice), and $P_q' = (P_q P_p^{-1} a_p^{-1} P_p) P_p^{-1} a_p P_p = P_q$.

Now let us check the invariance under the third Reidemeister move. Let β be a braid word, $\beta_1 = \beta\zeta_i\zeta_{i+1}\zeta_i$, and $\beta_2 = \beta\zeta_{i+1}\zeta_i\zeta_{i+1}$. Let p, q, r be the global numbers of strands occupying positions n, $n+1$, $n+2$ at the bottom of the braid.

Denote $\mathcal{F}(\beta)$ by $\{P_1, \ldots, P_n\}$, $\mathcal{F}(\beta_1)$ by $\{P_1^1, \ldots, P_n^1\}$, and $\mathcal{F}(\beta_2)$ by $P_1^2, \ldots, P_n^2$. Obviously, for any $i \neq p, q, r$ we have $P_i = P_i^1 = P_i^2$. Direct calculations show that $P_p^1 = P_p^2 = P_p \cdot t^2$, $P_q^1 = P_q^2 = P_q$ and $P_r^1 = P_r^2 = P_r \cdot t^{-2}$.

Now, let us consider the mixed move by using the same notation: $\beta_1 = \beta\zeta_i\zeta_{i+1}\sigma_i$, $\beta_2 = \sigma_{i+1}\zeta_i\zeta_{i+1}$. As before, $P_j^1 = P_j^2 = P_j$ for all $j \neq p, q, r$. Now, direct calculation shows that $P_p^1 = P_p t^2$, $P_q^1 =$

$P_q t^{-1}$, $P_r^1 = P_r t^{-1}(P_q t^{-1})^{-1} a_q^{-1}(P_q t^{-1}) = P_r P_q a_q^{-1} P_q t^{-1}$ and $P_p^2 = P_p t^2$, $P_q^2 = P_q t^{-1}$, $P_r^2 = P_r P_q^{-1} a_q^{-1} P_q$.

Finally, consider the "classical" case $\beta_1 = \beta\sigma_i\sigma_{i+1}\sigma_i, \beta_2 = \beta\sigma_{i+1}\sigma_i\sigma_{i+1}$; the notation is the same. Again, we have $P_j^1 = P_j^2 = P_j$ for any $j \neq p$, q, r. Besides this, since the pth strand forms two overcrossings in both cases then $P_p^1 = P_p^2 = P_p$. Then, $P_q^1 = P_q P_p^{-1} a_p^{-1} P_p$, $P_r^1 = (P_r P_p^{-1} a_p^{-1} P_p) \cdot (P_q P_p^{-1} a_p^{-1} P_p)^{-1} a_q^{-1} \ (P_q P_p^{-1} a_p^{-1} P_p) = P_r P_q^{-1} a_q^{-1} P_q P_p^{-1} a_p^{-1} P_p$; by construction we have $P_q^2 = P_q P_p^{-1} a_p^{-1} P_p$, $P_r^2 = P_r P_q^{-1} a_q^{-1} P_q P_p^{-1} a_p^{-1} P_p$.

As we see, the final results coincide and this completes the proof of the theorem. □

Thus, we have proved that $\mathcal{F}$ is a virtual braid invariant, i.e. for a given braid the value of $\mathcal{F}$ does not depend on the diagram representing a braid b. So, we can simply write $\mathcal{F}(b)$.

Remark 6.5. In fact, we can think of $\mathcal{F}$ as a function valued not in $\{E_1, \ldots, E_n\}$, but in n copies of G: All these invariances were proved for the general case of $(G, \ldots, G)$. The present construction of $\{E_1, \ldots, E_n\}$ is considered for the sake of simplicity.

6.5.2 *Representation of virtual braid group*

The above described invariant $\mathcal{F}$ gives a representation ψ of the group VB(n) into the group of automorphisms of the free group F_{n+1} with generators $a_1, \ldots, a_n, t$ by $(i = 1, \ldots, n-1)$:

$$\psi(\sigma_i) = \begin{cases} a_i \mapsto a_i a_{i+1} a_i^{-1}, \\ a_{i+1} \mapsto a_i, \\ a_l \mapsto a_l, l \neq i, i+1, \\ t \mapsto t, \end{cases} \quad \psi(\zeta_i) = \begin{cases} a_i \mapsto t a_{i+1} t^{-1}, \\ a_{i+1} \mapsto t^{-1} a_i t, \\ a_l \mapsto a_l, l \neq i, i+1, \\ t \mapsto t. \end{cases} \tag{6.2}$$

From the immediate checking we get the following theorem.

Theorem 6.7. *Let $\mathcal{F}(\beta) = \{x_1, \ldots, x_n\}$ for a braid word β realizing the permutation π. Then the mapping $a_i \mapsto x_{\pi(i)}^{-1} a_{\pi(i)} x_{\pi(i)}$, $t \mapsto t$ coincides with the action ψ described above.*

Thus the formula (6.2) *gives a pure representation of the classical braid group in the group of automorphisms of the free group.*

This theorem is a proof of the fact that $\mathcal{F}$ is a well-defined invariant.

6.5.3 *On completeness in the classical case*

In the case of classical braids let us define an *n-system* as the following simplification of the notion of a virtual n-system. Let us consider the free product G of n groups isomorphic to the group $\mathbb{Z}$ with generators $a_1, \dots, a_n$. Denote by E'_i the quotient set of left residue classes of the group G over the group generated by $\{a_i\}$, i.e. g_1, $g_2 \in G$ represent the same element in E'_i if and only if $g_1 = a_i^k g_2$ for some k.

Definition 6.7. An *n-system* is a set of elements $\{x_1 \in E'_1, \dots, x_n \in E'_n\}$.

Definition 6.8. An *ordered n-system* is a pair: $\langle$an n-system and a permutation from $S_n\rangle$.

In a natural way the invariant $\mathcal{F}$ is simplified to the invariant f taking values in ordered n-systems: we “forget” the generator t. Herewith for the case of classical braids there is no loss of information: For any classical braid given by a word β the expression $\mathcal{F}(\beta)$ does not contain occurrences of the variable t.

Thus, we associate with each classical braid given by a word β a set of elements $f(\beta) = \{x_1, \dots, x_n\}$. Each element x_i is defined up to left multiplication by some power of the generator a_i. Therefore, the *transformation of generators* $a_i \to x^{-1}_{\pi(i)} a_{\pi(i)} x_{\pi(i)}$ is well defined. Thus, we obtain an action of the braid group on a free group; this action is known and called the *Hurwitz action.* In [12] it was proved that the invariant f was complete. Therefore, the Hurwitz action is faithful. From here we get that the invariant f (and the invariant $\mathcal{F}$) is complete in the case of classical braids.

6.5.4 *First fruits*

Like classical knots, classical braids (i.e. braids without virtual crossings) can be considered up to two equivalences: classical (modulo only classical moves) and virtual (modulo all moves). Now, by using the invariant $\mathcal{F}$ we can prove that they are the same (as in the case of classical knots). This fact is not new. It follows from [87].

Setting $t = 1$ in the invariant $\mathcal{F}$ and passing to the invariant f, which is complete in the case of classical braids, we get a generalization of the complete invariant f of classical braids to the case of virtual braids.

Thus, the natural map of the group of classical braids into the group of virtual braids (taking σ_i to σ_i) is an inclusion. More precisely, we have the following theorem.

Theorem 6.8. *Two virtually equivalent classical braids b_1 and b_2 are classically equivalent.*

Proof. Since b_1 is virtually equivalent to b_2, we have $f(b_1) = f(b_2)$. Now, taking into account that f is a complete invariant on the set of classical braids, we have $b_1 = b_2$ (in the classical sense). □

Analogous to the case of virtual knots, in the case of virtual braids there exists a forbidden move, namely, $\sigma_i\sigma_{i+1}\zeta_i = \zeta_{i+1}\sigma_i\sigma_{i+1}$, see Fig. 6.11.

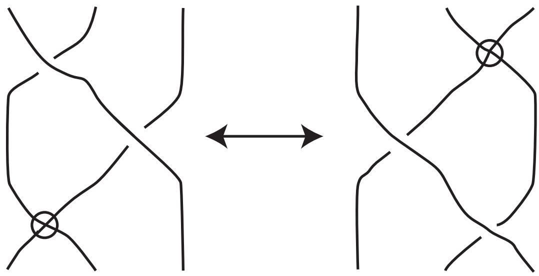


Fig. 6.11 A forbidden move for virtual braids.

Now, we are going to show that it cannot be represented by a finite sequence of the virtual braid group relations.

Statement 6.1. *A forbidden move (relation) cannot be represented by a finite sequence of legal moves (relations).*

Proof. Actually, let us calculate the values $\mathcal{F}(\sigma_1\sigma_2\zeta_1)$ and $\mathcal{F}(\zeta_2\sigma_1\sigma_2)$. In the first case we have

$$\{e,e,e\} \to \{e,a_1^{-1},e\} \to \{e,a_1^{-1},a_1^{-1}\} \to \{e,a_1^{-1}t,a_1^{-1}t^{-1}\}.$$

In the second case we have

$$\{e,e,e\} \to \{e,t,t^{-1}\} \to \{e,t,t^{-1}a_1^{-1}\} \to \{e,ta_1^{-1},t^{-1}a_1^{-1}\}.$$

As we see, the final results are not the same (i.e. they represent different virtual n-systems); thus, the forbidden move changes a virtual braid. □

Remark 6.6. When passing from the invariant $\mathcal{F}$ to the invariant f (if we put $t = 1$), we get $f(\sigma_1\sigma_2\zeta_1) = f(\zeta_2\sigma_1\sigma_2)$. Therefore, the variable t feels the forbidden move.

Here we give two more examples showing the advantages of the invariant $\mathcal{F}$.

Consider the 3-strand braid b given by the word

$$\beta = \zeta_2\sigma_2^{-1}\zeta_2\sigma_1\sigma_2\zeta_1\sigma_1\zeta_1\sigma_2^{-1}\sigma_1^{-1}.$$

Direct calculations show that for this braid $\mathcal{F}(\beta) \neq \mathcal{F}(\varepsilon)$. However, this braid is not distinguished by the virtual Jones–Kauffman polynomial proposed in [157]. More precisely, consider the link $\mathrm{Cl}(b)$ obtained as the closure of b and the Kauffman polynomial $X(\mathrm{Cl}(b))$ of this link. It is well known that the Kauffman polynomial does not distinguish links which differ from each other by the virtualization, see Fig. 4.2.

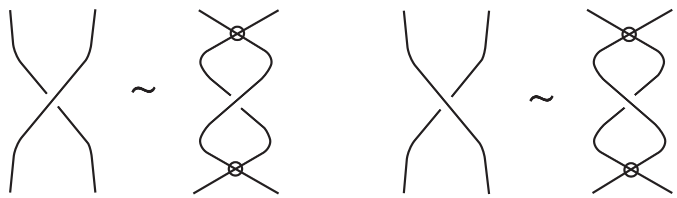


Fig. 6.12 A pair of diagrams not distinguished by the polynomial.

The virtualization is expressed in the language of virtual braids. Thus, it is easy to see that if for some braid we substitute $\sigma_i^{\pm 1}$ for $\zeta_i\sigma_i^{\pm 1}\zeta_i$, then the closures of both braids will have the same Kauffman polynomial, see Fig. 6.12.

So,

$$X(\mathrm{Cl}(b)) = X(\mathrm{Cl}(\sigma_2^{-1}\sigma_1\sigma_2\sigma_1\sigma_2^{-1}\sigma_1^{-1})).$$

The transformed braid is trivial, so $X(\mathrm{Cl}(b)) = X(\mathrm{Cl}(e))$.

Now, we give another example of the strength of the invariant $\mathcal{F}$. We showed that the braid $\beta = (\sigma_1^2\zeta_1\sigma_1^{-1}\zeta_1\sigma_1^{-1}\zeta_1)^2$ lay in the kernel of the Burau representation $\mathfrak{B}$. It is easy to see that the invariant $\mathcal{F}$ feels non-triviality of this braid.

6.5.5 *Completeness for the case of two-strand braids*

As we have shown above, the new invariant is stronger than the link coefficient, sometimes it recognizes virtual braids, which cannot be recognized by the Jones–Kauffman polynomial or by the Burau representation.

Besides this, the restriction of the invariant $\mathcal{F}$ for the case of classical braids (factorizing $t = 1$) coincides with the complete classical braid group invariant f.

The invariant $\mathcal{F}$ gives us an example of a map from one algebraic object (braid group) to another algebraic object (n copies of the free group or n residue classes in free groups). However, this map is not homomorphic. Nevertheless, it can be reconstructed to be the Hurwitz action, see (6.2).

Thus, in order to understand the strength of the invariant $\mathcal{F}$, we are going to establish some properties of this map.

From Theorem 6.7 we have the following theorem.

Lemma 6.1. *If $\mathcal{F}(\beta_1) = \mathcal{F}(\beta_2)$ for some braids b_1, b_2 given by words β_1 and β_2, then for any two braids a and c given by words α and γ we have $\mathcal{F}(\alpha\beta_1\gamma) = \mathcal{F}(\alpha\beta_2\gamma)$ (all braids are taken to have the same number of strands).*

The next step is how to describe all possible values of the invariant $\mathcal{F}$. In the general case, this problem is very difficult; we restrict ourselves only to the case of $n = 2$ strands and, thereby, we classify these braids. However, the group of virtual braids with two strands is organized very easily. It is isomorphic to the group $\mathbb{Z} * \mathbb{Z}_2$.

Notation change: Instead of generators a_1, a_2 we shall write a, b; instead of σ_1, ζ_1 we simply write σ, ζ.

Recall that the invariant f is obtained from $\mathcal{F}$ by "forgetting" the variable t. Denote the free group with the generators a, b by G', and let $E_1' = \{a\}\backslash G', E_2' = \{b\}\backslash G'$.

In the case of two strands, f is a map from VB(2) to (E_1', E_2') or, simply, to (G', G').

For a braid α, denote $f(\alpha)$ by $(P(\alpha), Q(\alpha))$.

First, let us consider some examples of virtual two-strand braid words and values of f on them:

(1) for the trivial word we have $\{e, e\}$;
(2) for σ we have $\{1, a^{-1}\}$;
(3) for σ^{-1} we have $\{b, 1\}$;
(4) for ζ we have $\{1, 1\}$.

It is not difficult to prove the following theorem.

Theorem 6.9. *Let β be a braid word (a virtual braid with two strands). Then $P(\beta)Q(\beta)^{-1} = a^k b^l$ for some k, l.*

Proof. We shall use the induction on the number of crossings. For zero crossings there is nothing to prove. Now, let β be a braid word with n crossings, $\beta' = \beta\alpha$, whence $\alpha = \zeta$, σ or σ^{-1}. Let $P(\beta)Q(\beta)^{-1} = a^n b^m$.

For $\alpha = \zeta$ we have $P(\beta') = P$, $Q(\beta') = Q(\beta)$. Thus $P(\beta')Q(\beta')^{-1} = a^n b^m$.

For $\alpha = \sigma$ we have $P \mapsto P$, $Q \mapsto QP^{-1}a^{-1}P$, $PQ^{-1} \mapsto a^{-1}PQ^{-1}$ if β is even, and we have $Q \mapsto Q$, $P \mapsto PQ^{-1}b^{-1}Q$, $PQ^{-1} \mapsto PQ^{-1}b^{-1}$ if β is odd.

For $\alpha = \sigma^{-1}$ we have $Q \mapsto Q$, $P \mapsto PQ^{-1}bQ$, $PQ^{-1} \mapsto PQ^{-1}b$ if β is even, and we have $P \mapsto P$, $Q \mapsto QP^{-1}aP$, $PQ^{-1} \mapsto aPQ^{-1}$ if β is odd.

Thus, we have made the induction step that completes the proof of the theorem. □

The condition on PQ^{-1} is, indeed, quite natural. It means that there exists an element $g \in G'$ such that $g \in [P] \in E'_1$ and $g \in [Q] \in E'_2$. Obviously, this element g is unique. Thus, g can be considered as an invariant of the group VB(2).

Obviously, for any braid b with two strands represented by a word β we have $f(\beta) = f(\beta\zeta)$. Besides, for each even virtual braid b in VB(2) there exists the unique braid $b\zeta$, corresponding to it. Thus, it is sufficient to consider only the even subgroup EVB(2) of the group VB(2).

Theorem 6.10. *The invariant g (as well as the invariant f) of the virtual braid group* EVB(2) *is complete.*

It suffices to prove that g is complete. To prove this theorem, we shall need an auxiliary lemma.

Lemma 6.2. *For any even two-strand braid words $\pi, \rho \in$ VB(2) we have $g(\pi\rho) = g(\rho)g(\pi)$ and $g(\pi)^{-1} = g(\pi^{-1})$, thus g is an antihomomorphism on* VB(2).

Proof. First, let us note that the group VB(2) is a free group with two generators $\zeta\sigma$ and $\zeta\sigma^{-1}$.

It can easily be checked that $g(\varepsilon) = e$, $g(\zeta\sigma) = b^{-1}$, $g(\zeta\sigma^{-1}) = a^{-1}$, $g((\zeta\sigma)^{-1}) = b$, $g((\zeta\sigma^{-1})^{-1}) = a$.

The assertion about antihomomorphism of the map g is evidently checked on the generators $\zeta\sigma$ and $\zeta\sigma^{-1}$. □

Let us return to the proof of Theorem 6.10.

***Proof of Theorem* 6.10.** Lemma 6.2 shows that g is an antihomomorphic map mapping the free group EVB(2) to the free group with generators a, b. This map takes generators $\zeta\sigma$, $\zeta\sigma^{-1}$ to the generators a, b^{-1}. Thus, it has no kernel. So, g is a complete invariant of EVB(2), and so is f. □

Certainly, $\mathcal{F}$ is a complete invariant of the group EVB(2) too. Besides, this invariant "feels" multiplication by ζ on the right side, thus $\mathcal{F}$ recognizes all elements of VB(2) as well. In order to recognize whether a pair of elements $\{x_1 \in E_1,\, x_2 \in E_2\}$ is a value of the invariant $\mathcal{F}$ on some braid, we just factorize them by the relation $t = 1$, take the preimage β (if this exists) of the obtained couple $\{x'_1,\, x'_2\}$ under f, and see whether $\mathcal{F}(\beta) = \{x_1,\, x_2\}$ or $\mathcal{F}(\beta\zeta) = \{x_1,\, x_2\}$.

The simplest example of $\{x_1 \in E_1,\, x_2 \in E_2\}$ that is not a value of $\mathcal{F}$ on a virtual braid is $\{b, a\}$. In this case $PQ^{-1} = ba^{-1}$ which is not equal to $a^k b^l$ for any integer numbers k, l.

Chapter 7

Vassiliev's Invariants and Framed Graphs

7.1 Introduction

In the end of 1980s, Vassiliev [304, 305] and, independently, Goussarov [113] gave a definition of a special sort of invariants of classical knots, which were, in the sequel, called *Vassiliev's knot invariants* or *finite-type invariants*. These definitions turned out to be extremely useful because all known polynomial invariants of knots [99, 141], as well as quantum invariants [259, 297], turned out to be expressible in terms of finite-type invariants, see [17, 28]. Vassiliev's invariants can be valued in any abelian group. In this chapter we deal with Vassiliev's knot invariants valued in $\mathbb{Q}$.

At the very beginning it turned out to be clear that Vassiliev's knot invariants are closely connected to functions on *chord diagrams* of some special sorts, so-called *weight functions* (see Sec. 7.2): with each invariant one associates such a function. Thus, the following natural question arose: Is it true that all weight systems generate Vassiliev's invariants?

The work by Kontsevich [184] allowed one to understand the structure of Vassiliev's knot invariants. The integral formula suggested by Kontsevich leads to the *universal Vassiliev–Kontsevich invariant* which restores the initial invariant of knots from a given weight system. In this sense the Kontsevich integral is equivalent to the collection of all Vassiliev's invariants. As it turned out (see, e.g. [27]) the Kontsevich integral was universal also for *quantum invariants* of knots. From this, one can see that quantum invariants *are not stronger* than Vassiliev's knot invariants (in the paper by Vogel [311], it is shown that they are *strictly weaker*, i.e. in this paper explicit examples of Vassiliev's knot invariants are given which are not expressible in terms of quantum invariants). The combinatorial approach to the Kontsevich integral turned out to be possible after the work by Le and Murakami [192], who gave a combinatorial definition of the Kontsevich integral for classical knots (nevertheless, these formulae are not completely explicit because they rely on Drinfeld's associator which needs additional calculations), for more details, see [259].

After this the combinatorial description was extended to a wider class of knots, in particular, to knots in thickened surfaces $M \times I$ with boundary; this was done in the paper by Lieberum [196] and in [8, 9].

From reviews devoted to the theory of Vassiliev's knot invariants (of classical) knots, we recommend the paper by Bar-Natan [17].

After Kontsevich's work, it became clear that there should be some combinatorial approach to *all* Vassiliev's knot invariants. This combinatorial approach was explicitly formulated in the paper by Polyak and Viro [266] for some particular cases; after that the question of the existence of such *combinatorial formulae* for *all* Vassiliev knot invariants of classical knots arose. A combinatorial formula is a formula counting a linear combination of the algebraic number of occurrences of some sample diagrams as subdiagrams in a given Gauss diagram.

The invention of virtual knot theory made a new prospective to the theory of finite-type invariants. One should especially mention the paper by Goussarov, Polyak and Viro [114]. The main result of this paper is Goussarov's theorem on the existence of combinatorial formulae for all Vassiliev invariants of classical knots, see also Sec. 7.3. It is worth to mention that among Gauss diagrams which appear in such formulae one cannot do it without *non-realizable* Gauss diagram, i.e. those Gauss diagrams not corresponding to any classical knot. In this paper, the authors were led to the notion of virtual knot in a way different from the usual Kauffman approach.

In the paper [306], Vassiliev made a justification of this result answering the question about the existence of combinatorial formulae with *integral coefficients*.

The existence of virtual knot diagrams in combinatorial formulae encouraged the authors of [114] to search for a construction of Vassiliev's knot invariants for virtual knots; in fact, this was another approach to the definition of virtual knots themselves, where a finite-type invariant was treated as an invariant expressible by an invariant having combinatorial formula counting occurrences of Gauss diagrams with appropriate coefficients.

In the same paper, the *Polyak algebra* naturally arose. It is the universal algebra from which all invariants of finite type for virtual links appear; elements of this algebra are, in fact, linear combinations of virtual links themselves (see, e.g. [60]).

Such a definition of the (Vassiliev) finite-type invariant for virtual knots turns out not to be very convenient, though such finite-type invariants can be easily classified. The point is that any finite-type invariant (in the sense of Polyak and Viro) of virtual knots restricts to a finite-type invariant of classical knots, but not every extension of classical finite-type invariants to virtual finite-type invariants is representable in the Polyak algebra. For example, any of the coefficients of the Jones polynomial are not representable [59].

As a reply to this work, one can consider the last version of the pioneering work by Kauffman [158] where he suggested a more formal approach to finite-type invariants, by using the notion of *rigid isotopy.*

It became clear that many polynomial invariants of rigid knots fitted into Kauffman's conception, and led to many finite-type invariants, see [8, 9].

However, the set of virtual knots is much larger than the set of classical knots, and as it turned out, on the set of virtual knots one could easily find counterexamples to some problems about Vassiliev's knot invariants, yet unsolved in the classical case.

For example, Vassiliev's knot invariants of virtual knots allow one to distinguish non-invertibility of knots (this was first mentioned by Sawollek [275]). For detecting non-invertibility by means of Vassiliev's invariants in the classical case, see, e.g. [57, 73]. Actually, in the virtual case even degree zero invariants detect non-invertibility, since flat knots (and even free knots) are non-invertible in the general case, see Chap. 8.

It is also worth to mention the following works related to Vassiliev's invariants.

It is known that there are no degree one Vassiliev invariants. For virtual knots, however, the space of degree one Vassiliev invariants is infinite dimensional. In [122] a sequence of degree one Vassiliev invariants of virtual knots is constructed.

In our book we consider the combinatorial description of Vassiliev's invariants. There is a topological description of these invariants. More precisely, in [312] it is shown that Vassiliev's invariants of classical knots can be identified with the Taylor towers of the configuration space of knots arising from Goodwillie–Weiss calculus.

The main object of the book is a virtual knot. A virtual knot can be considered as a knot in a thickened surface. Therefore, it would be interesting to define Vassiliev's invariant via topological point of view. In [91, 116] finite-type invariants for knots in thickened surface and combinatorial formulae for them are considered.

There is a connection between Vassiliev's invariants and Milnor's invariants. For example, in [117] a formula for computing the Milnor (concordance) invariants from the Kontsevich integral is obtained.

In Chap. 9 we consider free links and parities. Free links are a significant simplification of flat links considered in previous chapters. There are works related to finite-type invariants of flat links (see [136]) and free links (see [108]). In [60] it is shown that there are finite-type invariants arising from parity.

As we shall further see, a finite-type invariant gives rise to a functional on arrow diagrams. In [17, 18, 195] the authors consider Lie algebra weight systems for arrow diagrams.

The structure of the chapter is the following. First we give a definition of Vassiliev's invariants of classical knots and of J-invariants of planar curves, and also two definitions of the finite-type invariants for virtual knots, the one after Goussarov, Polyak and Viro, and the one after Kauffman. After that we formulate the results of [114] and then we discuss finite-type invariants after Kauffman.

In Sec. 7.5 we prove that the polynomial Ξ defined in Chap. 4 can be expanded as a power series, and leads to a triparametric series of the Vassiliev knot invariants (Theorem 7.5).

After that we discuss the connection between virtual knots, the Jones–Kauffman polynomial and finite-type invariants of classical knots (it turns out that the Kauffman bracket for virtual knots leads to a function on chord diagrams, which, in turn, leads to a series of finite-type invariants of classical knots).

We conclude the chapter by the description of a relation between finite-type invariants and framed 4-graphs. First, we give an important result due to the second-named author about the connection between rotating circuits and transverse circuits for framed 4-graphs. Rotating circuits play a central role in various combinatorial problems of knot theory. Using rotating circuits, we prove one of the central results of the chapter due to the first-named author: the proof of Vassiliev's conjecture on realizability of chord diagrams by planar singular knots (Theorem 7.13). This conjecture was motivated by the problem of existence of combinatorial formulae for Vassiliev's knot invariants, solved by Vassiliev in [306]. To prove the Vassiliev conjecture we use the combinatorial techniques described in previous chapters (atoms, d-diagrams, the source–sink structure). After that we investigate the embedding problem for framed 4-valent graphs in arbitrary 2-surfaces.

7.2 The Vassiliev invariants of classical knots and J-invariants of curves

In this chapter all links are assumed to be oriented, unless otherwise specified.

Definition 7.1. A *singular* k-component link of order n is an immersion of a disjoint collection of k oriented circles in $\mathbb{R}^3$, whose singularities are all simple transverse double points, and there are exactly n such points. Singular links are considered up to a *natural isotopy*, i.e. a map from the ambient space $\mathbb{R}^3$ to itself preserving the orientation.

With each singular k-component link we can naturally associate a chord diagram on k oriented circles. Every circle parametrizes an embedding of one of the link components. On these circles we connect by chords (or equivalently, mark by $S^0 \subset S^1$) those pairs of points having the same image in $\mathbb{R}^3$, see Fig. 7.1.

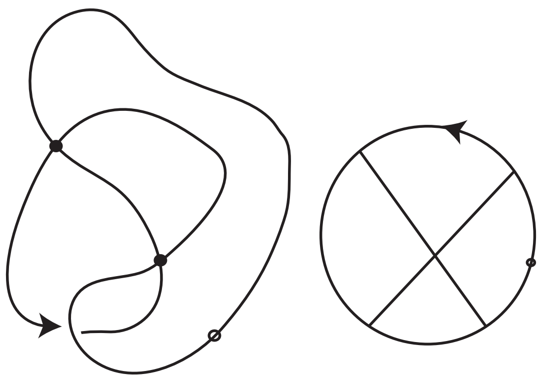
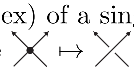
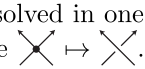
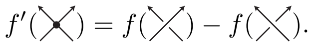


Fig. 7.1 A singular knot diagram and the corresponding chord diagram.

Every singular point (vertex) of a singular knot can be resolved in one of two ways: the positive one  and the negative one . Thus, with each singular link and a selected double point of it we associate two singular links of lower genus, herewith these two resolutions are well defined up to isotopy.

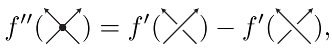
Let f be an invariant of classical links valued in a certain abelian group. We shall consider the formal linear space of links with coefficients in this abelian groups, and extend all invariant functions by linearity. We can extend the invariant f to singular links by the following formula:



(7.1)

This formula means that we define the *derivative* f' of the invariant f as a function on singular links of order one, where the value of the derivative on a singular link is decreed to be equal to the difference between the two values on two links obtained from the initial one by the positive and the negative resolutions, respectively. The formula (7.1) is called the *Vassiliev relation*.

If we have the derivative f' of the function f, we can define the second derivative f'' of f on singular links of the second order. We use the same formula (7.1) as



assuming that we deal with three singular links which coincide outside some small neighborhood of a singular crossing of one of these links; inside this neighborhood the second link is smoothed positively, the third link is resolved negatively, and outside this small neighborhood all three links have one more singular points. We can define the third derivative on singular links of order three, the fourth derivative on singular knots of order four, etc.

Remark 7.1. It is not difficult to show that the value of the extension, i.e. the definition of derivatives, is independent of the ordering in which the singular crossings are resolved.

Definition 7.2. One says that an invariant f is a (*Vassiliev*) *finite-type invariant* of order at most n if its $(n+1)$th derivative $f^{(n+1)}$ is identically equal to zero on singular links with $n+1$ singular crossings. Herewith, this invariant has order exactly n if it is an invariant of order at most n but is not an invariant of order at most $(n-1)$.

Denote the linear space of Vassiliev's invariants of order at most n by $\mathcal{V}_n$. There is an obvious inclusion, called the *Vassiliev filtration*, $\mathcal{V}_0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \cdots$.

Definition 7.3. For a Vassiliev invariant f of order n its nth derivative $f^{(n)}$ is called the *symbol* of f.

Since the Vassiliev relation deals with triples with (singular) links having the same number of components, it makes sense to speak separately about the Vassiliev invariants for every fixed number of components. For example, one may talk about the Vassiliev invariants of knots, about the Vassiliev invariants of two-component links, etc.

It turns out that the symbol contains an important piece of information about the invariant itself. Namely, the following theorem is evident.

Theorem 7.1 (Vassiliev). *Let v_1 and v_2 be two Vassiliev invariants of order n on k-component links having the same symbol. Then the difference $v_1 - v_2$ is a Vassiliev invariant of order at most $n-1$.*

Remark 7.2. This theorem is also valid for virtual links (for any of the definitions to be given ahead).

Thus, one can consequently study the structure of the space of the Vassiliev invariants $\mathcal{V}_n$, knowing $\mathcal{V}_{n-1}$ and the structure of the space of symbols.

Let v be a Vassiliev invariant of order n (on links with some fixed number of components k). Then $v^{(n+1)} = 0$; the latter means that for every two k-component link diagrams which differ by one crossing change  $\longleftrightarrow$  and having order n, we have $v^{(n)}($$) = v^{(n)}($$)$.

Thus, the value of the symbol of the invariant v does not depend on the structure of classical crossings of the corresponding singular link diagram (of order n). The only thing it depends on is the combinatorial structure of singular crossings on the link components.

The latter is represented by a *chord diagram* on k circles as follows (a chord diagram on one-dimensional manifolds was defined in Definition 4.2). A chord diagram consists of k free-standing oriented circles, preimages of the link components, herewith two preimages of the same singular crossing are connected by a chord.

We shall depict these chord diagrams on the plane, herewith we shall not point out orientations of circles by considering them to be oriented counterclockwise manner.

An example of a chord diagram on two circles is shown in Fig. 7.2.

Such a chord diagram is considered up to combinatorial equivalences of graphs on which there are k oriented cycles.

Thus, the symbol of an nth order Vassiliev invariant on links with k components is given by its values on chord diagrams on k circles with n chords or, more precisely, by a linear function on the linear space of these diagrams.

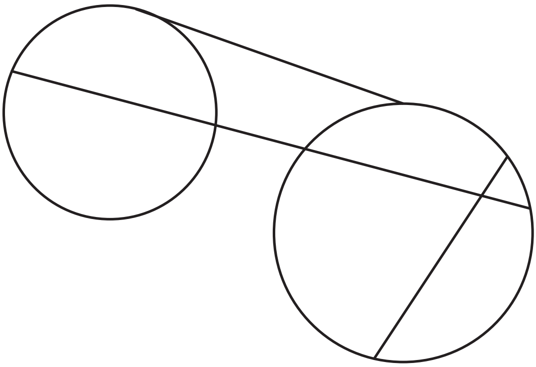


Fig. 7.2 A chord diagram on two circles.

It turns out that there are linear functions on this linear space which are not symbols of Vassiliev invariants.

Namely, each function which can play the role of a symbol has to satisfy the $1T$*-relation* (*one-term relation*) and the $4T$*-relation* (*four-term relation*).

The $1T$-relation is defined as follows. If we have a chord diagram $C =$  with any number of chords and a small solitary chord, then each symbol evaluated at this diagram equals zero. This relation follows directly from (7.1).

The $4T$-relation looks like as follows. For each symbol $v^{(n)}$ of an invariant v of order n the following relation holds:

$v^{(n)}($$) - v^{(n)}($$) - v^{(n)}($$) + v^{(n)}($$) = 0.$ (7.2)

This relation means that for any four diagrams having n chords, where $n-2$ chords (not shown in the formula (7.2)) are the same for all diagrams and the two others look as shown above, the above equality takes place. To see this relation one should consider one singular point and the two ways in which one can pass a vertical strand containing a singular point, through this point, see Fig. 7.3.

Analogously, the $4T$-relation is defined for links; in this case we have four diagrams on several circles, these four diagrams differ by a mutual position of a pair of chords, the ends of which lie in distinguished three parts (these parts, generally speaking, can belong to distinct circles).

Theorem 7.2 (Kontsevich–Vassiliev [184]**).** *A linear function f on the linear space of chord diagrams with n chords on k circles defines a symbol*

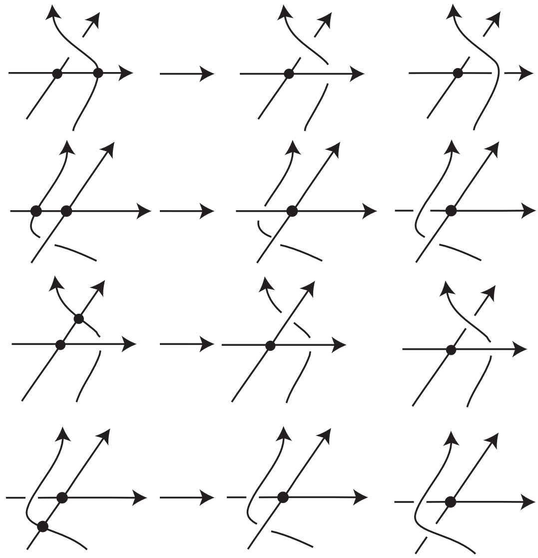


Fig. 7.3 The $4T$-relation.

of a Vassiliev invariant if and only if it satisfies the $1T$- *and* $4T$*-relations.*

This theorem gives a full description of the space $\mathcal{V}_n/\mathcal{V}_{n-1}$. There is an isomorphism $\mathcal{V}_n/\mathcal{V}_{n-1} \cong \mathcal{A}_n$, where $\mathcal{A}_n$ is the linear space of chord diagrams with n chords factored by the $1T$- and $4T$-relations.

Thus, any linear function on chord diagrams with a fixed number of circles, which satisfies the $1T$- and $4T$-relations, defines the symbol of a Vassiliev invariant.

The consideration of linear functions on chord diagrams which satisfy the $4T$-relation, but they possibly do not satisfy the $1T$-relation, is also important. These functions are called weight systems (see, e.g. [17]).

Sometimes by a *weight system* one means a function satisfying also the $1T$-relation. Any such weight system is the symbol of a Vassiliev invariant.

A weight system (without the $1T$-relation) is connected with finite-order invariants of *framed links* with even framing.

There is a theory parallel to the theory of Vassiliev invariants of knots, i.e. a theory of J-invariants of finite order of plane generic immersed curves. An idea of considering J-invariants belongs to Arnold [10, 11], and for more details see [190].

Note that before the paper by Lando [190] there were J-invariants corresponding to resolutions of direct self-tangency points (so-called J^+-invariants) and J-invariants corresponding to resolutions of inverse self-tangency points (J^--invariants). Goryunov [112] gave a complete description of J^+-invariants of finite order; he noted also that J^--invariants admitted the similar description.

In the given case curves immersed in the plane without self-tangency points are an analogue of knots, and sets of curves having a finite number of self-tangency points are an analogue of singular links. The combinatorial location of self-tangency points is also given by a chord diagram, see Fig. 7.4.

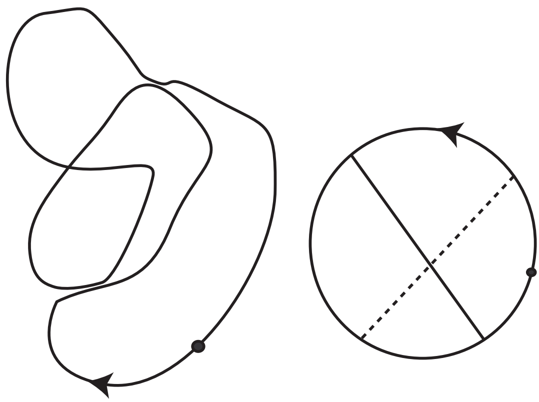


Fig. 7.4 A diagram with self-tangency points and the framed chord diagram corresponding to it.

Herewith there are essential different types of self-tangency points: *direct* and *inverse*, see Fig. 7.5.

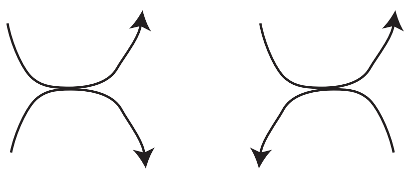


Fig. 7.5 Direct and inverse self-tangency points.

Let us set the sign plus to direct self-tangency points, and the sign minus to inverse self-tangency points. We shall also denote inverse self-tangency points by thick chords, and direct self-tangency points by dashed chords.

Therefore, for studying finite-order invariants of plane curves one should consider framed chord diagrams with two types of chords: *chords with framing* 0 (denoted by a thick line) and *chords with framing* 1 (denoted by a dashed line) (see Chap. 4).

As well as in the case of Vassiliev invariants, the value of the derivative of an invariant for a curve with a tangency point is defined by the difference of the values of the invariant at curves close to the initial one (see Fig. 7.6). The symbol of an invariant of nth order is analogously defined: It is a value of its nth derivative. This value is a weight function on framed chord diagrams. Functions on framed chord diagrams originating from J-invariants of finite order of plane curves have to satisfy some relations, called *generalized* $4T$*-relations*, see Fig. 7.7.

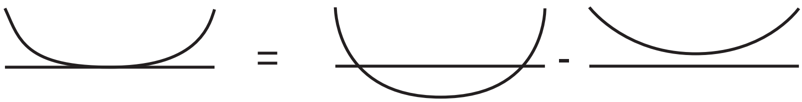


Fig. 7.6 Solution of a singularity for curves.

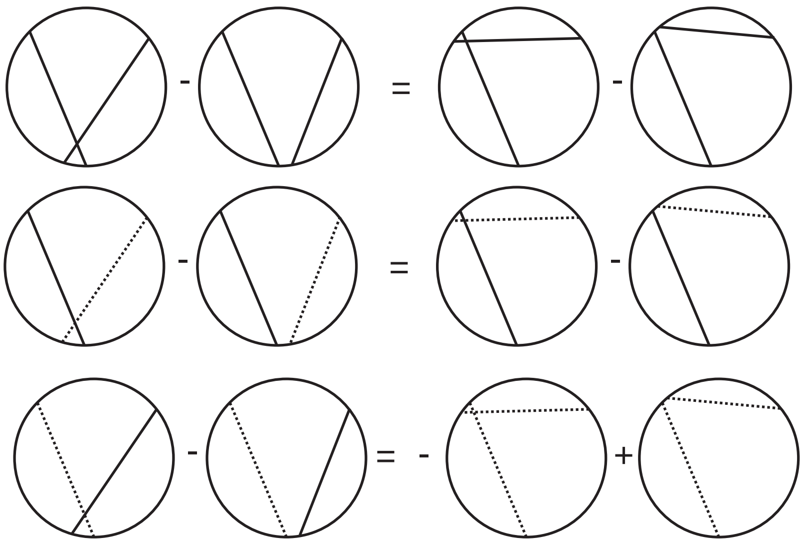


Fig. 7.7 Generalized $4T$-relation.

The generalized $4T$-relation consists of four framed chord diagrams having all chords, except two of them, being situated in the same manner, and the two chords are situated as it is shown in the $4T$-relation. Note that one of these two chords can be considered to be "immovable". Its framing and location are the same at each of the four diagrams taking part in the relation (this chord is shown in diagrams in Fig. 7.7 by an inclined line).

Herewith there are two essential different cases. In the first of them (the usual $4T$-relation without dashed chords is a particular case) the immovable chord is thick (with framing 0). In this case the movable chord has the same type in each of the four chord diagrams (it is either thick or dashed, see upper and middle parts of Fig. 7.7). In the second case the immovable chord is dashed. Under passing from the right part to the left part the movable chord changes its type, and all expressions change the sign, see the lower picture in Fig. 7.7.

Let us give more accurate definitions. We consider generic immersions of a circle in the plane up to an isotopy of the plane containing the curve, diffeomorphisms of the circle and passages of the curve through triple points. Invariants of curves under such conditions are called *J-invariants*.

Under an arbitrary deformation of the curve, besides a permitted deformation which allows the curve to pass triple points, there exists one more deformation which allows the curve to pass through a tangency point. This feature is analogous to the situation of passing through a singular crossing.

This is a starting point for defining J-invariants of finite order of plane curves. Namely, let us consider the set of all plane curves up to isotopies and passings through triple points. Consider *singular curves*, i.e. curves with finite number of tangency points. In this case one can define an equivalence relation allowing a curve to have tangency points on the set of singular curves, see Fig. 7.6.

Herewith each J-invariant of plane curves is extended to some function on curves having tangency points. A J-invariant is called an *invariant of order n* if its value on curves with $n+1$ tangency points equals zero.

A location of tangency points on a diagram is described by a framed chord diagram.

Analogously to the case of Vassiliev's invariants of knots, a *generalized weight function* on framed chord diagrams, i.e. a function satisfying the generalized $4T$-relation, is constructed for J-invariants of order n of plane curves.

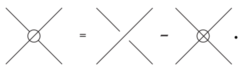
Theorem 7.3 (Lando [190]**).** *Any function on framed chord diagrams of* *order n, obtained from a J-invariant of order not bigger than n, satisfies the generalized $4T$-relation (to all versions).*

It is not known yet whether an analogous Kontsevich theorem is true for J-invariants of curves, i.e. whether it is true that for any generalized weight function one can restore a J-invariant of finite order. For more details see [190].

Thus, finding a weight system for chord diagrams or framed chord diagrams is an important problem. In the sequel we shall show that some particular solutions of these problems appear very unexpectedly by studying the Kauffman bracket polynomial of classical and virtual links, as well as by encoding knots with atoms and d-diagrams.

7.3 The Goussarov–Polyak–Viro approach to a definition of Vassiliev invariant for virtual knots

First, we shall give the definitions we are going to work with, see also [114]. We introduce the *semivirtual crossing*. This crossing still has an overpass and an underpass. In a diagram, a semivirtual crossing is shown as a classical one but encircled. Semivirtual crossings are related to the crossings of other types by the following formal relation:



Let K be a virtual knot diagram with n classical crossings, and let $\{v_1, v_2, \ldots, v_n\}$ be different classical crossings of it. For an n-tuple $\{\sigma_1, \ldots, \sigma_n\}$ of zeros and ones, let K_σ be the diagram obtained from K by switching all v_i's with $\sigma_i = 1$ to virtual crossings. Denote by $|\sigma|$ the number of ones in σ. The formal alternating sum

$$\sum_\sigma (-1)^{|\sigma|} K_\sigma$$

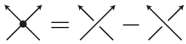
is called a *diagram with n semivirtual crossings.*

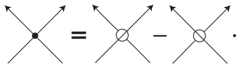
Denote by $\mathcal{K}$ the set of all virtual knots. Let $\nu\colon \mathcal{K} \to G$ be an invariant of virtual knots with values in an abelian group G. Extend this invariant to $\mathbb{Z}[\mathcal{K}]$ linearly.

The next definition is due to Goussarov, Polyak and Viro.

Definition 7.4. We say that ν is an *invariant of finite type* or a *finite-type invariant* if for some $n \in \mathbb{N}$, it vanishes on any virtual knot with more than

n semivirtual crossings, see [114]. The minimal such n is called *the degree* of the invariant ν.

The formal Vassiliev relation in the form  together with the relation defining a virtual crossing implies the relation



Remark 7.3. Note that singular knots are not considered here as independent objects having geometrical sense of knots with some singularities, but just as linear combinations of simpler objects.

It is obvious that for any finite-type invariant ν of the virtual theory in the sense described above, its restriction for the case of classical knots is a finite-type invariant in the ordinary sense.

However, not every classical finite-type invariant can be extended to a finite-type invariant in this virtual sense. For instance, there are no invariants of order two for (compact) virtual knots, see, for more details, [114, 259].

Starting from the formal relation defining a virtual crossing and the Vassiliev relation, Polyak constructed the Polyak algebra [114] that gave formulae for all finite-type invariants of virtual knots. Besides this, they give explicit diagrammatic formulae for some of them and also construct some finite-type invariants *for long virtual knots*.

Following Goussarov, Polyak and Viro [114], let us describe diagrammatic formulae for classical long knots. We need some definitions and constructions.

For a classical (virtual) long knot diagram we can construct the Gauss diagram, i.e. the line parametrizing the knot together with signed arrows connecting the preimages of each classical crossing.

Definition 7.5. An *arrow diagram* (on a circle) is an abstract diagram, which consists of an oriented circle with pairs of distinct points connected by dashed arrows. Each arrow is equipped with a sign. The *group of arrow diagrams* $\mathcal{A}$ is the free abelian group generated by all arrow diagrams.

Denote the set of all Gauss diagrams (non-realizable diagrams are allowed) by $\mathcal{D}$ (here all diagrams have thick arrows). Starting from any Gauss diagram we get an arrow diagram just by making all its arrows dashed. The extension of this map to $\mathbb{Z}[\mathcal{D}]$ defines a natural isomorphism $i\colon \mathbb{Z}[\mathcal{D}] \to \mathcal{A}$.

There is another important map $I\colon \mathcal{D} \to \mathcal{A}$, assigning to a Gauss diagram D the sum of all its subdiagrams and then making each of them dashed:

$$I(D) = \sum_{D' \subset D} i(D')$$

(here D' is a subdiagram of D if all the arrows of D' belong to D, and we write $D' \subset D$). Extend I to $\mathbb{Z}[\mathcal{D}]$ by linearity.

The following proposition is left to the reader as an exercise.

Proposition 7.1 ([114])**.** *There exists the inverse map $I^{-1}\colon \mathcal{A} \to \mathbb{Z}[\mathcal{D}]$ which is defined on the generators of $\mathcal{A}$ by the formula:*

$$I^{-1}(A) = \sum_{A' \subset A} (-1)^{|A-A'|} i^{-1}(A'),$$

where $|A - A'|$ is the number of arrows of A which do not belong to A'. Therefore, $I\colon \mathbb{Z}[\mathcal{D}] \to \mathcal{A}$ is an isomorphism.

Since the group $\mathcal{A}$ has a distinguished basis, consisting of arrow diagrams, there is a natural orthonormal scalar product $(\cdot,\cdot)$ on $\mathcal{A}$. Namely, on the generators of $\mathcal{A}$ we put (A_1, A_2) to be 1, if $A_1 = A_2$, and 0 otherwise, and then extend $(\cdot,\cdot)$ bilinearly. This allows us to define the pairing $\langle\cdot,\cdot\rangle\colon \mathcal{A} \times \mathcal{D} \to \mathbb{Z}$ by putting

$$\langle A, D\rangle = (A, I(D))$$

for any $D \in \mathcal{D}$ and $A \in \mathcal{A}$. Informally speaking, we count subdiagrams of D with weights, where the weight of a diagram D' is the coefficient of $i(D')$ in A.

Let us consider the case of (classical) long knots. The following theorem shows that any Vassiliev invariant can be calculated as a function of arrow polynomials evaluated on the knot diagram.

Theorem 7.4 (Goussarov et al. [114]). *Let G be an abelian group, and let ν be a G-valued invariant of degree n of classical long knots. Then there exists a function $\pi\colon \mathcal{A} \to G$ such that $\nu = \pi \circ I$ and π vanishes on any arrow diagram with more than n arrows.*

We immediately get the following corollary.

Corollary 7.1 ([114])**.** *Any integer-valued finite-type invariant of degree n of classical long knots can be presented as $\langle A, \cdot\rangle$, where A is a linear combination of arrow diagrams on a line with at most n arrows.*

Let us prove Theorem 7.4 by following along the lines of [114, Theorem 3.A]. Consider *long virtual singular knot diagrams*, i.e. we have three types of crossings. We equip each double point with a sign as follows. The branches at a double point are ordered and the sign is the intersection number of the branches (taken in this order).

On the Gauss diagram of a long singular knot, each double point is shown by a dashed chord equipped with the above sign.

Definition 7.6. A diagram D' is called a *subdiagram* of a diagram D if D' consists of all the chords and some arrows of D.

Definition 7.7. A diagram of a classical long knot is *descending* if when going along the knot in the positive direction we pass first along overcrossing and then undercrossing. In terms of Gauss diagrams it means that all the arrows are directed to the right.

Let us now extend this notion to virtual long knots with double points. We still require that all the arrows are directed to the right. There is also an additional condition: There is no chord whose left endpoint neighbors with an endpoint of an arrow from left. In Fig. 7.8 forbidden situations are shown.

Fig. 7.8 Forbidden situations.

It is not difficult to see that a classical long knot with a Gauss diagram of this type can be presented by a diagram such that

(1) all the double points are in the left half-plane,
(2) all the crossings are in the right half-plane,
(3) the intersection of the diagram with the left half-plane is an embedded tree,
(4) the intersection with the right half-plane is an ordered collection of arcs; each of them is descending and lies below all the previous ones.

An example of such a diagram is given in Fig. 7.9.

It is easy to see that the chord part of the Gauss diagram of a descending classical long knot diagram with singular crossings determines the isotopy

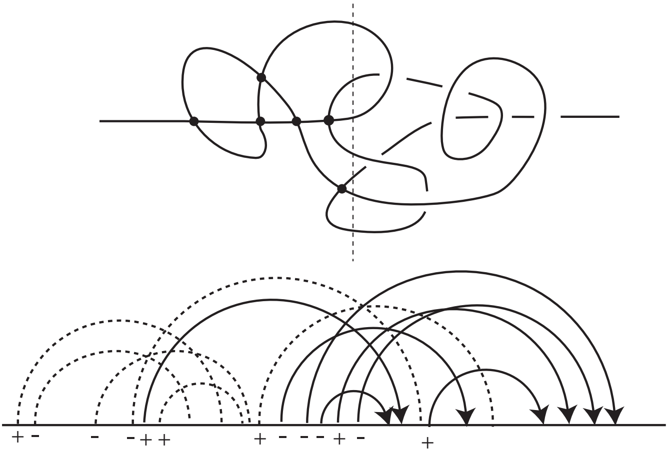


Fig. 7.9 A descending classical long knot diagram and its Gauss diagram.

class of the classical long knot completely. As a result, we get the following lemma.

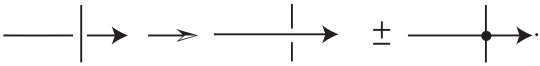
Lemma 7.1 ([114]). *Let D_1 and D_2 be Gauss diagrams of descending classical long knots* (*with singular crossings*), *and let ν be an invariant of long knots. If the chord parts of D_1 and D_2 coincide*, *then $\nu(D_1) = \nu(D_2)$.*

Remark 7.4. Lemma 7.1 is not true for descending virtual long knots with singular crossings. Namely, we cannot determine a virtual long knot with singular crossings by just knowing the chord part of the Gauss diagram of its descending diagram. Therefore, the proof of the Goussarov theorem given above cannot be straightforwardly generalized for the virtual case.

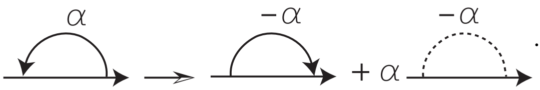
The next step of the proof is to show that the Gauss diagram of a classical long knot with double points can be represented as a linear combination of descending diagrams. There is an algorithm allowing us to do this. This algorithm consists of steps of two types. At each step, one inspects the Gauss diagram from the left to the right looking for the first fragment where the diagram fails to be descending. Such a fragment may either be a bad arrow or a bad chord.

Definition 7.8. An arrow is *bad* if it is directed to the left and a bad chord is depicted in Fig. 7.8.

In the case of a bad arrow the step of the algorithm is the replacement of the diagram with the sum of two diagrams according to the formula



In terms of Gauss diagrams this replacement is as follows:



In the case of a bad chord the step of the algorithm is the pulling of the crossing over or under the appropriate branch by isotopy, see Fig. 7.10 (knot diagrams) and Fig. 7.11 (Gauss diagrams).

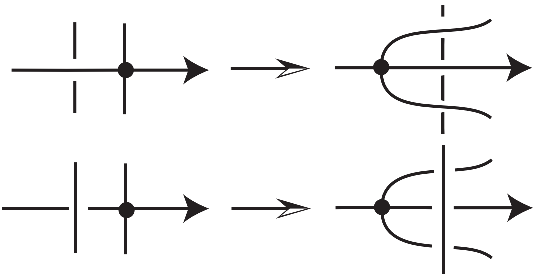


Fig. 7.10 The case of a bad chord.

Denote by $\mathcal{D}_n$ the free abelian group generated by Gauss diagrams of virtual long singular knots with at most n chords (note that $\mathbb{Z}[\mathcal{D}] = \mathcal{D}_0 \subset \mathcal{D}_n$). We shall think of a step of the algorithm as an operator acting on $\mathcal{D}_n$. Denote this operator by P. By the definition of P, for any descending Gauss diagram D we have $P(D) = D$.

Lemma 7.2 ([114]). *For any diagram $D \in \mathcal{D}_n$ there exists m such that $P^m(D)$ is a sum of descending diagrams.*

This lemma can be proved by considering the number $l(D)$ of chords of D which have one of the endpoints to the left of the first bad fragment. It is not easy to see that this number does not decrease after applying the operator P, and the number of such chords in a non-descending diagram is at most n. Recall that we deal with an invariant of degree n, the diagrams with more than n chords are disregarded. Thus, when one applies a step of the algorithm to a bad arrow in a diagram with n chords, the summand

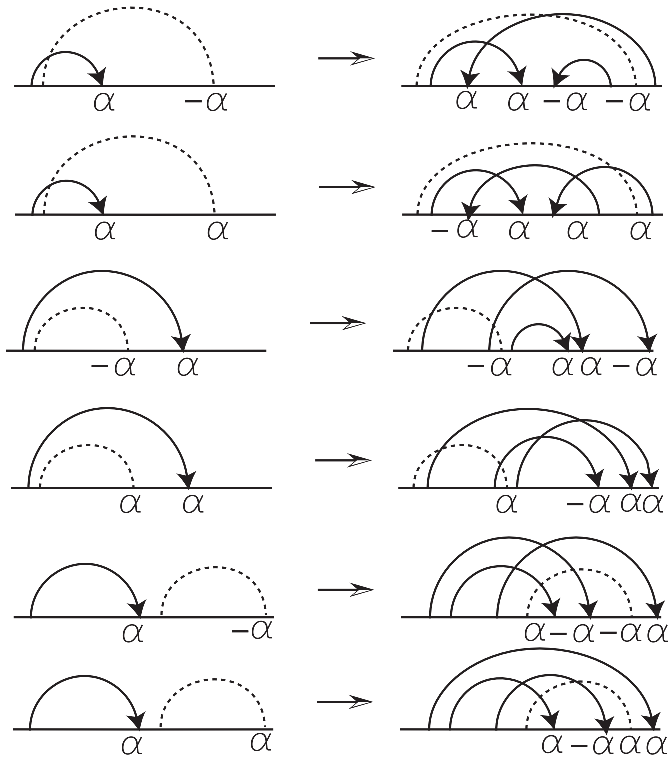


Fig. 7.11 The case of a bad chord in terms of Gauss diagrams.

with $n+1$ chords disappears. To complete the proof of the lemma one can show that the diagram cannot change infinitely many times in subsequent iterations of P without changing l.

Let us extend an invariant ν of degree at most n to all virtual knot diagrams.

Denote by $\mathcal{D}_n^{re}$ the subgroup of $\mathcal{D}_n$ generated by Gauss diagrams of *classical* long singular knots. Any finite-type invariant of classical knots of degree at most n extends to $\mathcal{D}_n^{re}$ by linearity. The next lemma is obvious.

Lemma 7.3 ([114]). *The operator $P\colon \mathcal{D}_n \to \mathcal{D}_n$ preserves $\mathcal{D}_n^{re}$. The restriction of P to $\mathcal{D}_n^{re}$ preserves any invariant of degree at most n.*

Let us first consider virtual descending long knot diagrams. By turning all the virtual crossings of such a diagram D into appropriate classical ones, we get a descending long classical knot diagram D^{re} with the same double points. Put $\nu(D) = \nu(D^{re})$ (note that this operation is not well defined on virtual knots, it is defined just on diagrams). By Lemma 7.1, $\nu(D^{re})$ does not depend on the choice of D^{re} for classical knots.

Using Lemma 7.2, for any virtual diagram D we can find a natural number m such that $P^m(D)$ is a sum of descending diagrams. Put $\nu(D) = \nu(P^m(D))$. Lemma 7.3 implies that for classical diagrams this definition coincides with the initial one. Since $P^{m+1}(D) = P^m(D)$ we get

Lemma 7.4 ([114]). *The operator* $P\colon \mathcal{D}_n \to \mathcal{D}_n$ *preserves* ν, *i.e.* $\nu \circ P = \nu$.

Let us construct the map $\pi\colon \mathcal{A} \to G$ by defining it as the composition

$$\mathcal{A} \xrightarrow{I^{-1}} \mathbb{Z}[\mathcal{D}] \subset \mathcal{D}_n \xrightarrow{\nu} G.$$

Then for any diagram D of a classical long knot we have

$$\nu(D) = \pi(I(D)) = \sum_{D' \subset D} \pi(i(D')).$$

In order to prove the Goussarov theorem, we must show that $\pi(A) = 0$ for any arrow diagram A with more than n arrows.

Denote by $\mathcal{A}_n$ the free abelian group generated by diagrams on the line containing signed dashed arrows and at most n dashed chords. The maps i, $I\colon \mathbb{Z}[\mathcal{D}] \to \mathcal{A}$ defined on Gauss diagrams without dashed chords extend to isomorphisms i, $I\colon \mathcal{D}_n \to \mathcal{A}_n$ (the chord parts of the diagrams remain untouched under both i and I).

Let us now define an operator $Q\colon \mathcal{A}_n \to \mathcal{A}_n$, which is an analogue of the operator P.

Definition 7.9. A diagram $A \in \mathcal{A}_n$ is called *descending* if $i^{-1}(A)$ is descending. A fragment of A is called *bad* if the corresponding fragment of $i^{-1}(A)$ is bad.

Put $Q(A) = A$ if A is descending. Otherwise, find the leftmost bad fragment of A. If it is a bad arrow, we define $Q(A) = iPi^{-1}(A)$. If it is a bad chord, put $Q(A) = \sum A'$ where the sum runs over all the subdiagrams of $iPi^{-1}(A)$, each of which contains all the arrows not shown in Fig. 7.11, all the chords and at least one more arrow. In other words, we sum up all seven subdiagrams of $iPi^{-1}(A)$ which contain all the arrows and chords also belonging to A plus at least one more arrow. Here we need that the number of arrows is not decreased by Q, though this map is not invariant.

The next lemma is obvious.

Lemma 7.5 ([114]). *For any diagram $A \in \mathcal{A}_n$, the total number of arrows and chords in each diagram appearing in $Q(A)$ is at least the total number of arrows and chords in A.*

The following lemma is analogous to Lemma 7.2.

Lemma 7.6 ([114]). *For any diagram $A \in \mathcal{A}_n$, there exists m such that $Q^m(A)$ is a sum of descending diagrams.*

Lemma 7.7 ([114]). *For any non-descending diagram $D \in \mathcal{D}_n$, there is a splitting $I(D) = U + V$ with $U, V \in \mathcal{A}_n$ such that*

$$I(P(D)) = Q(U) + V \tag{7.3}$$

and $U = i(D) + U'$, where U' is a sum of diagrams each of which has fewer arrows than D.

Proof. Let U be the sum of all the subdiagrams of $i(D)$ which include the first bad fragment of $i(D)$. These subdiagrams contain the same bad fragment as the whole diagram $i(D)$. Here $Q(U)$ is the sum of all subdiagrams of diagrams in $iP(D)$ which are not subdiagrams of $i(D)$. Then V is the sum of the subdiagrams of $i(D)$ which do not contain the arrow from the bad fragment and these subdiagrams of $i(D)$ remain unchanged, when one applies P to D. Thus $I(P(D)) = Q(U) + V$. □

Though the operator Q is not invariant under the Reidemeister moves (sometimes we remove one term from the summation), the following lemma holds.

Lemma 7.8 ([114]). *The operator $Q\colon \mathcal{A}_n \to \mathcal{A}_n$ preserves π, i.e. $\pi \circ Q = \pi$.*

Proof. Let $A \in \mathcal{A}_n$ be a diagram and $D = i^{-1}(A)$. Let us prove that $\pi(Q(A)) = \pi(A)$ by induction on the number of arrows in A.

The induction base. If this number equals 0, then A is descending and $Q(A) = A$ by definition of Q.

The induction step. Suppose inductively that the statement is correct for any diagram whose number of arrows is less than the number of arrows in A, and let us prove the statement for A. Apply π to the equality (7.3):

$$\pi \circ Q(U) + \pi(V) = \pi \circ I \circ P(D) = \nu \circ P(D).$$

Since the operator P preserves ν, we get

$$\nu \circ P(D) = \nu(D) = \pi \circ I(D) = \pi(U) + \pi(V).$$

Thus $\pi \circ Q(U) = \pi(U)$. By the induction assumption, $\pi \circ Q(U') = \pi(U')$, where $U' = U - A$, and we obtain the desired equality $\pi(Q(A)) = \pi(A)$. This completes the induction step. □

Lemma 7.9 ([114]). *Let $A \in \mathcal{A}_n$ be a descending diagram such that the total number of arrows and chords in A is greater than n. Then $\pi(A) = 0$.*

Proof. Let $D = i^{-1}(A)$. We have

$$\pi(A) = \nu \circ R(A) = \sum_{D' \subset D} (-1)^{|D-D'|} \nu(D').$$

Since any subdiagram D' of D is descending and has the same chord part, $\nu(D') = \nu(D)$ by the construction of ν. Therefore,

$$\pi(A) = \left(\sum_{D' \subset D} (-1)^{|D-D'|} \right) \nu(D).$$

As one can easily check by induction on the number of arrows in A, the sum in parentheses is equal to 1 if A has no arrows and is 0 otherwise. Since all the diagrams in $\mathcal{A}_n$ have at most n chords and the total number of arrows and chords in A is greater than n, it has at least one arrow. Hence $\pi(A) = 0$. □

Lemma 7.10 ([114]). *Let $A \in \mathcal{A}_n$ be a diagram such that the total number of arrows and chords in A is greater than n. Then $\pi(A) = 0$.*

Proof. Let m be the number which exists for A by Lemma 7.6. By Lemma 7.8, $\pi(A) = \pi(Q^m(A))$. By Lemma 7.5, the expansion of $Q^m(A)$ contains only descending diagrams with the total number of chords and arrows greater than n. Then by Lemma 7.9, $\pi(A) = 0$. □

The last lemma completes the proof of the Goussarov theorem.

7.4 Vassiliev invariants of virtual knots (due to Kauffman)

7.4.1 *Main definitions*

Kauffman starts from the formal definition of a singular virtual knot (link).

Definition 7.10. A *singular virtual link diagram* is a framed 4-graph in the plane endowed with orientations of unicursal curves and crossing structure: each crossing should be either classical, virtual or singular.

A singular crossing is depicted by a thick point.

Definition 7.11. A *singular virtual link* (*knot*) is an equivalence class of singular virtual link (knot) diagrams by generalized Reidemeister moves and *rigid vertex isotopy*, shown in Fig. 7.12.

Remark 7.5. All rigid isotopies except the last one, correspond to moves of three-dimensional space.

Definition 7.12. A *degree* of a singular virtual link is the number of its singular crossings.

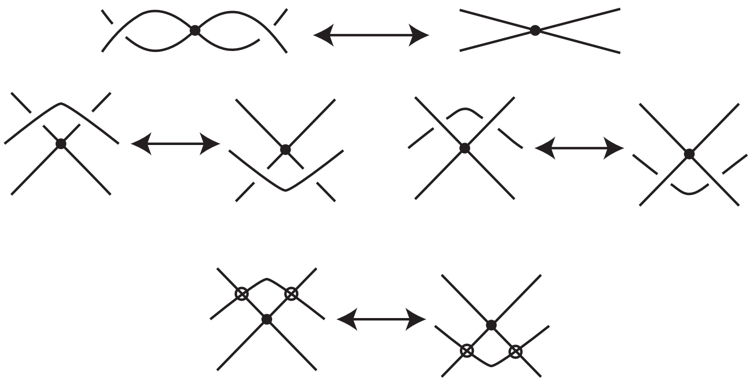


Fig. 7.12 Rigid vertex isotopy moves for virtual knots.

Now, the definition of Vassiliev's knot invariants is literally the same as in the classical case. For each invariant f of virtual links, one defines its formal derivatives f', f'', ..., by Vassiliev's rule

$$f^{(n)}(\diagup\!\!\!\!\bullet\!\!\!\!\diagdown) = f^{(n-1)}(\diagup\!\!\!\!\diagdown) - f^{(n-1)}(\diagdown\!\!\!\!\diagup)$$

and says that the invariant f has order less than or equal to n if $f^{(n+1)} \equiv 0$.

As it was noted in Chap. 1 the space of plane virtual knots is dual to the space of Vassiliev invariants of order zero.

7.4.2 *Invariants generated by the polynomial* Ξ

Two main constructions used in the definition (4.3) of the polynomial Ξ are the Jones–Kauffman polynomial and Vassiliev invariants of order zero.

Let us show that the invariant Ξ is stronger than the Jones–Kauffman polynomial and Vassiliev invariants of order zero, considered together.

Let us denote the diagram shown in Fig. 7.13 by K.

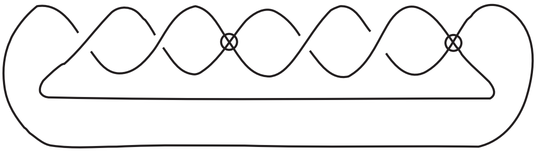


Fig. 7.13 The link diagram K.

It is obvious that the link generated by K has the same invariants of order zero as the trivial link with two components: By replacing types of classical crossings in two places and applying the move Ω_2 twice, we get the two-component link without classical crossings.

It is easy to check that the Jones–Kauffman polynomial of this link is the same as for the trivial two-component link. This trivial link is obtained from the diagram K by applying the virtualization twice and the detour move.

But the polynomial Ξ allows one to show that the link is not classical.

It is easy to see that all elements from the set S (see Sec. 4.3.2) obtained from the diagram K lie on the torus T^2. Indeed, the set of curves δ has the form shown in Fig. 7.14.

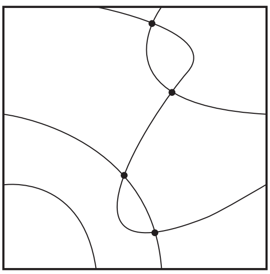


Fig. 7.14 The system of curves δ.

Two curves from δ on the torus are parallel.

Herewith for some states s the set $p(s)$ contains non-trivial curves not being parallel to curves from δ. It is not easy to check that the corresponding element $p \in \mathcal{S}$ has a non-zero coefficient in the decomposition (4.3). From this it follows that $\Xi(K)$ has a non-trivial coefficient at some elements from $\mathcal{S}$ (see Sec. 4.3.2) distinct from p.

Non-triviality of the link was proved by the first-named author in the year 2003, [217].

7.5 Vassiliev's invariants coming from the invariant Ξ

In this section we work with Vassiliev invariants due to Kauffman [217, 225].

The invariant Ξ (see (4.3)), whose coefficients at elements of $\mathcal{S}$ are Laurent polynomials in variable a, can be transformed into formal series after the variable change $a = e^x$ and the Taylor expansion. For the sake of simplicity, we shall use the same letter Ξ to denote the obtained invariant where all coefficients are series in x.

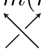
Let K be a virtual link diagram. Let $\Xi_m(K)$ be the coefficient of $\Xi_m(K)$ at x^m. It is just a linear combination of elements from $\mathcal{S}$ with rational coefficients.

Let us prove the following theorem.

Theorem 7.5. *For each $m \in \mathbb{N}_+$ the invariant Ξ_m is a Vassiliev invariant of virtual knots of order less than or equal to m.*

This theorem shows that the invariant Ξ is indeed weaker than Vassiliev's knot invariants of virtual links: For each element $W \in \mathcal{S}$, the coefficient of Ξ at W can be represented as a formal series of Vassiliev's invariants. On the other hand, the theorem allows one to derive infinitely many finite-type invariants of higher orders.

Remark 7.6. For the sake of simplicity, we shall denote oriented and non-oriented diagrams by the same letter. We shall write K instead of $|K|$.

Let us prove now Theorem 7.5. To do this, let us consider 2^n virtual link diagrams which differ only at n selected crossings. These diagrams represent all possible combinations of positive and negative classical crossings. Denote these diagrams by K_η, where $\eta \in \{0,1\}^n$. Let $m(\eta)$ be the number of ones in η (they correspond to negative crossings , whence zeros correspond to positive crossings ).

Now, we have to prove that

$$\sum_{\eta}(-1)^{m(\eta)}\Xi_{n-1}(K_\eta) = 0. \tag{7.4}$$

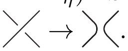
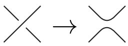
The first observation is that the manifold M is the same for all K_η by construction of the invariant Ξ (see Sec. 4.3.2). Moreover, by construction for each K_η we obtain the same family of elements $\delta(K_\eta)$. Besides, the set of possible $\delta'(s)$ considered under all distinct s (and, respectively, the set of all $p(s)$) is the same for all these diagrams, too. The only difference in the sum (4.3) for determining $\Xi(K_\eta)$ for different η's appears because coefficients at fixed $p(s) \in \mathcal{S}$ are different.

Now, we are going to prove that the corresponding difference of coefficients vanish in the formula (7.4). To do this, it is sufficient to prove that the corresponding coefficients (at a fixed $p(s)$) in

$$\sum_{\eta}(-1)^{m(\eta)}\Xi(K_\eta) = 0$$

are divisible by $(a-1)^n$ (that implies the divisibility by x^n).

So, let K_+ be the diagram K_η with all n positive crossings. Let us fix some state s of K_+. Consider the element $p(s)$. For each η, there naturally exists a state $s(\eta)$ of K_η with the same $p(s)$. Accidentally, there might be more states $\tilde{s}$ with $p(\tilde{s})$ equivalent to $p(s)$ in $\mathcal{S}$. But we shall work only with the "natural" ones, i.e. those obtained by "the same" smoothing as s for K_+.

We wish to calculate the coefficient at $p(s)$ (the alternating sum of 2^n coefficients corresponding to the diagrams K_η). Suppose in the state s of K_+ we have m crossings in position A: . Thus, the other $n-m$ crossings smoothed by B: . The other crossings (distinct from chosen n crossings) have the same state for each diagram. Suppose the sum of signs at the remaining classical crossings (which is the same for all η) is equal to w, the number of crossings (distinct from chosen n crossings) in position A is α and that in position B is β. Denote the number of circles in the state s of K_+ by γ.

Then the desired coefficient is

$$(-a)^{-3w}a^{(\alpha-\beta)}(-a^2-a^{-2})^{(\gamma-1)}(-a^{-2}+a^2)^m(-a^{-4}+a^4)^{n-m}.$$

The latter expression is divisible by $(a-1)^n$. All the other coefficients corresponding to different $p(s)$ are the same. This completes the proof of Theorem 7.5.

7.6 Infinity of the number of long virtual knots with a fixed closure

In this section we shall prove the following theorem by using Vassiliev invariants of order zero for virtual knots. The result given here was first proved by Silver and Williams in [281].

Theorem 7.6. *For each isotopic class of a knot K, the set of long virtual knots L such that* $\mathrm{Cl}(L) = K$ *is countable infinite.*

Proof. By having a long virtual knot L we can construct a new long virtual knot L' as shown in Fig. 7.15.

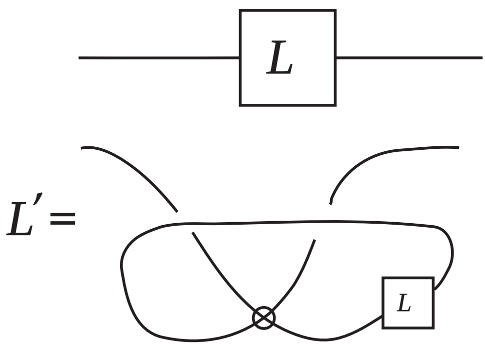


Fig. 7.15 The operation ′.

It is evident that $\mathrm{Cl}(L') = \mathrm{Cl}(L)$.

Let us consider a compact virtual knot K and a long virtual knot L obtained by breaking the knot K. Thus, we have $\mathrm{Cl}(L) = K$.

Now, let $L_0 = L$, $L_1 = (L_0)'$, $L_2 = (L_1)'$ and so on: for each natural i set $L_i = (L_{i-1})'$. It is obvious that for each natural i we have $\mathrm{Cl}(L_i) = \mathrm{Cl}(L_0) = K$.

Let us show that among the knots L_i there exist infinitely many distinct knots. For this it is sufficient to note that their *Kishino closures* are different (Kishino closure is defined and it is shown in Fig. 7.16).

It is easy to check that the Kishino closing operation is well defined: For equivalent long virtual knots we get equivalent compact knots under such closing. The fact that among Kishino closures of knots L_i there are infinite many different knots follows from the fact that these closures can have an arbitrary large preassigned genus.

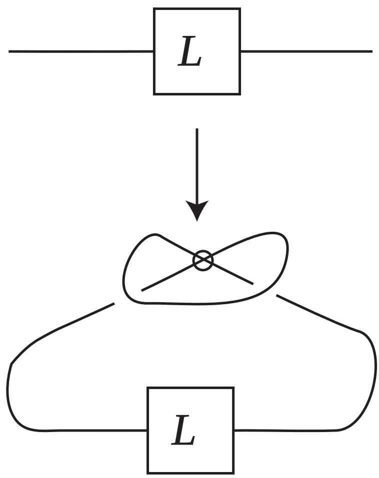


Fig. 7.16 Kishino closure.

The latter assertion follows from the fact that the corresponding virtual knots have an arbitrary large preassigned genus in the flat category, i.e. if we permit to apply the operation of swapping classical crossings besides the generalized Reidemeister moves (i.e. we shall work with virtual knots up to Vassiliev invariants of order zero). Namely, by using L_i we define the surface with a system of curves in it and we get the minimal realization (not having loops and bigons). □

7.7 Graphs, chord diagrams and the Kauffman polynomial

Let a chord diagram D with n chords be given.

Definition 7.13. The *intersection graph* (see [56]) $\Gamma(D)$ of D is a simple graph whose vertices are in one-to-one correspondence with chords of D, and two vertices are connected by an edge if and only if the corresponding chords are linked.

The *intersection graph* of a framed chord diagram D is the intersection graph of D (considered without framing), and all vertices are endowed with framing equal to the framing of the corresponding chord.

Definition 7.14. A simple graph Γ is *realizable*, if there exists a chord diagram D such that $\Gamma = \Gamma(D)$.

Note that not each graph is realizable. The criterion of realizability of graphs is given in [39].

Let a chord diagram D be given. Let us construct by using this diagram (more precisely, the atom with only black cell, for which D is an f-graph) a virtual link as described in Sec. 4.2.5. In this case, if D is a framed d-diagram, this construction can be well defined: we get a classical knot. Otherwise we can obtain an oriented (in the sense of atoms) virtual knot up to virtualization. In any case we get the object for which the Kauffman bracket polynomial (and the Khovanov complex due to Sec. 5.7) is well defined.

In the case, when a framed chord diagram is given, we can also construct a virtual link up to virtualization, see Fig. 4.2. Therefore, for the virtual diagram and the framed chord diagram, we can define the Kauffman bracket polynomial. The Kauffman bracket polynomial is also constructed for a chord diagram with several circles (by means of an atom), and it is invariant under virtualizations. Therefore, we have the map f associating with a chord diagram with one or several circles the polynomial.

It is easy to check the following theorem.

Theorem 7.7. *The map f satisfied the $4T$-relation, and also the generalized $4T$-relation.*

Thus, the *Kauffman bracket polynomial* defines infinite three-parametrical series of link invariants of finite order. The first parameter of this series is the degree of the monomial appearing in the Kauffman bracket polynomial. The second parameter is the degree of the chord diagram, and the third parameter is the number of components of the link, this number is equal to the number of circles of the chord diagram.

Note that the first part of this theorem (formulated in other terms not using virtual knots) was proved by Mellor [245] (and later, in terms given above, in [221]). Moreover, Mellor pointed out explicitly what Vassiliev invariants were obtained from this weight system; these are Vassiliev invariants originating from the Kauffman polynomial (in two variables). Thus, there is some interesting connection between the Kauffman bracket polynomial for virtual knots and the Kauffman polynomial for classical knots.

Definition 7.15. By a *surgery* of a framed chord diagram on a one-dimensional manifold M over a chord c we mean the following transformation. Let the chord c connect two points A and B. The surgery is the deletion of two small neighborhoods of A and B (not containing ends of other chords) and replace them with two segments. Namely, let us denote these neighborhoods by $(A-\varepsilon, A+\varepsilon)$ and $(B-\varepsilon, B+\varepsilon)$ due to the orientations of the circles containing the points A and B. Then as a result of the surgery instead of these two neighborhoods two segments $(A-\varepsilon, B+\varepsilon)$ and $(A+\varepsilon, B-\varepsilon)$ appear, if the chord has framing 0, and the segments $(A+\varepsilon, B+\varepsilon)$ and $(A-\varepsilon, B-\varepsilon)$, if the chord has framing 1. These surgeries appear under a passage from one state of the Kauffman bracket to a neighboring one.

Let us prove Theorem 7.7.

Proof of Theorem 7.7. Consider the following $2T$-*relations* (*two-term relations*), see Fig. 7.17, and *generalized* $2T$-*relations* (*generalized two-term relations*), see Fig. 7.18.

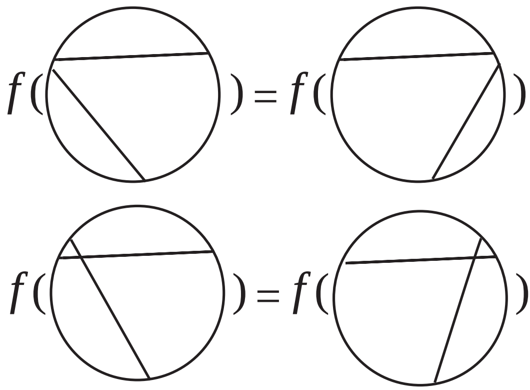


Fig. 7.17 $2T$-relations.

Each of them means the equality of values of some function on pair of diagrams differing locally as it is shown in figures.

Note that the (generalized) $2T$-relations imply the $4T$-relation. The latter decomposes as the sum of two relations.

Now, let us note that the function ϕ associating with a (framed) chord diagram the number of circles obtained by the surgery of the initial chord diagram along the set of all chords satisfies the $2T$-relations and generalized $2T$-relations. Thus, the function $(-a^2 - a^{-2})^{\phi-1}$ also satisfies the $2T$-relations.

Let us consider the function $f(D)$. It represents a linear combination

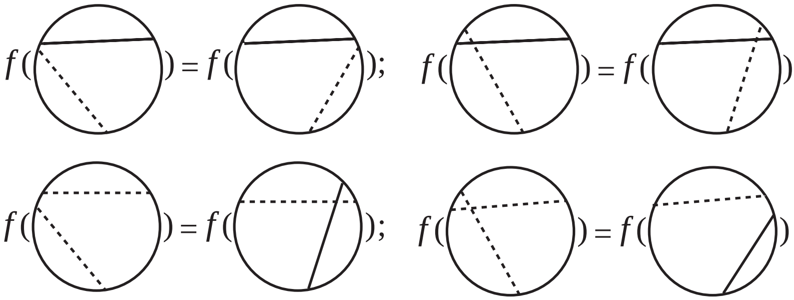


Fig. 7.18 Generalized $2T$-relations.

of values of the function $(-a^2 - a^{-2})^{\phi-1}$ at every possible subdiagrams of the chord diagram D.

Each (generalized) $4T$-relation can be written as $A_1 - B_1 - A_2 + B_2 = 0$, where $A_1 - A_2 = 0$ and $B_1 - B_2 = 0$ are the $2T$-relations, and the chord diagram A_i differs from the chord diagram B_i $(i = 1, 2)$ by only mutual position of two chords which take place in the relation.

Between chords of the diagrams A_1, B_1, A_2, B_2 there is a natural one-to-one correspondence; denote "movable" chords by α (for the diagrams A_i) and β (for the diagrams B_i).

Decompose $f(A_1 + B_1 - A_2 - B_2)$ as the sum over all subdiagrams. If a subdiagram of the diagram A_1 does not contain chords from α and β, then all four corresponding diagrams coincide identically. In the case, when a subdiagram contains exactly one chord from α or β, the subdiagram of the diagram A_i coincides with the corresponding subdiagram of the diagram B_i.

At last, in the case, when in a subdiagram there are two chords from α and β, we get four diagrams which can be split into two pairs forming the $2T$-relations.

Summing up, we get the proof of the theorem. □

7.8 Euler tours, Gauss circuits and rotating circuits

In this section we consider connected *framed* 4-graphs, i.e. 4-valent graphs with an A-structure specified and *Euler tours* on them. The main result of this section is an explicit formula connecting the adjacency matrices of *rotating circuits* and that of the *Gauss circuit*.

Each framed 4-graph G has an *Euler tour* U, i.e. a path traveling once along each edge of the graph. At each vertex we have two possibilities of running edges: we move from a half-edge to the half-edge opposite to it; we move from a half-edge to a half-edge non-opposite to it. There are two special types of Euler tours on the framed 4-graph G: In the first case only the first possibility occurs at each vertex, i.e. we move from a half-edge to the half-edge opposite to it, and in the second case only the second possibility of the passage from a half-edge to a half-edge occurs at each vertex. It is not difficult to see that the number of Euler tours of the first type on any framed 4-graph is less than or equal to 1. An Euler tour of the first type, if it exists, is called the *Gauss circuit*. Euler tours of the second type (*rotating circuits*, see [191, 224, 232, 238]) exist on any connected framed 4-graph and the number of them is more than one. If we consider a projection of a knot then it has the Gauss circuit, but a projection of a link with more than one component does not have a Gauss circuit.

In low-dimensional topology both approaches, the Gauss circuit approach and the rotating circuit approach, are very widely used. The Gauss circuit approach is applied in knot theory, namely in the construction of finite-type invariants, Vassiliev invariants [17, 56, 114], and in the planarity problem of immersed curves, see [47, 48, 270] and Secs. 7.9, 7.10 of the present chapter. However, for detecting planarity of a framed 4-graph it is more convenient to use the rotating circuit approach, see [224, 232, 238, 270] and Secs. 7.9 and 7.10. The criterion of the planarity of an immersed curve, which is a framed 4-graph, is formulated very easy: An immersed curve is planar if and only if the chord diagram obtained from a rotating circuit is a framed d-diagram [221, 270]. If we wish to extend the planarity problem to the problem of finding the minimum genus of a closed surface which a given curve can be immersed in, the rotating circuit approach is also more useful. There are criteria giving us the answer to the question what is the minimum genus for a given curve, see [232, 238] and Secs. 7.9 and 7.10.

Since there are many rotating circuits corresponding to the same Gauss circuit, then many properties of the Gauss circuit can be read out of any of these rotating circuits no matter which one is considered [129, 130]. Consequently, these properties do not depend on the particular choice of a rotating circuit. For instance, if one of the rotating circuits corresponds a framed d-diagram then the other ones do the same. Thus, it gives rise to the problem of obtaining an easy formula allowing us to get the Gauss circuit from a rotating circuit and vice versa. Of course, we can always understand whether a given framed 4-graph has a Gauss circuit or not. Traveling along our graph according to the definition of a Gauss circuit, at each vertex we have only one possibility of passing through it. Therefore, if when we return to the starting point we shall have passed every edge of our graph exactly once, then the graph has the Gauss circuit. But this method does not reflect explicit relations between topology and combinatorics of Euler tours if we have a 4-graph with many vertices.

In this section, we give an explicit formula which is expressed in terms of the adjacency matrices of Euler tours. Taking any Euler tour and constructing its adjacency matrix we can understand whether the framed 4-graph has the Gauss circuit and find the adjacency matrix of the Gauss circuit under the condition that it exists. The adjacency matrix of an Euler tour is a symmetric matrix, and there are symmetric matrices which are not adjacency matrices of any Euler tours. It turns out that the formula given below is also valid for all symmetric matrices. Investigating this formula we shall get some interesting facts about symmetric matrices.

7.8.1 4-*Graphs and Euler tours*

Let H be a connected 4-graph on the set of vertices $V(H)$, and let U be an Euler tour of H, i.e. a path traveling once along each edge of H. Let us describe a connection between two Euler tours on H.

Let us define a *k-transformation* on 4-graphs (Kotzig [185]). For every vertex $v \in V(H)$ there are precisely two closed paths P_v and Q_v on U having no common edges, starting and ending at v, and each of the paths containing at least one edge. There exists precisely one Euler tour distinct from U also connecting the paths P_v and Q_v (if we run along U in some direction, then in the new Euler tour we run along P_v according to the orientation of U, and run along Q_v according to the reverse orientation of U). Let us denote by $U * v$ the new Euler tour obtained from U. The transformation $U \mapsto U * v$ has been introduced by Kotzig [185] who called it a *k-transformation*, see Fig. 7.23.

Proposition 7.2 ([185]). *Any two Euler tours of a* 4*-graph are related by a sequence of k-transformations.*

Let $w = x_1x_2\ldots x_{k-1}x_k$ be a word, i.e. a sequence of letters from some finite alphabet X. The *mirror image of* w is the word $\widetilde{w} = x_kx_{k-1}\ldots x_2x_1$.

We shall consider the class of words: Each word from the class generated by a word $w = x_1x_2\ldots x_{k-1}x_k$ is either a cyclic permutation

$w_i = x_i x_{i+1} \ldots x_k x_1 \ldots x_{i-1}$, $1 \leqslant i \leqslant k$, of $x_1 \ldots x_k$ or the mirror image of a cyclic permutation w_i. We denote this class by $(x_1 \ldots x_k)$ and we call this class a *cyclic word*.

Definition 7.16. A word is called a *double occurrence word* if each of its letters occurs twice.

Example 7.1. The word *abccba* is a double occurrence word, but not the word *abccb*.

It is obvious that the mirror image and a cyclic permutation of a double occurrence word are double occurrence words. Then the notion of a cyclic double occurrence word is well defined. It is convenient to represent every double occurrence word by a "simple" chord diagram.

Let X be a finite set, and let m be a double occurrence cyclic word over X, i.e. a class of words. Then m has a chord diagram, which is constructed by placing successively the letters of m around a circle S^1, choosing a point of S^1 near each occurrence of a letter and joining by a chord each pair of points corresponding to the two occurrences of the same letter. It is not difficult to see that we get the one-to-one correspondence between the set of double occurrence cyclic words and the set of chord diagrams.

Example 7.2. Consider $m = (abacdbcd)$. The word m has the chord representation, which is depicted in Fig. 7.19.

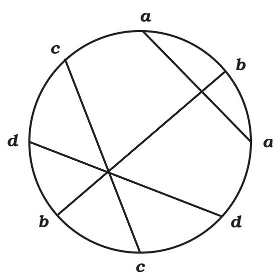


Fig. 7.19 A chord representation of (*abacdbcd*).

Define the operation $*$ on cyclic double occurrence words which will correspond to the k-transformation. Let $m = (vAvB)$ where A, B are subwords of m, and the letters belong to some finite alphabet. Then we define $m * v = (v\widetilde{A}vB)$, $\widetilde{A}$ is the mirror image of A. In Fig. 7.20 the transformation $m \mapsto m * v$ is depicted for chord diagrams (dashed arcs of chord diagrams contain the ends of all the chords distinct from v). Mostly for each transformation on a chord diagram we assume that only a fixed fragment of the chord diagram is being operated on. The pieces of the chord diagram not containing chords participating in this transformation are depicted by dashed arcs.

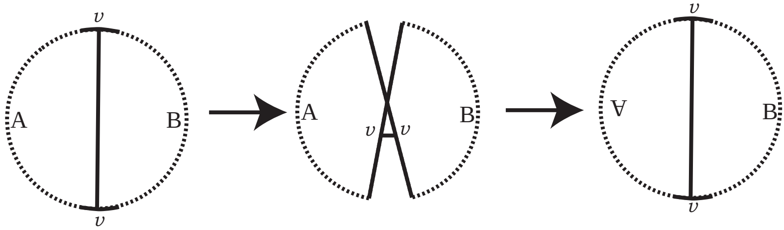


Fig. 7.20 The operation $*$ on chord diagrams.

Let U be an oriented Euler tour of a connected 4-graph H with the set of vertices $V(H) = \{v_1, \ldots, v_n\}$, which is also considered as an alphabet. When traveling along U we meet each vertex twice. Let us denote by $m(U)$ the cyclic word over $V(H)$ which equals the sequence of the vertices that are successively met along U. It is obvious that in the resulting word each vertex appears precisely twice then Euler tours are encoded by double occurrence cyclic words. It follows from the definition that $m(U * v) = m(U) * v$ and if we have a double occurrence cyclic word m or a chord diagram we can construct the 4-graph having an Euler tour U such that $m(U) = m$. We just "contract" every pair of vertices of the chord diagram labeled by a same letter (a chord) into a single vertex and identify the new vertex with this letter.

7.8.2 *Framed 4-valent graphs and Euler tours*

Let H be a framed 4-graph (see Definition 1.2), and let U be an Euler tour on H. Construct the *framed cyclic double occurrence word* $m(U)$ (the *framed chord diagram*) corresponding to U. At each vertex v of H we have the following three possibilities of running along U through v.

(1) We pass from a half-edge to the half-edge opposite to it, see Fig. 7.21(a). In this case, the vertex v is called a *Gaussian vertex for U* and the chord

corresponding to this vertex is called a *Gaussian chord*.

(2) We pass from a half-edge to a half-edge non-opposite to it, and the orientations of opposite edges are different, see Fig. 7.21(b). In this case, the vertex v is called a *non-Gaussian vertex for* U *with framing* 0 and the chord corresponding to this vertex is called a *non-Gaussian chord with framing* 0.

(3) We pass from a half-edge to a half-edge non-opposite to it, and the opposite edges have the same orientation, see Fig. 7.21(c). In this case, the vertex v is called a *non-Gaussian vertex for* U *with framing* 1 and the chord corresponding to this vertex is called a *non-Gaussian chord with framing* 1.

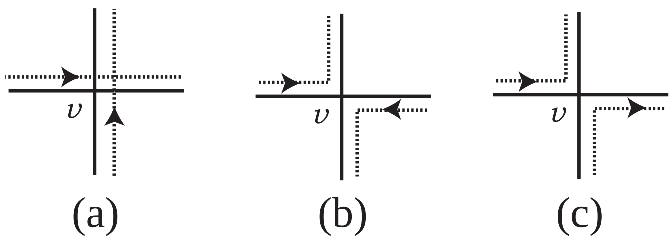


Fig. 7.21 Passing through a vertex.

Definition 7.17. An Euler tour having only Gaussian vertices is called a *Gauss circuit*. An Euler tour having only non-Gaussian vertices is called a *rotating circuit* (see [129, 130, 224, 232, 238, 270]).

Definition 7.18. Let us call a framed 4-graph having the Gauss circuit a *unicursal graph*.

When running along the Euler tour U we meet each vertex of H twice. Now we are ready to construct the *framed* cyclic double occurrence word $m(U)$ corresponding to the Euler tour U. Words will be constructed over the alphabet $X = V(H) \cup V(H)^{-1} \cup V(H)^G$, where the set $V(H)^{-1}$ consists of elements v^{-1} for each $v \in V(H)$, and $V(H)^G$ consists of elements v^G for each $v \in V(H)$. To each Gaussian vertex we assign two identical letters from the set $V(H)^G$ in $m(U)$, i.e. we attribute the superscript G to each appearance of the corresponding vertex. For example, $m(U) = (Av^G Bv^G)$ if v is a Gaussian vertex. To each non-Gaussian vertex with framing 0 we assign two identical letters from the set $V(H) \cup V(H)^{-1}$ in $m(U)$, i.e. we attribute either nothing or the superscript -1 to each appearance of the corresponding vertex (the superscripts are the same for both appearances). For example, $m(U) = (AvBv)$ or $m(U) = (Av^{-1}Bv^{-1})$ if v is a non-Gaussian vertex with framing 0 (we do not make any difference between these two words). To each non-Gaussian vertex with framing 1 we assign two distinct letters from the set $V(H) \cup V(H)^{-1}$ in $m(U)$, i.e. we attribute different superscripts to two appearances of the corresponding vertex. For example, $m(U) = (AvBv^{-1})$ or $m(U) = (Av^{-1}Bv)$ if v is a non-Gaussian vertex with framing 1 (we do not make any difference between these two words).

Thus, instead of just framed cyclic words we consider equivalence classes of framed cyclic words, where the equivalence is generated by automorphisms of the alphabet which exchange letters v and v^{-1} for some letter v. For the sake of simplicity we call these classes by just *framed cyclic words*.

Remark 7.7. The constructed framed word can be a non-double occurrence word. We can consider the projection $\pi\colon V(H) \cup V(H)^{-1} \cup V(H)^G \to V(H) \cup V(H)^G$ sending $v^{\pm 1} \mapsto v$ and $v^G \mapsto v^G$. The image (under this projection) of the word is already a double occurrence word. We call a framed word a *double occurrence word* if the image (under π) of the word is a double occurrence word (in the standard sense).

Depicting a cyclic double occurrence word by a framed chord diagram we shall use thick chords for vertices with framing 0, dashed chords for vertices with framing 1, and chords with the label G for Gaussian vertices.

Example 7.3. Consider the framed cyclic double occurrence word $m = (ab^{-1}acd^Ge^{-1}d^Gb^{-1}c^{-1}e)$. We have: d is a Gaussian letter, a, b are non-Gaussian letters with framing 0 and c, e are non-Gaussian letters with framing 1. The corresponding framed chord diagram is depicted in Fig. 7.22.

Let V be a finite set. Having a framed cyclic double occurrence word (a framed chord diagram) m over $V \cup V^{-1} \cup V^G$ we can construct the framed 4-graph having an Euler tour U such that the framed word $m(U)$ coincides with m. We construct the 4-graph and then define the type of each vertex.

Remark 7.8. When we consider a cyclic double occurrence word, it is important for us to know only the positions of two letter corresponding to a single vertex, but not their symbol, see [301].

Let us define the *framed star* operation on the set of framed cyclic double occurrence words. We denote this operation by the symbol $*$.

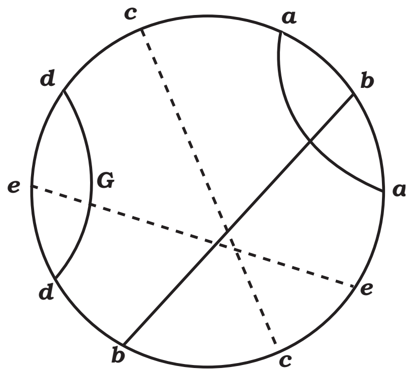


Fig. 7.22 A framed chord diagram for $(ab^{-1}acd^{G}e^{-1}d^{G}b^{-1}c^{-1}e)$.

Remark 7.9. We used the same notation for "just" double occurrence cyclic words. Further, we shall consider only framed double occurrence cyclic words and the same notations will not cause any confusion.

First, we construct the operation $\overline{w}$, where w is an arbitrary subword (not necessarily a double occurrence word) of a framed double occurrence cyclic word. Let $w = x_1^{\varepsilon_1} \dots x_k^{\varepsilon_k}$. Then $\overline{w} = \overline{x_k^{\varepsilon_k}} \dots \overline{x_1^{\varepsilon_1}}$, where $\overline{x_l^{\varepsilon_l}} = x_l^{\varepsilon_l}$ if $\varepsilon_l = G$, $\overline{a_l^{\varepsilon_l}} = a_l^{-\varepsilon_l}$ if $\varepsilon_l = \pm 1$. Further, $m = (a^{\varepsilon} m_1 a^{\varepsilon'} m_2)$ is a double occurrence cyclic word. We have $m * a = (a\overline{m_1}am_2)$ if $\varepsilon = \varepsilon' = G$ (Fig. 7.23(a)); $m * a = (a^G \overline{m_1} a^G m_2)$ if $\varepsilon = \varepsilon' \neq G$ (Fig. 7.23(a)); $m * a = (a\overline{m_1}a^{-1}m_2)$ if $\varepsilon = -\varepsilon'$ (Fig. 7.23(b)). Thus, by applying the framed star to a letter a we get: If a was a Gaussian letter, then in the new word it would be transformed to a non-Gaussian letter with framing 0; if a was a non-Gaussian letter with framing 0 (respectively, 1), then in the new word it would be transformed to a Gaussian letter (respectively, a non-Gaussian letter with the same framing 1).

Using Proposition 7.2, we immediately get the following proposition.

Proposition 7.3 ([129, 130, 232, 238]**).** *Any two framed cyclic double occurrence words obtained from a framed 4-graph are related to each other by a finite sequence of framed star operations.*

Taking an arbitrary Euler tour and applying the framed star operation we get the following proposition.

Proposition 7.4. *Every framed 4-graph has a rotating circuit.*

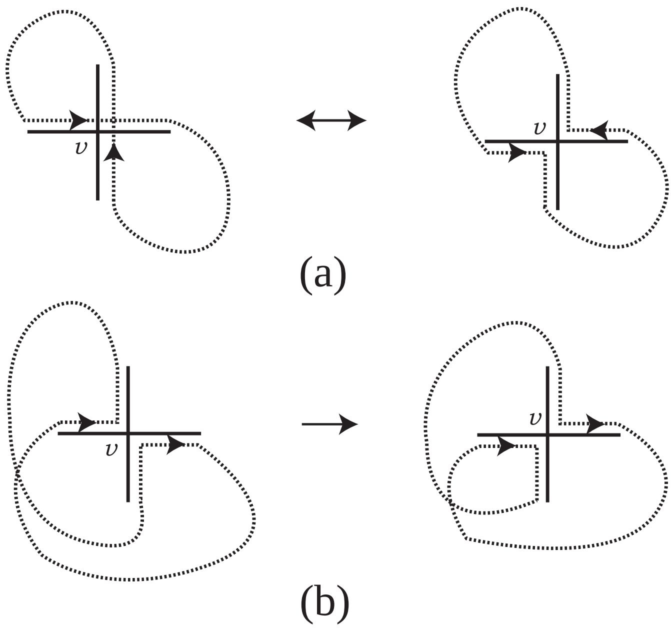


Fig. 7.23 The framed star operation.

Remark 7.10. It is not difficult to prove Proposition 7.4 using other methods, but we want to "get used" to the framed star operation.

It is obvious that there are many rotating circuits and that not every framed 4-graph has a Gauss circuit (if it has a Gauss circuit then this circuit is unique). Section 7.8.3 tells us whether or not there exists a Gauss circuit, and how to get it whenever it exists. The following theorem tells us how two rotating circuits are related.

Statement 7.1 ([129, 130, 232, 238])**.** *Any two rotating circuits given by framed cyclic double occurrence words are related by a sequence of the following two operations: The framed star operation applied to a non-Gaussian letter with framing* 1, *and* $(((m * a) * b) * a)$; *here* m *is a framed double occurrence cyclic word,* a, b *are non-Gaussian letters with framing* 0 *and they alternate in* m, *i.e.* $m = (\ldots a \ldots b \ldots a \ldots b)$.

7.8.3 *The existence of a Gauss circuit*

We need one notion for establishing whether the number of unicursal components of a connected framed 4-graph is 1, i.e. whether there exists a Gauss circuit: the *adjacency matrix* of a framed cyclic double occurrence word (a framed chord diagram).

Definition 7.19. The *adjacency matrix* of a chord diagram D with enumerated n chords is the $n \times n$ matrix $A(D) = (a_{ij})$ defined by

(1) a_{ii} is the framing of the chord with the number i, i.e. either G, or 0, or 1;
(2) $a_{ij} = 1$, $i \neq j$, if and only if the chords with the numbers i and j are linked;
(3) $a_{ij} = 0$, $i \neq j$, if and only if the chords with the numbers i and j are unlinked.

Remark 7.11. Adjacency matrices are considered over $\mathbb{Z}_2$ if we have no G on the diagonal.

Example 7.4. Let D be the framed chord diagram depicted in Fig. 7.22. Enumerate all the chords of D: the chord aa has the number 1, the chord bb has the number 2 etc. Then

$$A(D) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & G & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Assume we are given a chord diagram D with all the chords having framing 0 or 1 (with no Gaussian chords). Let us apply the surgery along a set of all chords of D as shown in Fig. 7.24 (see also Definition 7.15). By a small perturbation, the picture in $\mathbb{R}^2$ is transformed into a 1-manifold in $\mathbb{R}^3$. This manifold $M(D)$ is the *result of surgery*, see Fig. 7.25.

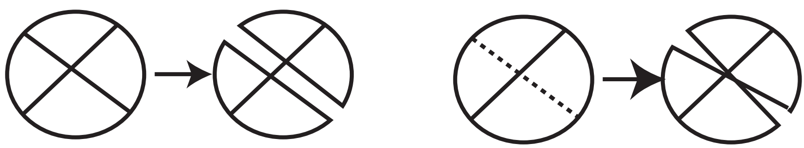


Fig. 7.24 A surgery along chords.

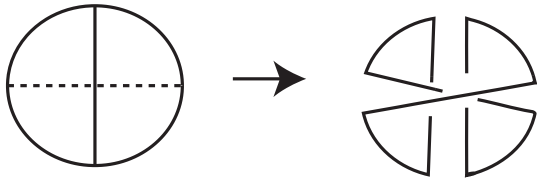


Fig. 7.25 The manifold $M(D)$.

Surprisingly, the number of connected components of $M(D)$ can be determined from the adjacency matrix $A(D)$ of D.

Theorem 7.8 ([21, 62, 251, 282, 289]). *Let D be a framed chord diagram not containing Gaussian chords. Then the number of connected components of $M(D)$ equals* $\operatorname{corank}_{\mathbb{Z}_2} A(D) + 1$, *where $A(D)$ is the adjacency matrix of D over $\mathbb{Z}_2$, and corank equals the difference between the size of the matrix and its rank and is calculated over $\mathbb{Z}_2$.*

Using Theorem 7.8, we can formulate whether the number of unicursal components is 1.

Let D be a framed chord diagram with the adjacency matrix $A(D)$. Construct the matrix $\widehat{A}(D)$ by deleting the rows and columns of $A(D)$ corresponding to Gaussian chords.

Theorem 7.9 ([129, 130]). *Let H be a framed 4-graph, and let U be an Euler tour on H. Then H has a Gauss circuit if and only if* $\operatorname{corank}_{\mathbb{Z}_2}(\widehat{A}(D)+E) = 0$, *where D is the framed chord diagram constructed from U and E is the identity matrix.*

Proof. The proof is illustrated in Fig. 7.26. In order for the Gauss circuit, i.e. a tour while traveling along it we pass from the half-edge e_3 to the half-edge e_1, to exist we have to delete all Gaussian chords and replace all the chords having the framing 0 with intersecting chords and all the chords having the framing 1 with parallel chords. □

7.8.4 *The Gauss circuit*

Let H be a framed 4-graph having the Gauss circuit, and let U be an Euler tour on H. Using Proposition 7.4, we can assume that $m(U)$ (respectively,

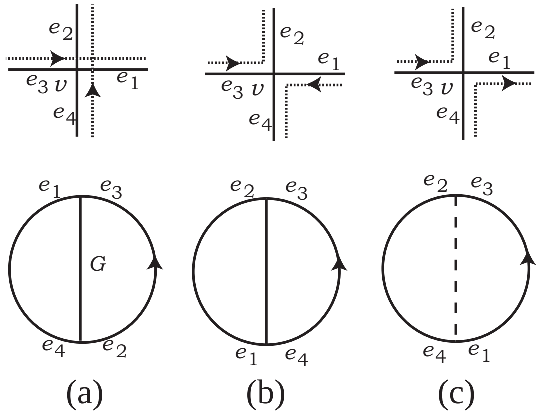


Fig. 7.26 The structure of a framed chord diagram.

the corresponding chord diagram D) has no Gaussian vertices (respectively, no Gaussian chords), i.e. U is a rotating circuit.

Definition 7.20. We say that two matrices $A = (a_{ij})$ and $B = (b_{kl})$ coincide (or equal to each other) *up to diagonal elements* if $a_{ij} = b_{ij}$, $i \neq j$.

The main result of this section is the following theorem.

Theorem 7.10. *The adjacency matrix of the Gauss circuit is equal to* $(A(D) + E)^{-1}$ *(over* $\mathbb{Z}_2$*) up to diagonal elements.*

Proof. Let $V(H) = \{v_1, \ldots, v_n\}$. It is obvious that the following two operations applied to D decrease the number of non-Gaussian chords:

(1) the framed star operation applied to a non-Gaussian chord having the framing 0;
(2) $m \mapsto (((m * a) * b) * a)$, where m is a framed cyclic double occurrence word, a, b are non-Gaussian letters having the framing 1 and they alternate in m, i.e. the corresponding chords are linked.

We call these operations *decreasing operations.* The decreasing operations change an Euler tour U on the 4-graph, and the new Euler tour has the number of non-Gaussian vertices smaller than U has, see Figs. 7.23(a), 7.27.

Let D be a framed chord diagram, and let $A(D)$ be its adjacency matrix. Let us apply decreasing operations. Without loss of generality we may assume that the decreasing operations are applied to the chords having the

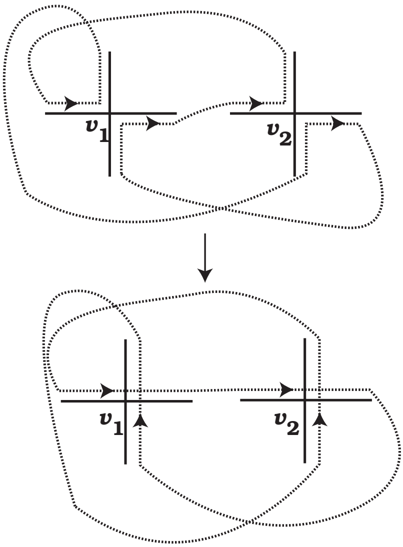


Fig. 7.27 The decreasing operations.

smallest numbers in our numeration. Then the first decreasing operation applied to the first element is

$$A(D) = \begin{pmatrix} 0 & \mathbf{0}^\top & \mathbf{1}^\top \\ \mathbf{0} & A_0 & A_1 \\ \mathbf{1} & A_1^\top & A_2 \end{pmatrix} \rightsquigarrow A(D') = \begin{pmatrix} G & \mathbf{0}^\top & \mathbf{1}^\top \\ \mathbf{0} & A_0 & A_1 \\ \mathbf{1} & A_1^\top & A_2 + (1) \end{pmatrix},$$

and the second one applied to the first two elements is

$$A(D) = \begin{pmatrix} 1 & 1 & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{0}^\top & \mathbf{1}^\top \\ 1 & 1 & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{1}^\top \\ \mathbf{0} & \mathbf{0} & A_0 & A_1 & A_2 & A_3 \\ \mathbf{1} & \mathbf{0} & A_1^\top & A_4 & A_5 & A_6 \\ \mathbf{0} & \mathbf{1} & A_2^\top & A_5^\top & A_7 & A_8 \\ \mathbf{1} & \mathbf{1} & A_3^\top & A_6^\top & A_8^\top & A_9 \end{pmatrix}$$

$$\rightsquigarrow A(D') = \begin{pmatrix} G & 1 & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{1}^\top \\ 1 & G & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{0}^\top & \mathbf{1}^\top \\ \mathbf{0} & \mathbf{0} & A_0 & A_1 & A_2 & A_3 \\ \mathbf{0} & \mathbf{1} & A_1^\top & A_4 & A_5 + (1) & A_6 + (1) \\ \mathbf{1} & \mathbf{0} & A_2^\top & A_5^\top + (1) & A_7 & A_8 + (1) \\ \mathbf{1} & \mathbf{1} & A_3^\top & A_6^\top + (1) & A_8^\top + (1) & A_9 \end{pmatrix},$$

where bold characters $\mathbf{0}$ and $\mathbf{1}$ indicate column vectors with all entries the same 0 and 1, respectively; (1) is a matrix consisting of ones, and A_i are matrices.

We shall successively apply these operations to D. Let us show that after applying these two decreasing operations to the framed chord diagram D having no Gaussian vertices we shall get the framed chord diagram with the adjacency matrix $(A(D)+E)^{-1}$ up to diagonal elements.

To get the matrix $(A(D)+E)^{-1}$ we shall perform elementary manipulations with rows of $B(D) = A(D) + E$ with $\det(A(D)+E) = 1$. Let us construct the matrix $(A(D)+E|E)$ with the size $n \times 2n$. We denote by $\widehat{M}_{ij\ldots k}$ the matrix obtained from M by deleting $i, j, \ldots, k$th rows and $i, j, \ldots, k$th columns.

As $\det B(D) = 1$, then either there is a diagonal element equal to 1 or there are two diagonal elements with the numbers i and j such that $b_{ii} = b_{jj} = 0$, $b_{ij} = b_{ji} = 1$.

In the first case, without loss of generality we assume that $b_{11} = 1$. Then after performing elementary manipulations on $B(D)$ by the first row, we get

$$B(D) = A(D) + E = \begin{pmatrix} 1 & \mathbf{0}^\top & \mathbf{1}^\top \\ \mathbf{0} & A_0 + E & A_1 \\ \mathbf{1} & A_1^\top & A_2 + E \end{pmatrix}$$

$$\rightsquigarrow B'(D) = \begin{pmatrix} 1 & \mathbf{0}^\top & \mathbf{1}^\top \\ \mathbf{0} & A_0 + E & A_1 \\ \mathbf{0} & A_1^\top & A_2 + E + (1) \end{pmatrix}$$

and

$$(B(D)|E) \rightsquigarrow (B'(D)|E') = \left(\begin{array}{ccc|ccc} 1 & \mathbf{0}^\top & \mathbf{1}^\top & 1 & \mathbf{0}^\top & \mathbf{0}^\top \\ \mathbf{0} & A_0 + E & A_1 & \mathbf{0} & E & 0 \\ \mathbf{0} & A_1^\top & A_2 + E + (1) & \mathbf{1} & 0 & E \end{array}\right).$$

After performing the first decreasing operation to D the chord corresponding to v_1 becomes a Gaussian chord and the adjacencies of non-Gaussian chords are defined by the matrix $\widehat{B'(D)}_1$ and the other adjacencies are defined by the first column of E' (up to diagonal elements).

In the second case, we may assume without loss of generality $b_{11} = b_{22} = 0$, $b_{12} = b_{21} = 1$. Then after performing elementary manipulations applied to the first two rows of $B(D)$, we get

$$B(D) = A(D) + E = \begin{pmatrix} 0\,1 & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{0}^\top & \mathbf{1}^\top \\ 1\,0 & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{1}^\top \\ \mathbf{0}\,\mathbf{0} & A_0 + E & A_1 & A_2 & A_3 \\ \mathbf{1}\,\mathbf{0} & A_1^\top & A_4 + E & A_5 & A_6 \\ \mathbf{0}\,\mathbf{1} & A_2^\top & A_5^\top & A_7 + E & A_8 \\ \mathbf{1}\,\mathbf{1} & A_3^\top & A_6^\top & A_8^\top & A_9 + E \end{pmatrix}$$

$$\rightsquigarrow B'(D) = \begin{pmatrix} 1\,0 & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{1}^\top \\ 0\,1 & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{0}^\top & \mathbf{1}^\top \\ \mathbf{0}\,\mathbf{0} & A_0 + E & A_1 & A_2 & A_3 \\ \mathbf{0}\,\mathbf{0} & A_1^\top & A_4 + E & A_5 + (1) & A_6 + (1) \\ \mathbf{0}\,\mathbf{0} & A_2^\top & A_5^\top + (1) & A_7 + E & A_8 + (1) \\ \mathbf{0}\,\mathbf{0} & A_3^\top & A_6^\top + (1) & A_8^\top + (1) & A_9 + E \end{pmatrix}$$

and

$$(B(D)|E) \rightsquigarrow (B'(D)|E')$$
$$= \left(\begin{array}{ccccc|cccccc} 1\,0 & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{1}^\top & 0 & 1 & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{0}^\top \\ 0\,1 & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{0}^\top & \mathbf{1}^\top & 1 & 0 & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{0}^\top \\ \mathbf{0}\,\mathbf{0} & A_0 + E & A_1 & A_2 & A_3 & \mathbf{0} & \mathbf{0} & E & 0 & 0 & 0 \\ \mathbf{0}\,\mathbf{0} & A_1^\top & A_4 + E & A_5 + (1) & A_6 + (1) & \mathbf{0} & \mathbf{1} & 0 & E & 0 & 0 \\ \mathbf{0}\,\mathbf{0} & A_2^\top & A_5^\top + (1) & A_7 + E & A_8 + (1) & \mathbf{1} & \mathbf{0} & 0 & 0 & E & 0 \\ \mathbf{0}\,\mathbf{0} & A_3^\top & A_6^\top + (1) & A_8^\top + (1) & A_9 + E & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & E \end{array}\right).$$

After performing the second decreasing operation to D the chords corresponding to v_1 and v_2 become Gaussian chords and the adjacencies of the non-Gaussian chords are defined by the matrix $\widehat{B'(D)}_{12}$ and the other adjacencies are defined by the first two columns of E'.

Let us suppose that we have performed k decreasing operations. After these operations the matrix $(B(D)|E)$ is transformed into a matrix

$$(B'(D)|E') = \left(\begin{array}{cc|cc} E & C & F & 0 \\ 0 & R & S & E \end{array}\right)$$

and F is an $l \times l$ matrix, R is a symmetric matrix. Then the new framed chord diagram contains l Gaussian chords, and the adjacencies of non-Gaussian chords are defined by R and the other adjacencies are defined by the first l columns of E'. As $\det B'(D) = 1$, then $\det R = 1$, and in the matrix R there is either a diagonal element equal to 1 or there are numbers p and q such that $r_{pp} = r_{qq} = 0$, $r_{pq} = r_{qp} = 1$.

Let us consider the first case. Without loss of generality we may assume that $r_{11} = 1$. In this case we apply the first decreasing operation. We shall get

$$(B'(D)|E') = \left(\begin{array}{cccc|cccc} E & C_1 & C_2 & C_3 & F & 0 & 0 & 0 \\ 0 & 1 & \mathbf{0}^\top & \mathbf{1}^\top & S_1 & 1 & \mathbf{0}^\top & \mathbf{0}^\top \\ \mathbf{0} & \mathbf{0} & R_1 & R_2 & S_2 & \mathbf{0} & E & 0 \\ \mathbf{0} & \mathbf{1} & R_2^\top & R_3 & S_3 & \mathbf{0} & 0 & E \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cccc|cccc} E & 0 & C_2' & C_3' & F_1' & F_2' & 0 & 0 \\ 0 & 1 & \mathbf{0}^\top & \mathbf{1}^\top & S_1 & 1 & \mathbf{0}^\top & \mathbf{0}^\top \\ \mathbf{0} & \mathbf{0} & R_1 & R_2 & S_2 & \mathbf{0} & E & 0 \\ \mathbf{0} & \mathbf{0} & R_2^\top & R_3 + (1) & S_3' & \mathbf{1} & 0 & E \end{array}\right)$$

$$= \left(\begin{array}{cc|cc} E & C' & F' & 0 \\ 0 & R' & S' & E \end{array}\right) = (B''(D)|E''),$$

where F' is a $(l+1)\times(l+1)$ matrix, R' is a symmetric matrix. The number of Gaussian vertices is $l+1$, and the adjacencies of non-Gaussian vertices are defined by R' and the other adjacencies are defined by the first $l+1$ columns of E''. The second case is considered analogously to the first one.

We end up with the matrix

$$\left(E \,\middle|\, (A(D)+E)^{-1} \right)$$

and the framed chord diagram having only Gaussian vertices. The adjacency matrix of this chord diagram is $(A(D)+E)^{-1}$ up to diagonal elements.

We have proved the theorem for non-diagonal elements. But we know that all the diagonal elements are G. □

Remark 7.12. Let H be an arbitrary (connected and containing at least one vertex) oriented 4-graph, and at each vertex exactly two half-edges are incoming to it and two other edges are emanating from it (orientations of half-edges corresponding to one edge coincide). It is easy to see that there is an oriented Euler tour U on the graph H. Define a framing on H in such a way that U is a rotating circuit on the new framed 4-graph H, and at each vertex every pair of opposite edges consists of one incoming and one emanating edge. If there exists an oriented Gauss circuit on the framed oriented 4-graph H, then Theorem 7.10 gives the formula for the adjacency matrix of the Gauss circuit. Thus, the last claim about the existence of a Gauss circuit is Theorem 3.4 from [143], and, therefore, Theorem 3.4 from [143] is a particular case of Theorem 7.10.

From the last theorem we immediately get the following corollary.

Corollary 7.2. *Let U_1 and U_2 be two rotating circuits, and let D_1 and D_2 be their framed chord diagrams such that* $\det(A(D_i) + E) = 1$. *Then the matrices $(A(D_1)+E)^{-1}$ and $(A(D_2)+E)^{-1}$ equal up to diagonal elements.*

Example 7.5. Consider the framed 4-graph having 4 vertices v_i, Fig. 7.28. Let U_1 and U_2 be two rotating circuits given by the framed double occurrence cyclic words $m(U_1) = (v_1v_4v_2v_1^{-1}v_2v_3v_4v_3)$ and $m(U_2) = (v_1v_4v_3v_4v_2v_3v_1v_2^{-1})$, respectively. Then

$$A(m(U_1)) = \begin{pmatrix} 1\,1\,0\,1 \\ 1\,0\,0\,0 \\ 0\,0\,0\,1 \\ 1\,0\,1\,0 \end{pmatrix}, \qquad A(m(U_2)) = \begin{pmatrix} 0\,1\,0\,0 \\ 1\,1\,1\,0 \\ 0\,1\,0\,1 \\ 0\,0\,1\,0 \end{pmatrix}.$$

We get

$$(A(m(U_1)) + E)^{-1} = \begin{pmatrix} 0\,0\,1\,1 \\ 0\,1\,1\,1 \\ 1\,1\,0\,1 \\ 1\,1\,1\,1 \end{pmatrix}, \qquad (A(m(U_2)) + E)^{-1} = \begin{pmatrix} 1\,0\,1\,1 \\ 0\,0\,1\,1 \\ 1\,1\,1\,1 \\ 1\,1\,1\,0 \end{pmatrix}$$

and

$$A = \begin{pmatrix} G & 0 & 1 & 1 \\ 0 & G & 1 & 1 \\ 1 & 1 & G & 1 \\ 1 & 1 & 1 & G \end{pmatrix}$$

is the adjacency matrix of the Gauss circuit given by $(v_1v_4v_3v_1v_2v_4v_3v_2)$.

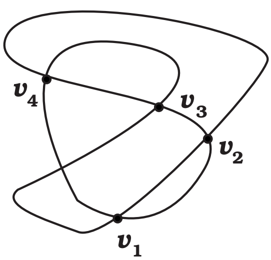


Fig. 7.28 A framed 4-graph.

The next corollary immediately follows from the criterion of the planarity of an immersed curve, see Lemma 7.18, and from atom theory, see Chap. 4.

Corollary 7.3. *Let D be a framed chord diagram, all the chords of D are non-Gaussian chords with framing* 0 *and* $\det(A+E)=1$. *If there are numbers* $\lambda_1,\ldots,\lambda_n\in\mathbb{Z}_2$ *such that*

$$\det\big((A(D)+E)^{-1}+\mathrm{diag}(\lambda_1,\ldots,\lambda_n)\big)=1,$$

then the matrix $\big((A(D)+E)^{-1}+\mathrm{diag}(\lambda_1,\ldots,\lambda_n)\big)^{-1}$ *has all diagonal elements equal to* 1. *Moreover, if D is a framed d-diagram* (*with all framings* 0), *then the matrix* $\big((A(D)+E)^{-1}+\mathrm{diag}(\lambda_1,\ldots,\lambda_n)\big)^{-1}$ *is the adjacency matrix of a framed d-diagram. Here* $\mathrm{diag}(\lambda_1,\ldots,\lambda_n)$ *is the diagonal matrix with* $\lambda_1,\ldots,\lambda_n$ *on its diagonal.*

From a geometric point of view the first part of Corollary 7.3 means the following. Having a framed 4-graph H and a rotating circuit U on it, we can define an orientation on the graph H. We orient a rotating circuit U in an arbitrary way and define the orientation of H by means of it. It turns out that if any rotating circuit provides the source–sink condition (see Definition 5.7) on a framed 4-graph, then any other rotating circuit also gives the source–sink condition, since the corresponding atom is orientable. The second part of the corollary is treated with the planarity question. If we have a planar framed 4-graph, i.e. a graph embedded into the plane with the structure of opposite edges preserved, then all rotating circuits are represented by d-diagrams (with all framings 0).

7.8.5 *Adjacency matrices*

In Chap. 9 we shall "generalize" the notion of a virtual link and construct a new theory, the *theory of graph-links*: We consider not only intersection graphs of chord diagrams and moves on them but all simple graphs and moves on them, which are induced by moves on intersection graphs. Graph-links can be constructed in different ways. We shall construct two theories of graph-links: in the first one we use Gauss circuits and in the second one we use rotating circuits. The results of this section will be needed to prove the equivalence of these two approaches, see Chap. 9.

It is well known that there are symmetric matrices over $\mathbb{Z}_2$ which cannot be realized by chord diagrams [39] (here a symmetric matrix is the adjacency matrix of the corresponding graph), and symmetric matrices which can be realized by different chord diagrams. In the proof of Theorem 7.10, we have just used elementary manipulations and adjacency matrices. It turns out that Theorem 7.10 and Corollary 7.3 can be reformulated for arbitrary symmetric matrices, see also Sec. 9.2.

In this section, all the matrices are over $\mathbb{Z}_2$ with diagonal elements equal to either 0 or 1. Consider the following operation over the set of symmetric matrices.

Definition 7.21. Let $A = (a_{ij})$ be a symmetric $n \times n$ matrix. Let us fix an arbitrary element $a_{kk} = 1$ and construct the matrix $\mathrm{Loc}(A, k) = (\widetilde{a}_{ij})$, where $\widetilde{a}_{pq} = a_{pq} + 1$, p, $q \neq k$, if both $a_{pk} = 1$ and $a_{kq} = 1$, and $\widetilde{a}_{pq} = a_{pq}$ otherwise. We call the transformation $A \mapsto \mathrm{Loc}(A, k)$ the *local complementation of the matrix A at the element a_{kk}* (this operation is analogous to the framed star operation).

It is not difficult to show that, if $a_{ij} = 1$, then the matrices $\mathrm{Loc}(\mathrm{Loc}(\mathrm{Loc}(A, i), j), i)$ and $\mathrm{Loc}(\mathrm{Loc}(\mathrm{Loc}(A, j), i), j)$ coincide up to the diagonal elements with the numbers i, j.

Definition 7.22. Let A be a symmetric matrix with $a_{ii} = a_{jj} = 0$, $a_{ij} = a_{ji} = 1$ for some pair i, j. A *pivot* operation is the transformation $A \mapsto \widetilde{A}$, where the diagonal elements of $\widetilde{A}$ coincide with the diagonal elements of A and the other elements of $\widetilde{A}$ coincide with the corresponding elements of $\mathrm{Loc}(\mathrm{Loc}(\mathrm{Loc}(A, i), j), i)$.

Let $\mathrm{Sym}(n, \mathbb{Z}_2)$ be the set of all symmetric $n \times n$ matrices over $\mathbb{Z}_2$. Consider two equivalence relations on $\mathrm{Sym}(n, \mathbb{Z}_2)$. The first relation is the coincidence of two matrices up to diagonal elements, denote this equivalence relation by $\sim_D$. The second equivalence is defined as follows: Two matrices A and B are said to be *obtained from each other by changing a circuit*, denote the second relation by $A \sim_C B$, if A and B are related by a finite sequence of local complementations and pivot operations.

Remark 7.13. In the realizable case the second equivalence relation corresponds to the "real" change of a rotating circuit on a framed 4-graph.

The proof of the main result of this subsection is based on the following five lemmas. These lemmas are very technical, but they have geometric interpretations in the case where the matrices are realizable. These interpretations are given in the remarks which follow each of the lemmas. The proofs of these lemmas can be skipped on a first reading.

Lemma 7.11 **([129, 130]).** *If* $\det(A+E)=1$ *and* $B \sim_C A$, *then* $\det(B+E)=1$, *where* E *is the identity matrix.*

Remark 7.14. If two symmetric matrices A and B are realizable by chord diagrams, then Lemma 7.11 means that after applying appropriate framed star operations to a rotating circuit on a virtual knot diagram we get a new rotating circuit on a virtual knot diagram (not on a link diagram).

Let $\mathrm{Sym}_+(n,\mathbb{Z}_2) \subset \mathrm{Sym}(n,\mathbb{Z}_2)$ be the subset of the set of symmetric matrices consisting of matrices A with $\det(A+E)=1$.

Lemma 7.12. *The relation* $\sim_C$ *is an equivalence relation on* $\mathrm{Sym}_+(n,\mathbb{Z}_2)$.

Consider two sets

$$\mathfrak{L}(n) = \mathrm{Sym}(n,\mathbb{Z}_2)/\sim_D \quad \text{and} \quad \mathfrak{G}(n) = \mathrm{Sym}_+(n,\mathbb{Z}_2)/\sim_C .$$

Lemma 7.13. *Every element of* $\mathfrak{L}(n)$ *has a representative with the determinant equal to* 1.

Proof. Let us prove this lemma by induction on the size of a matrix.

The induction base. For $n=1$ the claim of the lemma is evident.

The induction step. Assume the claim of the lemma holds for $n-1$, and let A be an $(n\times n)$-matrix. By the induction hypothesis, we can assume that $\det A^{11}=1$, where A^{ij} is the cofactor of a_{ij}. Then either

$$\det A = a_{11}A^{11} + \sum_{j=2}^{n} a_{1j}A^{1j} = a_{11} + \sum_{j=2}^{n} a_{1j}A^{1j} = 1$$

or

$$\det \widetilde{A} = (a_{11}+1)A^{11} + \sum_{j=2}^{n} a_{1j}A^{1j} = a_{11} + 1 + \sum_{j=2}^{n} a_{1j}A^{1j} = 1,$$

where the matrix $\widetilde{A}$ differs from A only by the element $\widetilde{a}_{11}$. □

Lemma 7.14. *Let* B *and* $\widetilde{B}$ *be two symmetric matrices over* $\mathbb{Z}_2$ *with* $\det B = \det \widetilde{B} = 1$, *and let* B *and* $\widetilde{B}$ *coincide up to one element on the diagonal. Then the matrices* $B^{-1}+E$ *and* $\widetilde{B}^{-1}+E$ *are related by a local complementation at an element being equal to* 1.

Proof. Without loss of generality we may assume that $B=(b_{ij})$, $\widetilde{B} = (\widetilde{b}_{ij})$ are $(n\times n)$-matrices and $\widetilde{b}_{nn} = b_{nn}+1 = 0$, $\widetilde{b}_{ij} = b_{ij}$, $i \neq n$ or $j \neq n$. We shall perform elementary manipulations with rows of B and $\widetilde{B}$ to get two identity matrices. Then we shall apply these elementary manipulations to two identity matrix to get the inverse matrices.

Using the equality $\det B = \det \widetilde{B} = 1$, we have

$$1 = \det \widetilde{B} = \det B + \det \widehat{B}_n^n = 1 + \det \widehat{B}_n^n, \qquad \det \widehat{B}_n^n = 0,$$

$$\mathrm{rank} B = n, \qquad \mathrm{rank} \widehat{B}_n^n = n-2;$$

here $\widehat{B}_n^n$ is the matrix obtained from B by deleting the nth row and nth column. Since $\widehat{B}_n^n$ is a symmetric matrix, without loss of generality we may assume that $\det C = 1$, where C is the matrix obtained from $\widehat{B}_n^n$ by deleting the $(n-1)$th row and $(n-1)$th column.

Performing elementary manipulations with rows of B and $\widetilde{B}$ (the first $(n-1)$ rows of $\widetilde{B}$ are the same as the ones of B), we get

$$B \rightsquigarrow \left(\begin{array}{ccc|ccc} E & \mathbf{u} & \mathbf{v} & F & \mathbf{0} & \mathbf{0} \\ \mathbf{0}^\top & 0 & 1 & \mathbf{a}^\top & 1 & 0 \\ \mathbf{0}^\top & 1 & 1 & \mathbf{b}^\top & 0 & 1 \end{array}\right), \widetilde{B} \rightsquigarrow \left(\begin{array}{ccc|ccc} E & \mathbf{u} & \mathbf{v} & F & \mathbf{0} & \mathbf{0} \\ \mathbf{0}^\top & 0 & 1 & \mathbf{a}^\top & 1 & 0 \\ \mathbf{0}^\top & 1 & 0 & \mathbf{b}^\top & 0 & 1 \end{array}\right);$$

here F is an $((n-2)\times(n-2))$-matrix, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{u}$, $\mathbf{v}$ are $(n-2)$-column vectors. Further, performing elementary manipulations with rows of $\widetilde{B}$ and B, we have

$$\widetilde{B} \rightsquigarrow \left(\begin{array}{ccc|ccc} E & \mathbf{u} & \mathbf{v} & F & \mathbf{0} & \mathbf{0} \\ \mathbf{0}^\top & 0 & 1 & \mathbf{a}^\top & 1 & 0 \\ \mathbf{0}^\top & 1 & 0 & \mathbf{b}^\top & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} E & \mathbf{u} & \mathbf{v} & F & \mathbf{0} & \mathbf{0} \\ \mathbf{0}^\top & 1 & 0 & \mathbf{b}^\top & 0 & 1 \\ \mathbf{0}^\top & 0 & 1 & \mathbf{a}^\top & 1 & 0 \end{array}\right)$$
$$\rightsquigarrow \left(\begin{array}{ccc|ccc} E & \mathbf{u} & \mathbf{0} & F_1 & \mathbf{b} & \mathbf{0} \\ \mathbf{0}^\top & 1 & 0 & \mathbf{b}^\top & 0 & 1 \\ \mathbf{0}^\top & 0 & 1 & \mathbf{a}^\top & 1 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} E & \mathbf{0} & \mathbf{0} & F_2 & \mathbf{b} & \mathbf{a} \\ \mathbf{0}^\top & 1 & 0 & \mathbf{b}^\top & 0 & 1 \\ \mathbf{0}^\top & 0 & 1 & \mathbf{a}^\top & 1 & 0 \end{array}\right),$$

i.e. $\mathbf{u} = \mathbf{a}$, $\mathbf{v} = \mathbf{b}$ (the inverse matrix to a symmetric matrix is symmetric),

$$B \rightsquigarrow \left(\begin{array}{ccc|ccc} E & \mathbf{u} & \mathbf{v} & F & \mathbf{0} & \mathbf{0} \\ \mathbf{0}^\top & 0 & 1 & \mathbf{a}^\top & 1 & 0 \\ \mathbf{0}^\top & 1 & 1 & \mathbf{b}^\top & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} E & \mathbf{a} & \mathbf{b} & F & \mathbf{0} & \mathbf{0} \\ \mathbf{0}^\top & 1 & 0 & \mathbf{a}^\top + \mathbf{b}^\top & 1 & 1 \\ \mathbf{0}^\top & 0 & 1 & \mathbf{a}^\top & 1 & 0 \end{array}\right)$$
$$\rightsquigarrow \left(\begin{array}{ccc|ccc} E & \mathbf{a} & \mathbf{0} & F_1 & \mathbf{b} & \mathbf{0} \\ \mathbf{0}^\top & 1 & 0 & \mathbf{a}^\top + \mathbf{b}^\top & 1 & 1 \\ \mathbf{0}^\top & 0 & 1 & \mathbf{a}^\top & 1 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} E & \mathbf{0} & \mathbf{0} & F_3 & \mathbf{a}+\mathbf{b} & \mathbf{a} \\ \mathbf{0}^\top & 1 & 0 & \mathbf{a}^\top + \mathbf{b}^\top & 1 & 1 \\ \mathbf{0}^\top & 0 & 1 & \mathbf{a}^\top & 1 & 0 \end{array}\right).$$

It is not difficult to see that F_3 is obtained from F_2 by adding $\mathbf{a}^\top$ to the rows of F_2 corresponding to the rows of B^{-1} having the last element equal to 1. Therefore, the matrix $B^{-1}+E$ is obtained from $\widetilde{B}^{-1}+E$ by the local complementation at the element corresponding to $\widetilde{b}_{nn}$. □

Lemma 7.15. *Let B and $\widetilde{B}$ be two symmetric matrices over $\mathbb{Z}_2$ with* $\det B = \det \widetilde{B} = 1$, *and let B and $\widetilde{B}$ coincide up to two elements on the diagonal with numbers i and j. Suppose that* $\det \widehat{B}_i^i = \det \widehat{B}_j^j = 1$, *where $\widehat{B}_k^k$ is the matrix obtained from B by deleting the kth row and kth column. Then the matrices $B^{-1} + E$ and $\widetilde{B}^{-1} + E$ are related by a pivot operation.*

Proof. Without loss of generality, we may assume that $B = (b_{ij})$, $\widetilde{B} = (\widetilde{b}_{ij})$ are two $(n \times n)$-matrices and $\widetilde{b}_{(n-1)(n-1)} = b_{(n-1)(n-1)} + 1$, $\widetilde{b}_{nn} = b_{nn} + 1$, $\widetilde{b}_{ij} = b_{ij}$ for $(i,j) \neq (n-1, n-1), (n,n)$. We shall perform elementary manipulations with rows of B and $\widetilde{B}$ to get two identity matrices. Then we shall apply these manipulations to two identity matrices to get their inverse matrices.

Using the equality $\det B = \det \widetilde{B} = 1$, we have

$$1 = \det \widetilde{B} = \det B + \det \widehat{B}_{n-1}^{n-1} + \det \widehat{B}_n^n + \det \widehat{B}_{(n-1)n}^{(n-1)n} = 1 + \det \widehat{B}_{(n-1)n}^{(n-1)n},$$

$$\det \widehat{B}_{(n-1)n}^{(n-1)n} = 0, \quad \mathrm{rank} \widehat{B}_{n-1}^{n-1} = \mathrm{rank} \widehat{B}_n^n = n-1, \quad \mathrm{rank} \widehat{B}_{(n-1)n}^{(n-1)n} = n-3;$$

here $\widehat{B}_{(n-1)n}^{(n-1)n}$ is the matrix obtained from $\widehat{B}_n^n$ by deleting the $(n-1)$th row and $(n-1)$th column. Since $\widehat{B}_{(n-1)n}^{(n-1)n}$ is a symmetric matrix, without loss of generality we may assume that $\det C = 1$, where C is the matrix obtained from $\widehat{B}_{(n-1)n}^{(n-1)n}$ by deleting the $(n-2)$th row and $(n-2)$th column. It is not difficult to show that the two matrices obtained from $\widetilde{B}$ by deleting the nth row, nth column and the $(n-1)$th row, $(n-1)$th column, respectively, are both non-degenerate.

Performing elementary manipulations with rows of B and $\widetilde{B}$ (the first $(n-2)$ rows of $\widetilde{B}$ are the same as the ones of B), we get

$$B \rightsquigarrow \left(\begin{array}{cccc|cccc} E & \mathbf{u} & \mathbf{v} & \mathbf{w} & F & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0}^\top & 0 & 1 & 1 & \mathbf{a}^\top & 1 & 0 & 0 \\ \mathbf{0}^\top & 1 & 1 & l & \mathbf{b}^\top & 0 & 1 & 0 \\ \mathbf{0}^\top & 1 & l & 0 & \mathbf{c}^\top & 0 & 0 & 1 \end{array}\right), \qquad \widetilde{B} \rightsquigarrow \left(\begin{array}{cccc|cccc} E & \mathbf{u} & \mathbf{v} & \mathbf{w} & F & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0}^\top & 0 & 1 & 1 & \mathbf{a}^\top & 1 & 0 & 0 \\ \mathbf{0}^\top & 1 & 0 & l & \mathbf{b}^\top & 0 & 1 & 0 \\ \mathbf{0}^\top & 1 & l & 1 & \mathbf{c}^\top & 0 & 0 & 1 \end{array}\right);$$

here F is a $((n-3) \times (n-3))$-matrix, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are $(n-3)$-column vectors, and $l \in \{0, 1\}$. Further, performing elementary manipulations with rows, we have

$$B \rightsquigarrow \left(\begin{array}{cccc|cccc} E & \mathbf{u} & \mathbf{v} & \mathbf{w} & F & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0}^\top & 0 & 1 & 1 & \mathbf{a}^\top & 1 & 0 & 0 \\ \mathbf{0}^\top & 1 & 1 & l & \mathbf{b}^\top & 0 & 1 & 0 \\ \mathbf{0}^\top & 1 & l & 0 & \mathbf{c}^\top & 0 & 0 & 1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cccc|cccc} E & \mathbf{u} & \mathbf{v} & \mathbf{w} & F & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0}^\top & 1 & 0 & 0 & l\mathbf{a}^\top + l\mathbf{b}^\top + (1+l)\mathbf{c}^\top & l & l & 1+l \\ \mathbf{0}^\top & 0 & 1 & 0 & l\mathbf{a}^\top + \mathbf{b}^\top + \mathbf{c}^\top & l & 1 & 1 \\ \mathbf{0}^\top & 0 & 0 & 1 & (1+l)\mathbf{a}^\top + \mathbf{b}^\top + \mathbf{c}^\top & 1+l & 1 & 1 \end{array}\right) \rightsquigarrow (\, E \,|\, B^{-1}\,);$$

here

$$B^{-1} = \begin{pmatrix} F_1 & l(\mathbf{a}+\mathbf{b}+\mathbf{c})+\mathbf{c} & l\mathbf{a}+\mathbf{b}+\mathbf{c} & (1+l)\mathbf{a}+\mathbf{b}+\mathbf{c} \\ l\mathbf{a}^\top + l\mathbf{b}^\top + (1+l)\mathbf{c}^\top & l & l & 1+l \\ l\mathbf{a}^\top + \mathbf{b}^\top + \mathbf{c}^\top & l & 1 & 1 \\ (1+l)\mathbf{a}^\top + \mathbf{b}^\top + \mathbf{c}^\top & 1+l & 1 & 1 \end{pmatrix}$$

and

$$\widetilde{B} \rightsquigarrow \left(\begin{array}{cccc|cccc} E & \mathbf{u} & \mathbf{v} & \mathbf{w} & F & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0}^\top & 0 & 1 & 1 & \mathbf{a}^\top & 1 & 0 & 0 \\ \mathbf{0}^\top & 1 & 0 & l & \mathbf{b}^\top & 0 & 1 & 0 \\ \mathbf{0}^\top & 1 & l & 1 & \mathbf{c}^\top & 0 & 0 & 1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cccc|cccc} E & \mathbf{u} & \mathbf{v} & \mathbf{w} & F & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0}^\top & 1 & 0 & 0 & l(\mathbf{a}^\top + \mathbf{b}^\top + \mathbf{c}^\top) + \mathbf{b}^\top & l & 1+l & l \\ \mathbf{0}^\top & 0 & 1 & 0 & (1+l)\mathbf{a}^\top + \mathbf{b}^\top + \mathbf{c}^\top & 1+l & 1 & 1 \\ \mathbf{0}^\top & 0 & 0 & 1 & l\mathbf{a}^\top + \mathbf{b}^\top + \mathbf{c}^\top & l & 1 & 1 \end{array}\right) \rightsquigarrow (\, E \,|\, \widetilde{B}^{-1}\,),$$

$$\widetilde{B}^{-1} = \begin{pmatrix} F_2 & l(\mathbf{a}+\mathbf{b}+\mathbf{c})+\mathbf{b} & (1+l)\mathbf{a}+\mathbf{b}+\mathbf{c} & l\mathbf{a}+\mathbf{b}+\mathbf{c} \\ l(\mathbf{a}^\top + \mathbf{b}^\top + \mathbf{c}^\top) + \mathbf{b}^\top & l & 1+l & l \\ (1+l)\mathbf{a}^\top + \mathbf{b}^\top + \mathbf{c}^\top & 1+l & 1 & 1 \\ l\mathbf{a}^\top + \mathbf{b}^\top + \mathbf{c}^\top & l & 1 & 1 \end{pmatrix}.$$

Let us investigate the matrices F_1 and F_2 more carefully. We have four cases depending on the last two elements of the rows.

(1) If a row of the matrix B^{-1} has the last two elements equal to 0, then the corresponding row of $\widetilde{B}^{-1}$ also has the last two elements equal to 0. We have the following two cases: The rows of the matrices F_1 and F_2 are either obtained from F by adding the sum of the two rows $l\mathbf{a}^\top + \mathbf{b}^\top + \mathbf{c}^\top$ and $(1+l)\mathbf{a}^\top + \mathbf{b}^\top + \mathbf{c}^\top$ to the corresponding row of F or are equal to the corresponding row of F. In both cases, we have the equality of rows of B^{-1} and $\widetilde{B}^{-1}$ having the last two elements equal to 0.

(2) If a row of the matrix B^{-1} has the last two elements equal to 1, then the corresponding row of $\widetilde{B}^{-1}$ also has the last two elements equal to 1. We have the following two cases: The rows of the matrices F_1 and F_2 are obtained from F either by adding $l\mathbf{a}^\top + \mathbf{b}^\top + \mathbf{c}^\top$ to the corresponding row of F or by adding $(1+l)\mathbf{a}^\top + \mathbf{b}^\top + \mathbf{c}^\top$ to the corresponding row of F. In both cases, we have the equality of rows of B^{-1} and $\widetilde{B}^{-1}$ having the last two elements equal to 1.

(3) If a row of the matrix B^{-1} has the penultimate element equal to 0 and the last one equal to 1, then the corresponding row of $\widetilde{B}^{-1}$ has the penultimate element equal to 1 and the last one equal to 0. We have the following two cases: The rows of F_1 and F_2 are obtained from F either by adding the sum of two rows

$$l\mathbf{a}^\top + l\mathbf{b}^\top + (1+l)\mathbf{c}^\top, \qquad l(l\mathbf{a}^\top + \mathbf{b}^\top + \mathbf{c}^\top)$$

for F_1 and the sum of two rows

$$l(\mathbf{a}^\top + \mathbf{b}^\top + \mathbf{c}^\top) + \mathbf{b}^\top, \qquad l((1+l)\mathbf{a}^\top + \mathbf{b}^\top + \mathbf{c}^\top)$$

for F_2, or by adding the sum of the three rows

$$l\mathbf{a}^\top + l\mathbf{b}^\top + (1+l)\mathbf{c}^\top, \qquad (1+l)(l\mathbf{a}^\top + \mathbf{b}^\top + \mathbf{c}^\top), \qquad (1+l)\mathbf{a}^\top + \mathbf{b}^\top + \mathbf{c}^\top$$

for F_1 and the sum of the three rows

$$l(\mathbf{a}^\top + \mathbf{b}^\top + \mathbf{c}^\top) + \mathbf{b}^\top, \qquad (1+l)((1+l)\mathbf{a}^\top + \mathbf{b}^\top + \mathbf{c}^\top), \qquad l\mathbf{a}^\top + \mathbf{b}^\top + \mathbf{c}^\top$$

for F_2 to the corresponding row of F. In both cases, the sum of the corresponding rows of B^{-1} and $\widetilde{B}^{-1}$ is $l\mathbf{a}^\top + \mathbf{b}^\top + \mathbf{c}^\top$.

(4) If a row of the matrix B^{-1} has the penultimate element equal to 1 and the last one equal to 0, then the corresponding row of $\widetilde{B}^{-1}$ has the penultimate element equal to 0 and the last one equal to 1. We have the following two cases: The rows of F_1 and F_2 are obtained from F either by adding the sum of the rows

$$l\mathbf{a}^\top + l\mathbf{b}^\top + (1+l)\mathbf{c}^\top, \qquad (1+l)(l\mathbf{a}^\top + \mathbf{b}^\top + \mathbf{c}^\top)$$

for F_1 and the sum of the rows

$$l(\mathbf{a}^\top + \mathbf{b}^\top + \mathbf{c}^\top) + \mathbf{b}^\top, \qquad (1+l)((1+l)\mathbf{a}^\top + \mathbf{b}^\top + \mathbf{c}^\top)$$

for F_2, or by adding the sum of the three rows

$$l\mathbf{a}^\top + l\mathbf{b}^\top + (1+l)\mathbf{c}^\top, \qquad l(l\mathbf{a}^\top + \mathbf{b}^\top + \mathbf{c}^\top), \qquad (1+l)\mathbf{a}^\top + \mathbf{b}^\top + \mathbf{c}^\top$$

for F_1 and the sum of rows

$$l(\mathbf{a}^\top + \mathbf{b}^\top + \mathbf{c}^\top) + \mathbf{b}^\top, \qquad l((1+l)\mathbf{a}^\top + \mathbf{b}^\top + \mathbf{c}^\top), \qquad l\mathbf{a}^\top + \mathbf{b}^\top + \mathbf{c}^\top$$

for F_2 to the corresponding row of F. In both cases, the sum of the corresponding rows of B^{-1} and $\widetilde{B}^{-1}$ is $(1+l)\mathbf{a}^\top + \mathbf{b}^\top + \mathbf{c}^\top$.

Therefore, the matrices $B^{-1}+E$ and $\widetilde{B}^{-1}+E$ are related by a pivot operation. □

Remark 7.15. If a symmetric matrix B is realizable by a chord diagram, then the matrix $\widetilde{B}$ being equal to B up to diagonal elements is also realizable by a chord diagram. In this case Lemmas 7.14 and 7.15 mean that rotating circuits, obtained from the Gauss circuit by different ways, are connected with each other by the transformations mentioned in Statement 7.1.

Theorem 7.11 ([127]). (1) *The map* $\upsilon\colon \mathfrak{G}(n) \to \mathfrak{L}(n)$ *given by the formula*:

$$\upsilon[A]_C = [(A+E)^{-1}]_D,$$

is well defined.
(2) *There exists the inverse map* $\upsilon^{-1}\colon \mathfrak{L}(n) \to \mathfrak{G}(n)$.

Proof. Let E_{ij}, $i \neq j$, be the matrix with ones along the diagonal, and the element in the intersection of the ith row and jth column is 1, the others are 0.

(1) Let $A \sim_C \widetilde{A}$.

If A and $\widetilde{A}$ are related by the pivot operation for the first two elements, then

$$B = A+E = \begin{pmatrix} 1\ 1 & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{0}^\top & \mathbf{1}^\top \\ 1\ 1 & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{1}^\top \\ \mathbf{0}\ \mathbf{0} & A_0+E & A_1 & A_2 & A_3 \\ \mathbf{1}\ \mathbf{0} & A_1^\top & A_4+E & A_5 & A_6 \\ \mathbf{0}\ \mathbf{1} & A_2^\top & A_5^\top & A_7+E & A_8 \\ \mathbf{1}\ \mathbf{1} & A_3^\top & A_6^\top & A_8^\top & A_9+E \end{pmatrix},$$

$$\widetilde{B} = \widetilde{A}+E = \begin{pmatrix} 1\ 1 & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{1}^\top \\ 1\ 1 & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{0}^\top & \mathbf{1}^\top \\ \mathbf{0}\ \mathbf{0} & A_0+E & A_1 & A_2 & A_3 \\ \mathbf{0}\ \mathbf{1} & A_1^\top & A_4+E & A_5+(1) & A_6+(1) \\ \mathbf{1}\ \mathbf{0} & A_2^\top & A_5^\top+(1) & A_7+E & A_8+(1) \\ \mathbf{1}\ \mathbf{1} & A_3^\top & A_6^\top+(1) & A_8^\top+(1) & A_9+E \end{pmatrix}$$

$$= BE_{1k_1}\dots E_{1k_p}E_{2(k_p+1)}\dots E_{2k_q}E_{1(k_q+1)}\dots E_{1n}$$
$$\cdot E_{2(k_q+1)}\dots E_{2n}E_{12}E_{21}E_{12} = BM;$$

here $k_1 > 2,\dots,k_p$ are the numbers of those columns which have 1 in the first row and 0 in the second row, $k_p+1,\dots,k_q$ are the numbers of those columns which have 0 in the first row and 1 in the second row, and $k_q+1,\dots,n$ are the numbers of those columns which have 1 in the first two rows.

We get $\widetilde{B}^{-1} = M^{-1}B^{-1}$. The last matrix is obtained from B^{-1} by adding some rows to the first and second rows of it. Since the matrices $\widetilde{B}^{-1}$ and B^{-1} are symmetric, then $\widetilde{B}^{-1}$ might differ from B^{-1} only by the four elements located in the first two rows and columns. So we have to prove the equality $b^{12} = \widetilde{b}^{12}$, $B^{-1} = (b^{ij})$, $\widetilde{B}^{-1} = (\widetilde{b}^{ij})$. We have

$$b^{12} = \det\begin{pmatrix} 1 & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{1}^\top \\ \mathbf{0} & A_0+E & A_1 & A_2 & A_3 \\ \mathbf{1} & A_1^\top & A_4+E & A_5 & A_6 \\ \mathbf{0} & A_2^\top & A_5^\top & A_7+E & A_8 \\ \mathbf{1} & A_3^\top & A_6^\top & A_8^\top & A_9+E \end{pmatrix}$$

$$= \det\begin{pmatrix} 1 & \mathbf{0}^\top & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{1}^\top \\ \mathbf{0} & A_0+E & A_1 & A_2 & A_3 \\ \mathbf{0} & A_1^\top & A_4+E & A_5+(1) & A_6+(1) \\ \mathbf{0} & A_2^\top & A_5^\top & A_7+E & A_8 \\ \mathbf{0} & A_3^\top & A_6^\top & A_8^\top+(1) & A_9+E+(1) \end{pmatrix}$$

$$= \det\begin{pmatrix} A_0+E & A_1 & A_2 & A_3 \\ A_1^\top & A_4+E & A_5+(1) & A_6+(1) \\ A_2^\top & A_5^\top & A_7+E & A_8 \\ A_3^\top & A_6^\top & A_8^\top+(1) & A_9+E+(1) \end{pmatrix},$$

$$\widetilde{b}^{12} = \det\begin{pmatrix} 1 & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{0}^\top & \mathbf{1}^\top \\ \mathbf{0} & A_0+E & A_1 & A_2 & A_3 \\ \mathbf{0} & A_1^\top & A_4+E & A_5+(1) & A_6+(1) \\ \mathbf{1} & A_2^\top & A_5^\top+(1) & A_7+E & A_8+(1) \\ \mathbf{1} & A_3^\top & A_6^\top+(1) & A_8^\top+(1) & A_9+E \end{pmatrix}$$

$$= \det\begin{pmatrix} 1 & \mathbf{0}^\top & \mathbf{1}^\top & \mathbf{0}^\top & \mathbf{1}^\top \\ \mathbf{0} & A_0+E & A_1 & A_2 & A_3 \\ \mathbf{0} & A_1^\top & A_4+E & A_5+(1) & A_6+(1) \\ \mathbf{0} & A_2^\top & A_5^\top & A_7+E & A_8 \\ \mathbf{0} & A_3^\top & A_6^\top & A_8^\top+(1) & A_9+E+(1) \end{pmatrix}$$

$$= \det\begin{pmatrix} A_0+E & A_1 & A_2 & A_3 \\ A_1^\top & A_4+E & A_5+(1) & A_6+(1) \\ A_2^\top & A_5^\top & A_7+E & A_8 \\ A_3^\top & A_6^\top & A_8^\top+(1) & A_9+E+(1) \end{pmatrix} = b^{12}.$$

We have proved that $B^{-1} \sim_D \widetilde{B}^{-1}$.

If A and $\widetilde{A}$ are related by the local complementation at the first element, then

$$B = A + E = \begin{pmatrix} 0 & \mathbf{0}^\top & \mathbf{1}^\top \\ \mathbf{0} & A_0 + E & A_1 \\ \mathbf{1} & A_1^\top & A_2 + E \end{pmatrix}$$

and

$$\widetilde{B} = \begin{pmatrix} 0 & \mathbf{0}^\top & \mathbf{1}^\top \\ \mathbf{0} & A_0 + E & A_1 \\ \mathbf{1} & A_1^\top & A_2 + (1) + E \end{pmatrix} = (A(G_1) + E)E_{1m}E_{1(m+1)} \dots E_{1n};$$

here the numbers $m, m+1, \dots, n$ correspond to the numbers of columns containing 1 in the first row.

We get $\widetilde{B}^{-1} = E_{1n} \dots E_{1m} B^{-1}$. The matrix $\widetilde{B}^{-1}$ is obtained from B^{-1} by sum of the rows with numbers from m to n to the first row of it. Since the matrices $\widetilde{B}^{-1}$ and B^{-1} are both symmetric, then $\widetilde{B}^{-1}$ might differ from B^{-1} only by the first diagonal element. So we have proved $B^{-1} \sim_D \widetilde{B}^{-1}$.

If A and $\widetilde{A}$ are related by pivot operations and local complementations, then, by applying two preceding cases consequently, we get $(A+E)^{-1} \sim_D (\widetilde{A}+E)^{-1}$.

(2) If $B \sim_D \widetilde{B}$ and $\det B = \det \widetilde{B} = 1$, then, by using Lemmas 7.14 and 7.15, we get $B^{-1} + E \sim_C \widetilde{B}^{-1} + E$. Using Lemma 7.13, we see that there exists some B with $\det B = 1$ in each class $[C]_D$. So we can define the inverse map $\upsilon^{-1} \colon \mathfrak{L}(n) \to \mathfrak{G}(n)$ by $\upsilon^{-1}([C]_D) = [B^{-1} + E]_C$. $\square$

Remark 7.16. The map υ gives rise to an equivalence between the set of homotopy classes of looped graphs, see [293], and the set of graph-knots, see [129, 130]. We shall address this question in Chap. 9.

If we consider symmetric matrices realizable by chord diagrams, then the corresponding elements from $\mathfrak{G}(n)$ and $\mathfrak{L}(n)$ are just framed 4-graphs up to *mutation*, see [58, 106]. Our isomorphism υ gives us a correspondence between different ways to define framed 4-graphs.

7.9 A proof of Vassiliev's conjecture

Each Vassiliev invariant of order n of classical knots is a Vassiliev invariant of order one for singular knots with $(n-1)$ intersection points (the inverse statement, generally speaking, is not true). As it turned out [306], the investigation of Vassiliev invariants of order one of $(n-1)$-singular knots gives an important information about combinatorial formulae for invariants of finite type of classical knots.

The famous Goussarov theorem [114] asserts that combinatorial formulae exist for all invariants of finite order of classical knots; herewith combinatorial formulae can have fractional coefficients; for more details see [114, 306].

For defining the structure of the cohomology of the space of singular knots and solving the problem of whether there exists Polyak–Viro combinatorial formulae with integer coefficients for a given Vassiliev invariant, Vassiliev [306] formulated the following conjecture, which is formulated here as a theorem.

Theorem 7.12. *A framed* 4*-graph is not embedded in the plane with preserving the* A*-structure if and only if it contains two cycles without common edges and with precisely one crossing point.*

Remark 7.17. Here by a *cycle* we mean a sequence of pairwise distinct edges $e_0, e_1, \ldots, e_n = e_0$ such that edges e_{i-1}, e_i have a common vertex v_i and the vertices v_i, v_{i+1} (which may coincide) are connected by the edge e_i, i.e. we do not require for a cycle to be an Euler tour.

In other words, the theorem asserts that for a given A-structure a B-structure giving a flat atom exists if and only if the *Vassiliev obstruction* (two cycles with exactly one crossing point) is absent.

We shall write down cycles by a set of sequence of edges.

Definition 7.23. By a *crossing point* of two cycles $e_0, e_1, \ldots, e_n = e_0$ and $e'_0, e'_1, \ldots, e'_m = e'_0$ without common edges we mean a common vertex $v_i = v'_j$ for some i, j such that the edges e_{i-1}, e_i are opposite (it follows that the edges e'_{j-1}, e'_j are also opposite).

A *self-crossing point* of a cycle $e_0, e_1, \ldots, e_n = e_0$ of length n is a vertex $v_i = v_j$ $(i \neq j \pmod{n})$ of this cycle such that the edges e_{i-1}, e_i are opposite. Later on the subscripts of edges and vertices of a cycle of length n are taken modulo n.

The goal of the present section is a proof of Theorem 7.12, see also [224]. Note that the methods applied here are closely connected with those methods which we use under the construction of the Khovanov complex for virtual knots. Namely, we use graphs with the A-structure and B-structure (atoms), their orientability (the source–sink structure), and d-diagrams.

In what follows, all framed 4-graphs are assumed to be connected. The case of disconnected graphs can obviously be reduced to this.

We note that one half of the conjecture is obvious: If a graph is embedded in the plane (with preserving the A-structure), then it cannot have two cycles with only one crossing. This follows from the fact that the intersection number of two smooth generic curves in the plane is even.

Furthermore, if there are cycles U_1 and U_2 with precisely one crossing point, then there are simple cycles U_1' and U_2' with the same unique crossing point. These can be obtained from U_1 and U_2 as follows. Suppose, for example, that U_1 is non-simple. Consider a self-crossing point v of U_1. It is easy to see that U_1 is divided by the vertex v into two cycles U_1' and U_1'' such that one of them (say, U_1') has precisely one crossing point with U_2 (the same one as U_1). A simplifying transformation replaces U_1 by U_1' (see Fig. 7.29). We can go on simplifying until both U_1 and U_2 become simple.

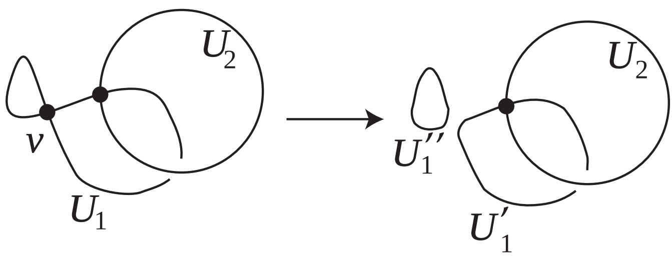


Fig. 7.29 Simplifying the cycle U_1.

We shall prove the non-obvious half of the conjecture without assuming that the graphs are simple. Namely, we shall prove the following theorem.

Theorem 7.13. *Let H be a framed 4-graph not embedded in the plane with preserving the A-structure. Then H contains two cycles U_1, U_2 without common edges such that the number of crossing points of U_1 and U_2 is equal to unity.*

In what follows, it will be important for us whether a framed 4-graph has the source–sink structure. We shall define orientations of edges of the graph and investigate whether this orientation gives the source–sink structure at each vertex.

An important property of d-diagrams is the possibility for embedding them in the plane (as graphs).

Lemma 7.16. *Suppose that a framed 4-graph H does not satisfy the source–sink condition. Then H contains two cycles U_1 and U_2 without common edges and with precisely one crossing.*

Proof. Consider a rotating circuit U of H and orient an edge e_i of U from the vertex v_i to the vertex v_{i+1}, $i = 1, 2, \ldots$. We suppose that the graph H does not satisfy the source–sink condition. The circuit U defines an orientation of every edge of H. By hypothesis, there is a vertex where the source–sink condition does not hold for this orientation. Let $v_j = v_k$ be this vertex. We consider two cycles $U_1 = e_j, e_{j+1}, \ldots, e_{k-1}$ and $U_2 = e_k, e_{k+1} \ldots, e_{j-1}$ obtained by splitting U at the point $v_j = v_k$. Both cycles are rotating at any other vertex (since they coincide locally with U). Thus the number of crossing points between these two cycles does not exceed 1. The only point where a crossing may occur is $v_j = v_k$. We claim that it is a crossing point. To prove this, we must check that the cycle U_1 passes from an edge to the opposite edge at the point $v_j = v_k$. At this point the cycle U_1 passes from the half-edge e_{k-1} to the half-edge e_j. At this point four half-edges e_{j-1}, e_j, e_{k-1}, e_k meet. The edge e_k is not opposite to the edge e_{k-1} by the construction of the circuit U. Furthermore, the edge e_{j-1} cannot be opposite to the edge e_{k-1} since otherwise the source–sink condition holds at $v_j = v_k$: we would have two opposite incoming edges e_{k-1} and e_{j-1} and two emanating edges e_k and e_j. This contradicts our assumption. Thus, the point $v_j = v_k$ is the unique crossing point of the cycles U_1 and U_2. □

The following lemma is central.

Lemma 7.17. *Suppose that H is a framed 4-graph and, for any two cycles U_1 and U_2 without common edges, the number of transversal crossing points (in the sense of the A-structure) is not equal to 1. Then for any rotating circuit U of the graph H the chord diagram corresponding to this circuit is a framed d-diagram.*

Remark 7.18. The claim of Lemma 7.17 means the absence of the obstruction formulated in the Vassiliev conjecture.

Proof of Lemma 7.17. We start with the following fact [203]. A chord diagram is a d-diagram if and only if it has no subdiagram Δ_{2n+1} of "$(2n+1)$-gon type", i.e. a chord diagram with $2n+1$ chords ($n > 0$) such that the chord j is linked only with chords $j+1$ and $j-1$ (the enumeration being taken modulo $2n+1$), see Fig. 7.30. We note that for every n there is a unique chord diagram with these properties.

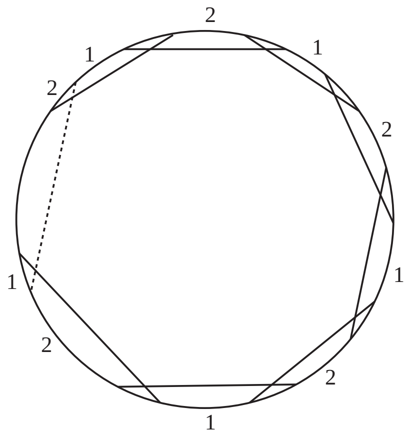


Fig. 7.30 The chord diagram Δ_{2n+1} and its cycles 1 and 2.

Suppose that the hypotheses of the lemma hold. By Lemma 7.16, the framed 4-graph H satisfies the source–sink condition.

Thus it suffices to prove that if H satisfies the source–sink condition and has a $(2n+1)$-gon as a subdiagram, then it has two cycles without common edges and with precisely one crossing point.

Arcs of the chord diagram correspond to edges of the graph (see Definition 4.4), and chords correspond to vertices. If the graph satisfies the source–sink condition, then, for each chord with endpoints X, Y, the arc entering X is opposite to the arc entering Y. In what follows we say that an arc of the chord diagram belongs to a cycle if the corresponding edge of the framed 4-graph belongs to the cycle.

The further proof of Lemma 7.17 is the following. For every n we explicitly construct two cycles U_1, U_2 consisting of arcs of the chord diagram Δ_{2n+1}. If the chord diagram D contains Δ_{2n+1} as a subdiagram, then every arc of Δ_{2n+1} is divided into arcs of D by endpoints of chords. In this case, our cycles are also subdivided. The edges corresponding to arcs of Δ_{2n+1} are divided into edges corresponding to arcs of D. This splitting gives rise to new vertices corresponding to endpoints of chords belonging to D but not to Δ_{2n+1}. However, these vertices do not affect the number of crossing points of the cycles in question because the cycles do not pass from an edge to the opposite edge at these vertices (by construction). Thus, even if such a vertex occurs in both cycles, it cannot be a crossing point.

An example of two such cycles for a $(2n+1)$-gon is shown in Fig. 7.30.

The arcs corresponding to the first cycle are marked by 1, and those corresponding to the second one by 2. The chord corresponding to the only intersection point of these cycles is shown by a dotted line. □

The following lemma is proved in the paper [158] but in another formulation.

Lemma 7.18. *Let a framed* 4*-graph* H *satisfy the source–sink condition. Suppose that the chord diagram* D *corresponding to some circuit of* H *is a d-diagram. Then the graph* H *admits an embedding in the plane which preserves the* A*-structure.*

The proof is illustrated in Fig. 7.31. In the middle part of the figure, it is shown how to recover the neighborhood of a vertex of the framed 4-graph from a chord. In the lower part we show that, in the case of a d-diagram, the corresponding framed 4-graph is embeddable in the plane together with the A-structure.

Let us prove Lemma 7.18.

***Proof of Lemma* 7.18.** We consider a d-diagram D and embed it in the plane as follows. Split the set of chords of D into two subsets F_1, F_2 in such a way that no two chords of the same subset are linked. (We choose an arbitrary splitting with this property.) Then we consider the standard embedding of the circle in the plane and distribute the endpoints of the chords in such a way that no two of them form the endpoints of a diameter. We locate the chords of the first (respectively, second) subset inside (respectively, outside) the circle. This can be done without intersections if we map the chords of the first subset to intervals and those of the second to the images of such intervals under inversion in the circle.

We orient the circle in counterclockwise manner. Every chord c connects two points X and Y of the circle. We consider the following points of the circle: $X_1 = X - \varepsilon$, $X_2 = X + \varepsilon$, $Y_1 = Y - \varepsilon$, $Y_2 = Y + \varepsilon$. Here ε is a small number (angle) and the operations $+$ and $-$ mean rotations through $\pm\varepsilon$. We now delete the chord c and the arcs $[X_1, X_2]$ and $[Y_1, Y_2]$. We replace these arcs by two (curvilinear) intervals $[X_1, Y_1]$ and $[X_2, Y_2]$ in such a way that they intersect transversely at exactly one point Z (say, at the midpoint of the deleted chord). Having done for all the chords of D, we get a 4-valent graph G.

By construction, the graph G is isomorphic to the graph H. It remains to show that they are isomorphic as framed 4-graphs.

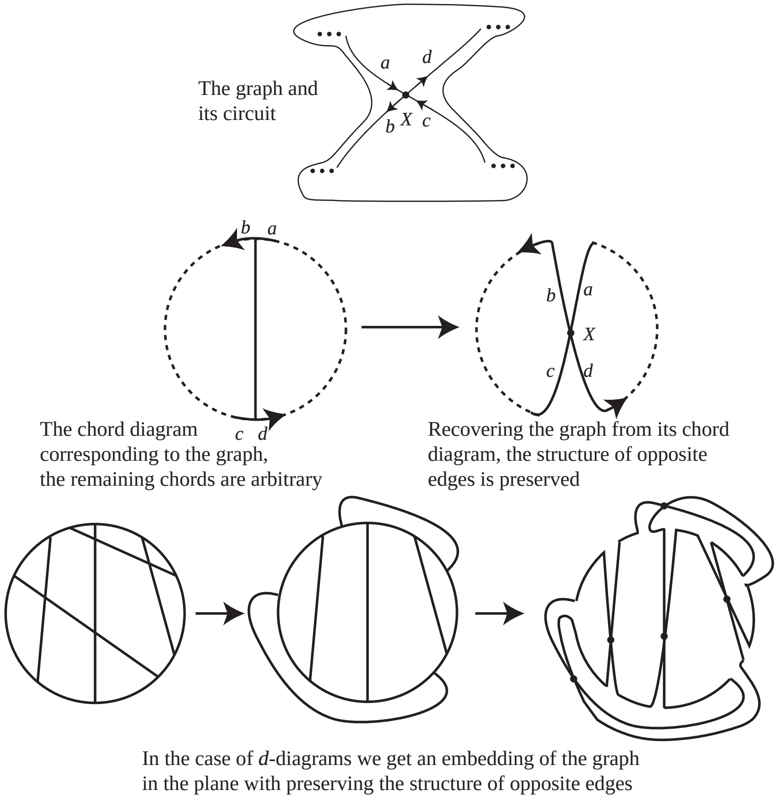


Fig. 7.31 Recovering the graph from a d-diagram.

Since H satisfies the source–sink condition, we see that the resulting embedding is a realization, that is, the edge corresponding to $[X_1, Z]$ is opposite to the edge corresponding to $[Z, Y_1]$ in the framed 4-graph H. Indeed, since H satisfies the source–sink condition, every circuit of H can be oriented with respect to this structure: Every edge entering a vertex is followed by the (non-opposite) edge emanating from it.

It remains to mention that, for any rotating circuit of G, the edge $[Z, X_2]$ follows directly after $[X_1, Z]$ if and only if $[Z, Y_2]$ follows $[Y_1, Z]$. This follows from the fact that every rotating circuit of a 4-valent graph is approximated by an embedding of the circle. □

By combining Proposition 7.4 and Lemmas 7.16–7.18, we obtain a proof of the Vassiliev conjecture.

Lemma 7.17 shows that if there are no obstructions (pairs of cycles with only one crossing point), then the chord diagram corresponding to *any circuit* is a d-diagram. On the other hand, Lemma 7.18 requires only the source–sink condition and the existence of a d-diagram corresponding to *some circuit*. We actually have the following assertion for framed 4-graphs satisfying the source–sink condition (see also Corollary 7.3).

Statement 7.2. *If the chord diagram corresponding to some rotating circuit of a framed* 4*-graph is a d-diagram, then the same holds for chord diagrams corresponding to all rotating circuits.*

As an example, let us consider the framed 4-graph shown in the upper-left corner of Fig. 7.32. This graph is not embedded in the plane with preserving the A-structure. The rotating circuit e_1, e_2, e_3, e_4 of its edges (shown in the upper-right corner of the same figure) generates a chord diagram with two linked chords. This is a d-diagram (but not a framed d-diagram!). It follows that the embedded graph (the lower-left corner of Fig. 7.32) is isomorphic to the original framed 4-graph. However, the circuit e_1, e_2, e_3, e_4 does not satisfy the source–sink condition at the vertex v_1 since the edges e_1 and e_4 are opposite: e_1 goes into v_1 and e_4 goes out. Therefore, the graph in the lower-left corner has another framing at v_1: the edges e_1 and e_3 are opposite (in contrast to the original graph, where e_1 and e_4 are opposite).

Note that the criterion given above provides a fast (quadratic) algorithm to determine whether a framed 4-graph is embeddable in the plane with preserving the A-structure. If this is not the case, then the algorithm gives two cycles with precisely one crossing point. Here we mean complexity with respect to the number of vertices. Here are the main steps of the algorithm.

Enumerate the edges and vertices of the graph arbitrarily. For each vertex, write down all edges incident to it and indicate which edges are opposite.

The edge enumeration has linear complexity, and so does the enumeration of the vertices and memorizing the information about the edges incident to each vertex.

Then we construct a rotating circuit of the graph. This operation is also linear.

Having a rotating circuit, we can check whether it satisfies the source–sink condition at every vertex. If this is not the case, then we get two cycles with a unique crossing point by Lemma 7.16. This operation is linear.

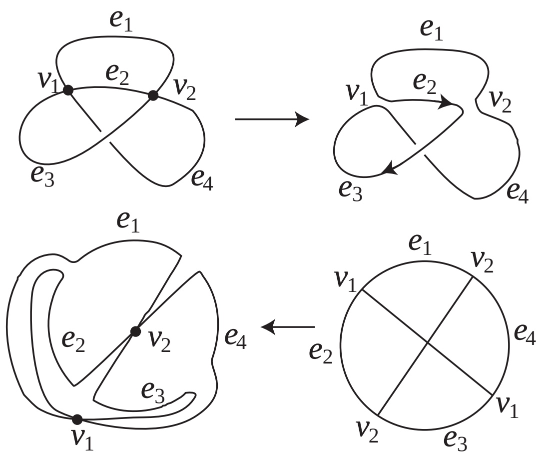


Fig. 7.32 Non-existence of a source–sink structure yields non-embedding with the A-structure.

Suppose that the rotating circuit defines a source–sink structure. We construct the corresponding diagram. The operation of constructing the chord diagram is linear. The operation of determining which pairs of chords are linked has quadratic complexity (one must consider all pairs of chords). Thus we obtain the adjacency matrix of the chord diagram or, equivalently, the intersection graph of the chord diagram. Having done all this, we can restate the original question (whether the original diagram is a d-diagram) as follows: Is the intersection graph bipartite?

This question can be answered in quadratic time. Moreover, if the graph is not bipartite, then quadratic time is sufficient to find a cycle of odd length, that is, in terms of chord diagrams, a $(2n+1)$-gon. Having done this, we can construct a pair of cycles with a unique crossing point according to Lemma 7.17. This operation has linear complexity. There are other criteria of embeddability of a framed 4-graph in the plane with preserving the A-structure, see, e.g. [47, 48].

7.10 Embeddings of framed 4-graphs into 2-surfaces

In this section we consider *framed* 4-graphs and investigate the question about an estimate of the genus of 2-surfaces (by the genus of a non-orientable closed surface we mean $(2-\chi)/2$, where χ is the Euler characteristic), which framed 4-graphs can be embedded in. By an *embedding* we mean an embedding such that the formal structure of opposite edges coincides with the structure of opposite edges induced by the embedding. We restrict the case when the complement to the graph in the surface represents a disjoint union of 2-cells. Later on, all graphs are assumed to be connected.

Among all embeddings the class of $\mathbb{Z}_2$-homology trivial embeddings is natural, this class is characterized in the following way: If a framed 4-graph H is $\mathbb{Z}_2$-homology trivial embedded in the surface S, then cells from $S \setminus \Gamma$ admit a checkerboard coloring. Any embedding of a framed 4-graph can be reduced to a checkerboard colorable embedding by means of an embedding of the corresponding covering.

Determining the minimal genus of a surface which the framed 4-graph is embedded in (by checkerboard manner) can be decided by considering all possible gluings of black and white cells (it is done at 2^n times, where n is the number of vertices). In [92] an algorithm is given, which defines the genus of an embedding for an arbitrary graph in a polynomial time over the number of edges. This algorithm can be easily adopted for embeddings of framed 4-graphs, however, constants are very large and, therefore, the algorithm is ineffective in this case.

In the section we both reformulate an embeddability criterion into the language of matrices for three cases: $\mathbb{R}^2$, $\mathbb{R}P^2$, K^2 and also consider *generating functions* describing all embeddings of a given framed 4-graph. It turns out that this has an interesting connection to knot theory.

Let H be a connected framed 4-graph, and U be a rotating circuit of H. Denote by $A(D_U)$ the adjacency matrix of the chord diagram D_U corresponding to U.

The main result is the following theorem.

Theorem 7.14 ([232, 238]). *A framed 4-graph H with n vertices admits a checkerboard colorable embedding in a minimal (with respect to genus) surface with genus g if and only if for a rotating circuit U of H the set of indices of the matrix $A(D_U)$ can be split, $\{1, \dots, n\} = I \sqcup J$, such that the sum of ranks of the square block matrices is equal to $2g$, i.e. $\mathrm{rank}_{\mathbb{Z}_2} A(D_U)_I + \mathrm{rank}_{\mathbb{Z}_2} A(D_U)_J = 2g$ (a block matrix is defined by choosing rows and columns corresponding to the chosen set of indices). Moreover, all surfaces, which the graph H can be embedded in with checkerboard coloring, are orientable if all diagonal elements of the matrix $A(D_U)$ are zero, and non-orientable if there exists an element on the diagonal equal to one.*

Proof. Using the language of atoms, the main problem is reformulated in the following manner. Let a framed 4-graph H be given. We should choose at each vertex a rule of pasting black and white cells such that the genus of the obtained surface is minimal. For a framed 4-graph with n vertices there exist 2^n atoms.

Consider the set $\mathfrak{A}$ of all atoms corresponding to H, and denote by $f(H)$ the generating function:

$$\sum_{\mathrm{At}\in\mathfrak{A}} t^{g(\mathrm{At})},$$

where $g(\mathrm{At})$ is the genus of an atom At.

Theorem 7.14 follows from the following more general theorem.

Theorem 7.15. *For any rotating circuit U of H the equality $f(H) = F(D_U)$ holds, where F is the function on chord diagrams (not depending on a circuit) defined by the adjacency matrix of the chord diagram D as follows:*

$$F(D) = \sum_{I\sqcup J=\{0,\dots,n\}} t^{(\mathrm{rank}\,A(D)_I+\mathrm{rank}\,A(D)_J)/2}.$$

It turns out that the function F defined on framed chord diagrams is connected with *Vassiliev invariants of knots* [304]. Namely, *the restriction of F on chord diagrams with framing* 0 *satisfies the $4T$-relation* (see, e.g. [17]). This claim follows from straightforward checking.

Theorem 7.15 follows from the following arguments. Assume that there exists a checkerboard colorable embedding of H in a surface S. Fix a rotating circuit U of H. It defines the map $u\colon S^1 \to S$ which is approximated by the embedding (one should correct it in neighborhoods of ends of chords). This embedding is obviously *splitting*: White cells lie on one side with respect to the circle image, and black cells lie on the other side. Thereby chords of D_U are partitioned into two sets: chords lying in the "black side" and chords lying in the "white side". Thus, we get two chord diagrams D_{b} and D_{w} having the same circle as the chord diagram D_U, herewith black chords are related to D_{b}, and white chords are related to D_{w}. To determine the genus of the surface S we have to sum two genera of the surfaces with boundary which are obtained from S by dividing it with the circle U. It is clear that these surfaces are defined by the framed chord diagrams D_{b} and D_{w}. Therefore, the genus of the surface is defined as the sum of values of some functions on chord diagrams, and the generating function for determining all the genera of all the surfaces is obtained by summing over all partitioning of chords into two sets.

To obtain the "half" D_{b} (or D_{w}) of the desired surface we have to attach chords to a disc, paste boundary components of the obtained manifold by discs and delete discs. Actually, boundary components correspond to black and white cells of the atom which we are constructing.

After that the claim of Theorem 7.14 immediately follows from the argument given above: We determine the Euler characteristic of the atom by means of Theorem 7.8 by remembering that the surgery is performed along two sets of chords corresponding to the partition of the chord diagram. □

From the theorem one can easily get fast (polynomial) criteria of checkerboard colorable embeddability of a graph in the sphere S^2, the projective plane $\mathbb{R}P^2$ and the Klein bottle K^2. Another approach (connected with solving equations) for solving the same problems was suggested in [197] (see also [47, 48, 64, 198, 224, 270]).

Let us describe how one can detect whether a given graph is embedded in the Klein bottle or projective plain. Without loss of generality we shall consider *connected* chord diagrams, i.e. those with connected intersection graphs. It is obvious that the genus of a surface is additive with respect to the connected sum of chord diagrams, and the corresponding generating function is multiplicative.

In the case of projective plane we have to find a chord with framing 1. Further, all chords with framing 1 have to be linked with it in order that the rank of the corresponding submatrix is not bigger than one. After that two sets of chords have to be constructed as follows. Chords from the first set contain all chords with framing 1 linked with each other, and also chords with framing 0 not linked with each other and with chords having framing 1. The second set contains the remaining chords with framing 0 which are not linked pairwise. Further algorithm repeats verbatim the algorithm for recognizing d-diagrams. In the case of recognizing embeddability in the Klein bottle we shall need the following obvious lemma.

Lemma 7.19. *Let a framed* 4*-graph H be checkerboard colorable and embedded in the Klein bottle, and let U be a rotating circuit of H. Then either the circuit U partitions the Klein bottle into Möbius bands, or there exists a chord with framing* 1 *such that the circuit U' obtained from U by changing it at the vertex corresponding to the chord partitions the Klein bottle into Möbius bands.*

After that it is necessary to construct a partition of the adjacency matrix for one of the chord diagrams D_U and $D_{U'}$ into two sets, each of them has rank one. The algorithm of the partition and verification repeats verbatim the algorithm of recognizing d-diagrams, only in the case of two chords with framing 1 it is necessary to replace the word "intersection" with the word "non-intersection" and vice versa. One has to change the adjacency of the intersection graph for vertices with framing 1. The case when the intersection graph is disconnected is easily reduced to the checking of each of its components.

Remark 7.19. Note that so far no good algorithm has been found for determining embeddability of a graph in the torus.

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Chapter 8

Parity in Knot Theory: Free-Knots: Cobordisms

8.1 Introduction

It has been known from Gauss' time that a knot can be encoded by a chord diagram with some additional information, the Gauss diagram. Let a knot and its Gauss diagram be given. We call a chord of the Gauss diagram *even* if the number of chords linked with it, is even, and *odd* otherwise (we consider a chord as unlinked with itself). In this case, we say that we have the *Gaussian parity* (see the general definition of a parity below). For chords a, b we write $\langle a, b\rangle = 0 \in \mathbb{Z}_2$ if a, b are unlinked, and $\langle a, b\rangle = 1 \in \mathbb{Z}_2$ if they are linked.

As was mentioned by Gauss, the Gauss diagram of a classical knot diagram has only even chords [47, 48, 270].

The Reidemeister moves for classical diagrams are naturally rewritten in the language of Gauss diagrams. Besides, there are Gauss diagrams not realizing classical knots. In particular, Gauss diagrams having odd chords do not realize classical knots.

A virtual knot [158] appears as a natural generalization of a classical knot. This is an equivalence class of Gauss diagrams modulo formal Reidemeister moves. It turns out that the existence of odd chords for virtual knots allows one to prove important structural theorem.

Odd crossings (odd chords) were first used for constructing invariants of virtual knots by Kauffman [161]: In Chap. 1 *odd self-linking index* was constructed for virtual knots; this index was equal to 0 for classical knots.

The set of classical knots is a proper subset of the set of virtual knots [114]. It turns out that there exists thorough simplification of virtual knots, *free knots* (see Definition 8.2) which are obtained by forgetting the information about directions and signs of chords on Gauss diagrams of virtual diagrams. Free knots are the main object of this chapter for studying; they are closely connected with graph-links studied in [129, 130, 132, 293]. Chapter 9 is devoted to graph-links.

Note that free knots and links are also called *homotopy classes of Gauss words and phrases* [107, 301, 302]. This is due to the fact that Gauss diagrams can be encoded by words. But if we consider a link with many components, then the Gauss diagram on many circles is associated with it where the number of the circles of the diagram is equal to the number of the components of the link. Gauss diagrams on many circles are in turn encoded by Gauss phrases. Thus, any statement about Gauss diagrams can be rewritten in the language of words and phrases.

Let a framed 4-graph K be introduced by its Gauss diagram $D_G(K)$. Having a Gauss diagram we can construct the abstract framed 4-graph, where edges of the graph are associated with arcs of the chord diagram, and vertices are associated with chords of the chord diagram. Each vertex is incident to four half-edges corresponding to arcs attaching to two ends of a chord. Arcs incident to the same chord end will correspond to the (formally) opposite half-edges. We say that a Gauss diagram is *odd*, if all its chords are odd. We say that a Gauss diagram is *irreducible* if for any two distinct chords a, b there exists a chord c such that $\langle a, c\rangle \neq \langle b, c\rangle$. The same terminology (odd, irreducible) will be applied to the framed 4-graph corresponding to a Gauss diagram.

One can easily check that we cannot apply Reidemeister moves to an odd irreducible diagram such that these moves decrease the number of crossings (the first and the second ones), and the third Reidemeister move, i.e. we can only complicate this diagram for "one step".

It turns out that the following theorem is true.

Theorem 8.1. *An odd irreducible Gauss diagram D is minimal, i.e. any Gauss diagram D' representing the free knot generated by D has more crossings than D does.*

This theorem disproves *Turaev's conjecture* [302] stating that all free knots are trivial. The first proof of Theorem 8.1 and the development of parity theory appeared in the series of preprints [233–235] (see also [133, 134]).

Examples of non-trivial free knots were independently (a little later) obtained by Gibson in [107].

Moreover, due to Theorem 8.1 one can construct an infinite number of odd irreducible Gauss diagrams such that free knot minimal diagrams corresponding to these Gauss diagrams differ from each other. Therefore, one can prove that the number of equivalence classes of free knots is infinite.

Later we shall prove a stronger statement.

The parity is also used for studying *cobordisms of free links*, see [239]. The parity arguments allowed us to prove the existence of free knots which are non-cobordant to the trivial free knot from combinatorially point of view [131] and from topologically point of view [236]. In [236] the first-named author extended the notion of parity from vertices of framed 4-graphs to double lines of 2-surfaces. A survey of parity can be found in [240] (see also [147, 148] for other applications of parity). It is worth mentioning that for classical knots there exists only the trivial parity.

This chapter is organized as follows. In the next section, we shall define free knots and describe a natural set of properties of odd and even crossings and their behavior under the Reidemeister moves. This set of properties can be reformulated as *axiomatics of parity*, i.e. the set of requirements which are satisfied by any abstract parity in knot theory; also, the definition of parity can be reformulated in terms of homology (we shall need this construction in the following section). We shall give other examples of parities satisfying those axioms (different from the parity obtained by counting linking number of chords of Gauss diagrams). Then we shall show that in the case of free knots there exists a unique non-trivial parity, the Gaussian parity.

In Sec. 8.3 we construct a functorial map f: For each knot theory having some parity we shall construct a well-defined map on the set of equivalence classes of knots by "forgetting" odd crossings. By means of f we construct a projecting map setting all virtual knots to virtual knots with orientable atoms and setting a virtual knot with an orientable atom to itself. Some invariants of virtual knots are directly defined for virtual knots with orientable atoms but it is hard to define them for all virtual knots. Examples of such invariants are Khovanov homology, the signature of a virtual knot defined by using the Göritz matrix. Due to the projecting map, we can lift invariants from virtual knots with orientable atoms to all virtual knots.

In Sec. 8.4 we shall construct the invariants $[\cdot]$ and $\{\cdot\}$ which allow one, in particular, to prove a refined version of Theorem 8.1. This allows one to prove non-invertibility of free knots.

Section 8.5 is devoted to Goldman's bracket [110] and Turaev's cobracket [296], that first appeared in the study of homotopy classes of curves on surfaces, and these brackets are, in a natural way, generalized to a simplification of homotopy classes of curves, free knots and links (see also [295]).

Using the results from Secs. 8.4.3 and 8.5 about non-invertibility of free links and Turaev's cobracket, in Sec. 8.6 we prove non-invertibility of free knots. In this section, we also construct different analogues of Goldman's bracket and Turaev's cobracket.

In Sec. 8.7 we use parity and free knots for refining the Kauffman bracket polynomial.

The aim of Sec. 8.8 is to prove that the minimal number of virtual crossings for some families of virtual knots grows quadratically with respect to the minimal number of classical crossings. All previously known estimates for the virtual crossing number [1, 23, 77, 274] were principally no more than linear in the number of classical crossings (or, what is the same, in the number of edges of a virtual knot diagram), and no virtual knot was found with the virtual crossing number greater than the classical crossing number.

Section 8.9 is devoted to cobordisms of free knots. We shall construct a simple invariant which provides a sliceness obstruction for free knots. This obstruction provides a new point of view to the problem of studying cobordisms of curves immersed in 2-surfaces, a problem previously studied by Carter, Turaev, Orr, and others.

8.2 Free knots and parity axioms

Throughout this chapter by a 4-*graph* we mean the following generalization of a 4-valent graph: a 1-cell complex, whose each connected component is homeomorphic either a circle, or a 4-valent graph; by *vertices* we mean only vertices of those components which are homeomorphic to a 4-valent graph, and by *edges* we mean both edges of the 4-valent graph and circle-components (the latter we call *cyclic edges*).

Let us consider a (non-oriented) chord diagram D. Then the corresponding 4-graph $G(D)$ with a unique unicursal component is constructed as follows (see Definition 1.5, we consider the Gauss circuit approach). If the set of chords of D is empty, then the corresponding graph will be G_0. Otherwise, the edges of the graph are in one-to-one correspondence with arcs of the chord diagram, and vertices are in one-to-one correspondence with chords of the chord diagram. The arcs incident to the same chord end, correspond to the half-edges which are formally opposite at the vertex corresponding to the chord.

The inverse procedure (the construction of a chord diagram from a framed 4-graph with one unicursal component) is obvious. In this case each framed 4-graph with one unicursal component can be represented by a quotient set obtained from a circle by identifying some pairs of points. Connecting points which are identified by a chord we get the chord diagram.

Further, for a framed 4-graph G with a unique unicursal component we define a pairing of vertices v_1, v_2 (over $\mathbb{Z}_2$): $\langle v_1, v_2\rangle = \langle a(v_1), a(v_2)\rangle$, where $a(v_i)$ are the chords corresponding to vertices v_i.

8.2.1 *Free links*

A framed 4-graph is a serious simplification of the notion of a knot (link) diagram: At each classical crossing we do not indicate the over/undercrossing structure, moreover, we do not take into consideration the sign of a crossing (or equivalently we forget about the over/undercrossing structure and the cyclic order of half-edges, and remember only the structure of opposite edges).

Our first aim is to study equivalence classes of framed 4-graphs modulo some moves corresponding to Reidemeister moves for knots.

Each of these moves is a transformation of one fragment of a framed 4-graph.

Definition 8.1. *The first Reidemeister move* is an addition/removal of a loop, see Fig. 8.1.

The second Reidemeister move is an addition/removal of a bigon formed by a pair of edges which are adjacent (not opposite) at each of the two vertices, see Fig. 8.2.

The third Reidemeister move is shown in Fig. 8.3.

Definition 8.2. A *free link* is an equivalence class of framed 4-graphs modulo the Reidemeister moves.

It is evident that the number of components of a framed 4-graph does not change after applying a Reidemeister move, so, it makes sense to talk about *the number of components* of a free link.

By a *free knot* we mean a free link with one unicursal component.

The *free unknot* or *free trivial knot* (respectively, the *free n-component trivial link*) is the free knot (respectively, link) represented by G_0 (respectively, by n disjoint copies of G_0).

Remark 8.1. Free knots can be treated as equivalence classes of Gauss diagrams by moves corresponding to the Reidemeister moves.

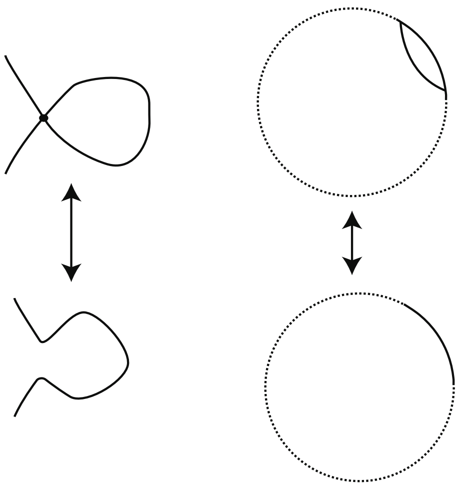


Fig. 8.1 The first Reidemeister move for knots and chord diagrams.

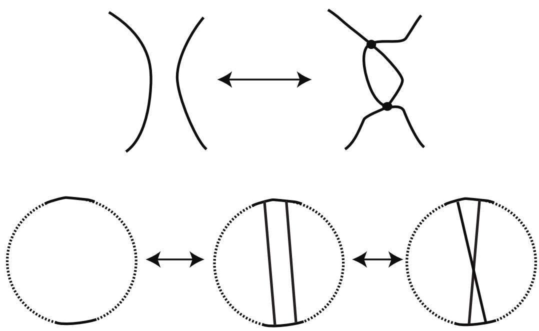


Fig. 8.2 The second Reidemeister move for knots and chord diagrams.

Analogously, one defines *long free knots*; here one should break the only unicursal component and pull its ends to "infinity". For free links (and graphs representing them) moves are considered only in finite domains. More formally, we may consider a framed 4-graph with one unicursal component with a marked edge (say this edge is marked by a vertex situated in the middle of the edge). Defining an equivalence relation we allow only those moves which are performed away from the marked point. If we consider chord diagrams, then the marked point is situated on an arc, and we

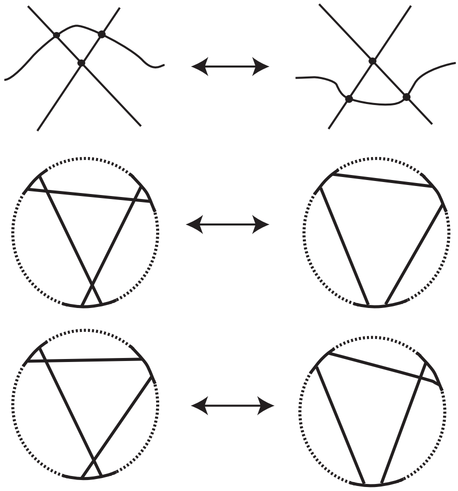


Fig. 8.3 The third Reidemeister move for knots and chord diagrams.

forbid moves on chord diagrams under which ends of chords go through the marked vertex.

A natural question is: What are free knots? Note that there is a classification of flat knots but there is not that of free knots. If we consider a planar framed 4-graph (which originates from a classical knot), then this planar graph can be easily reduced to the graph G_0. It can be easily shown that a one-component framed 4-graph embeddable in the torus (in a way that the original framing coincides with the framing induced from the torus) is also reducible to the graph G_0 (without vertices).

Free knots are closely connected to *flat virtual knots*, i.e. equivalence classes of virtual knots modulo the transformation swapping overcrossing and undercrossing mutually. The latter are equivalence classes of immersed curves in orientable 2-surfaces modulo homotopy and stabilization.

Nevertheless, the equivalence of free knots is even stronger than the equivalence of flat virtual knots: Our free knots do not require any surface. Every time one applies a Reidemeister move to a regular 4-graph, one embeds this graph into a 2-surface arbitrarily (with framing preserved), apply this Reidemeister move inside the surface and then forget the surface again.

Example 8.1. Consider the flat virtual *Kishino knot*, see Fig. 8.4.

It is known that this knot is not trivial as a flat virtual knot. Nevertheless, the corresponding framed 4-graph considered by itself has a bigon formed by two edges which are adjacent in two vertices. Thus, the free knot represented by the flat Kishino knot is trivial.

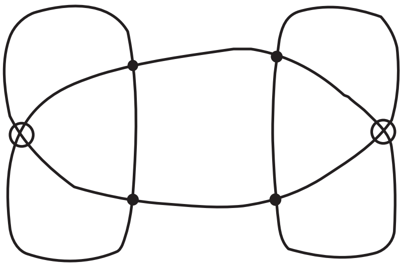


Fig. 8.4 The flat Kishino knot.

The exact statement connecting virtual knots and free knots sounds as follows.

Lemma 8.1. *A free knot is an equivalence class of virtual knots modulo two transformations: crossing switches and virtualizations.*

One may think of a virtualization as a way of changing the immersion of a framed 4-graph in the plane such that the cyclic order of half-edges changes but the structure of opposite half-edges does not.

In contrast to free knots, virtual knots have a *cyclic order* of half-edges at a vertex: We can say not only which half-edges are locally opposite but in which order they appear when we go around a vertex in counterclockwise manner. Moreover, virtual knots have a natural over/undercrossing structure at vertices.

A natural map of "forgetting" the crossing structure takes the set of virtual knots (links) to the set of free knots (links). Therefore, all invariants of free knots give some invariants of virtual knots.

Note that in the case of *free links* it is much easier to find a non-trivial example, i.e. a link which cannot be transformed to the framed 4-graph without vertices.

Indeed, consider the framed 4-graph with one vertex v and two edges a, b, each connecting v to v in such a way that the edge a is opposite to a, and the edge b is opposite to b. This free link is not equivalent to the trivial one because of the following simple *invariant* of two-component free links.

It is easy to see that the parity of the number of crossings formed by the two components is invariant under the Reidemeister moves. Since the number of such vertices is odd for our framed 4-graph, then the free link is not equivalent to the free trivial link.

8.2.2 *The parity axiomatics*

Let $\mathcal{K}$ be a knot. We use the notion of *knot* (in a broad sense) for an equivalence class of diagrams where the equivalence relation is given by moves, i.e. local transformations from some list. In each case, we indicate which list of transformations is under consideration. As a rule, such transformations will correspond to isotopies of knots in a three-dimensional manifold or homotopies of curves on a 2-surface.

Each of the diagrams represents some class of graphs (maybe with an additional structure at vertices), and some 4-valent vertices have a framing. An equivalence relation is given by the Reidemeister moves described above with respect to framed 4-valent vertices. In some cases (for example, in classical knot theory) we impose some restrictions on the moves (for example, we consider moves having a definite over/undercrossing structure).

Examples of such theories are the theory of classical and virtual knots, the theory of flat knots and the theory of braids and tangles (classical or virtual).

Recall that in the case of virtual knots we use the notion of "crossing" only for classical crossings unless otherwise indicated. We ignore the detour move since this move does not change a mutual position of crossings.

Let us define the category $\mathfrak{K}$ of diagrams of the knot $\mathcal{K}$. The objects of $\mathfrak{K}$ are diagrams of $\mathcal{K}$ and morphisms of the category $\mathfrak{K}$ are (formal) compositions of elementary morphisms. By an *elementary morphism* we mean the following:

- an isotopy;
- a Reidemeister move.

Since the number of vertices of a diagram may change under Reidemeister moves (under the first (respectively, second) Reidemeister move the number of vertices is changed by one (respectively, two)), therefore, there is no bijection between the sets of vertices of two diagrams connected by a sequence of the Reidemeister moves. But we want to have any "correspondence" between the vertices after applying moves. We have the natural bijection only for the vertices of diagrams connected by the third Reidemeister move. To construct any connection between two sets of vertices for all cases of the moves we shall introduce the notion of a partial bijection which means just a bijection between the subsets of vertices, corresponding to each other in the two diagrams.

Definition 8.3. A *partial bijection* of sets X and Y is a triple $(\widetilde{X}, \widetilde{Y}, \phi)$, where $\widetilde{X} \subset X$, $\widetilde{Y} \subset Y$ and $\phi\colon \widetilde{X} \to \widetilde{Y}$ is a bijection.

Let us denote by $\mathcal{V}$ the vertex functor on $\mathfrak{K}$, i.e. a functor from $\mathfrak{K}$ to the category, objects of which are finite sets and morphisms are partial bijections. For each diagram K we define $\mathcal{V}(K)$ to be the set of classical crossings of K, i.e. the vertices of the underlying framed 4-graph. Any elementary morphism $f\colon K \to K'$ naturally induces a partial bijection $f_*\colon \mathcal{V}(K) \to \mathcal{V}(K')$.

Now we shall define a parity with coefficients in an arbitrary abelian group. In [233, 234, 237, 240] the parity with coefficients in $\mathbb{Z}_2$ was defined. We extend that notion to the case with an abelian group. Note that one can define a parity valued in a non-abelian group.

Let A be an abelian group.

Definition 8.4. A *parity p on diagrams of a knot $\mathcal{K}$ with coefficients in A* is a family of maps $p_K\colon \mathcal{V}(K) \to A$, $K \in \mathrm{ob}(\mathfrak{K})$ is an object of the category, such that for any elementary morphism $f\colon K \to K'$ the following holds:

(1) $p_{K'}(f_*(v)) = p_K(v)$ provided that $v \in \mathcal{V}(K)$ and there exists $f_*(v) \in \mathcal{V}(K')$;
(2) $p_K(v_1) + p_K(v_2) = 0$ if f is a decreasing second Reidemeister move and v_1, v_2 are the disappearing crossings;
(3) $p_K(v_1) + p_K(v_2) + p_K(v_3) = 0$ if f is a third Reidemeister move and v_1, v_2, v_3 are the crossings participating in this move.

Remark 8.2. The first condition in Definition 8.4 means that a parity respects the partial bijection, i.e. the corresponding crossings of two diagrams are of the same parity.

Of course, a parity has to be the same for isomorphic graphs. For example, if a framed 4-graph has a symmetry (i.e. an isomorphism preserving framing), then crossings obtained from each other under this symmetry should have the same parity.

Note that each knot can have its own group A. So, one can speak of a parity for a given knot.

The initial definition of parity, see [233, 234, 237, 240], imposed some restrictions on the first Reidemeister move. It turns out that this restriction follows from the second and third conditions of Definition 8.4. Namely, we have the following lemma.

Lemma 8.2. *Let p be any parity and K be a diagram. Then $p_K(v) = 0$ if f is a decreasing first Reidemeister move applied to K and v is the disappearing crossing of K.*

Proof. Let us apply the second Reidemeister move g to the diagram K as shown in Fig. 8.5. We have

$$p_{K'}(v_1) + p_{K'}(v_2) = 0, \qquad p_{K'}(g_*(v)) + p_{K'}(v_1) + p_{K'}(v_2) = p_K(v) = 0.$$

□

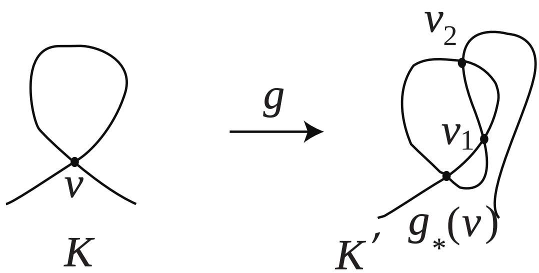


Fig. 8.5 Reduction of the first Reidemeister move to the second and third Reidemeister moves.

Definition 8.5. A *weakened parity* p on diagrams of a knot $\mathcal{K}$ is a family of maps $p_K : \mathcal{V}(K) \to \mathbb{Z}_2$, $K \in \mathrm{ob}(\mathfrak{K})$ is an object of the category, such that for any elementary morphism $f : K \to K'$ the following holds:

(1) $p_{K'}(f_*(v)) = p_K(v)$ provided that $v \in \mathcal{V}(K)$ and there exists $f_*(v) \in \mathcal{V}(K')$;
(2) $p_K(v_1) + p_K(v_2) = 0 \pmod 2$ if f is a decreasing second Reidemeister move and v_1, v_2 are the disappearing crossings;
(3) the number of vertices v_i with $p_K(v_i) = 1$ is not equal to one if f is a third Reidemeister move and v_1, v_2, v_3 are the crossings participating in this move.

Definition 8.6. Let p be a parity with coefficients with $\mathbb{Z}_2$ or a weakened parity on diagrams of a knot $\mathcal{K}$, and let K be any diagram of $\mathcal{K}$. We call a vertex v of K *even* for p if $p_K(v) = 0$, and *odd* otherwise.

Remark 8.3. The weakened parity was also introduced by Gibson in his Ph.D. thesis which is not published. Gibson told us that he had some ideas with respect to the map f, "killing odd crossings" (the definition of f see later).

Let us consider some examples of parities for some knot theories.

8.2.3 *Gaussian parity for free, flat and virtual knots*

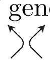
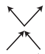
Let $A = \mathbb{Z}_2$ and K be a virtual (flat) knot diagram (respectively, a framed 4-graph with one unicursal component).

To the diagram K we assign the Gauss diagram $D_G(K)$. Recall that a crossing is *even* (respectively, *odd*) if the corresponding chord is linked with even (respectively, odd) number of chords.

Define the map $\mathrm{gp}_K : \mathcal{V}(K) \to \mathbb{Z}_2$ by putting $\mathrm{gp}_K(v) = 0$ if v is an even crossing, and $\mathrm{gp}_K(v) = 1$ otherwise (an odd crossing).

Lemma 8.3 ([233]). *The map* gp *is a parity for free, flat and virtual knots.*

Definition 8.7. The parity gp is called the *Gaussian parity.*

Another way to define the Gaussian parity is the following. Let us consider a virtual (flat) diagram (respectively, a framed 4-graph with one unicursal component) K and its (classical) crossing v. Let us orient K. We can *smooth* the diagram K at the crossing v (without loss of generality we assume that v is ) in two different ways: the way  $\to$  which respects the orientation of K and the way  $\to$  which does not respect the orientation of K. It is easy to see (Fig. 8.6) that the first way of smoothing gives a link with two components, and the second one gives a knot.

In Fig. 8.6 the part of the knot diagram lying outside a neighborhood of the crossing is depicted by a dashed line.

Note that the number of components obtained after a smoothing of the diagram K at the crossing v in a different way does not depend on the orientation of K.

Let us consider a diagram obtained from K by the smoothing with

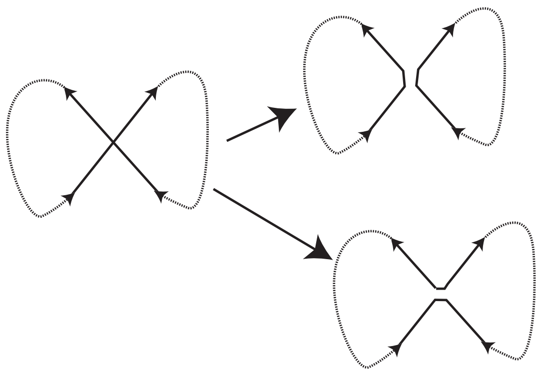


Fig. 8.6 Two smoothings of a crossing.

respect to the orientation (i.e. we get a two-component link). Let us denote the obtained link by $K_1 \cup K_2$. Let us consider those classical crossings of the diagram K at which the component K_1 intersects the component K_2.

If the number of such crossings is even, then we call the crossing v *even*; otherwise we call the crossing v *odd*.

It is evident that the definition of parity in such a manner coincides with the initial definition of the Gaussian parity.

Indeed, the smoothing of the crossing v with respect to the orientation can be treated as a *surgery* of the Gauss diagram along the chord. After that all crossings formed by two components K_1 and K_2 of the link $K_1 \cup K_2$ correspond to chords on the smoothed Gauss diagram on two circles, which connect two circles. Chords of the initial chord diagram corresponding to chords connecting two circles are just chords linked with the chord corresponding to v.

Remark 8.4. If we apply the rule described above to classical knots, then we shall get that each crossing of a knot is even. This does not give us any new information necessary for constructing further invariants and invariant mappings.

Note that chord diagrams having all even chords correspond to virtual knots with orientable atoms, see Theorem 8.6.

In particular, as we shall see later, the property of an atom to be oriented does not depend on the over/undercrossing structure at classical crossings of a virtual diagram, but does depend on the frame, 4-graph, of an atom.

8.2.4 *Two-component classical and virtual links*

Here we call the theory of two-component virtual links just knot theory.

The parity is defined as follows. Let $A = \mathbb{Z}_2$ and $K = K_1 \cup K_2$ be a virtual link diagram with two components. For K we call a classical crossing formed by a single component (K_1 or K_2) *even*, and a classical crossing formed by two components K_1 and K_2 *odd*.

Define the map $p_K : \mathcal{V}(K) \to \mathbb{Z}_2$ by putting $p_K(v) = 0$ if v is an even crossing, and $p_K(v) = 1$ otherwise (an odd crossing).

It is not difficult to show the following lemma.

Lemma 8.4. *The map p is a parity for two-component links.*

8.2.5 *Knots in the solid torus, curves on 2-surfaces*

Let us consider a knot $\mathcal{K}$ in the solid torus B represented as the thickened annulus $S^1 \times I^1 \times I^1$. Knots are represented by their projections on $S^1 \times I^1$ which are obtained by "forgetting" the last factor I^1. We restrict ourselves to the consideration of those knots whose homology class in $H_1(S^1 \times I^1 \times I^1, \mathbb{Z}_2) = \mathbb{Z}_2$ is equal to zero.

For the knot $\mathcal{K}$ (all classical knots lying inside some ball $D^3 \subset B$ are also attributed to this class), we define a parity in the following manner.

Let $A = \mathbb{Z}_2$, and let K be a diagram, v be its crossing. Let us smooth the diagram K at the crossing v with respect to the orientation (see above). We obtain the link L. The following equality of homology classes in $H_1(S^1 \times I^1 \times I^1, \mathbb{Z}_2) = \mathbb{Z}_2$: $[K] = [L] = [K_1 \cup K_2] = [K_1 + K_2]$ is evident. Thus, taking into consideration the equality $[K] = 0$, we get $[K_1] = [K_2] \in \mathbb{Z}_2$.

If $[K_1] = [K_2] = 1 \in \mathbb{Z}_2$, then we call the crossing v *odd*, otherwise we call the crossing *even*.

Define the map $p_K : \mathcal{V}(K) \to \mathbb{Z}_2$ by putting $p_K(v) = 0$ if v is an even crossing, and $p_K(v) = 1$ otherwise (an odd crossing).

From a straightforward check we get the following lemma.

Lemma 8.5. *The map p is a parity for knots in the solid torus.*

Remark 8.5. As a particular case of the parity described above we may consider the parity for the theory of closed braids consisting of even number of strands up to an isotopy of braids and a conjugation.

Each braid can be represented in a natural way by a *tangle*, i.e. a diagram inside $I^1 \times I^1$, and a closed braid can be represented as a diagram in $S^1 \times I^1$.

A method using $\mathbb{Z}_2$-homology can be used in a more general setup.

It is known that virtual knots represent knots in thickened surfaces $S_g \times I$ considered up to isotopy and stabilization.

Let us forget about stabilization and consider two classes of objects: curves in S_g up to homotopy and knots in $S_g \times I$ up to isotopy.

In both cases objects can be encoded by diagrams in general position which represent framed 4-graphs on S_g (in the case of knots, of course, the structure of over/undercrossing and a cyclic order specified at each crossing). The equivalence relation is defined with the help of the Reidemeister moves.

Let us fix a homology class α from $H^1(S_g, \mathbb{Z}_2) = H^1(S_g \times I, \mathbb{Z}_2)$. We shall consider only knots (curves) γ such that $\alpha(\gamma) = 0$.

Then for such a knot (a curve) $\mathcal{K}$ and for its diagrams on S_g we can define the parity of a crossing by smoothing it with respect to the orientation and taking the class α of one of the "halves" K_1 or K_2 obtained after this smoothing.

It is easy to check that we get a parity.

8.2.6 *Parity and homology*

In this section, we consider parity valued in $\mathbb{Z}_2$ for free links and homology of framed 4-graphs. We show how one can define a parity from homology and vice versa, cf. [240]. This reformulation will be useful later for understanding the way in a two-dimensional context: We shall define a parity for double lines of 2-manifolds, see Sec. 8.9.5.

Consider a framed 4-graph K with one unicursal component. The homology group $H_1(K, \mathbb{Z}_2)$ is generated by "halves" corresponding to vertices. For every vertex v we have two halves of the graph, $K_{v,1}$ and $K_{v,2}$, obtained from K by smoothing at this vertex, see Fig. 8.7. If the equivalence class of K (possibly, with some further decorations at vertices) is endowed with a *parity* p with coefficients from $\mathbb{Z}_2$, we may assume that we are given the following cohomology class h (over $\mathbb{Z}_2$): For each of the two halves $K_{v,1}$, $K_{v,2}$ we set $h(K_{v,1}) = h(K_{v,2}) = p(v)$, where $p(v)$ is the parity of the vertex v. Taking into account that for every vertex, the sum of the two corresponding halves is the cycle generated by the whole graph, we have defined a "characteristic" cohomology class h from $H_1(K, \mathbb{Z}_2)$ equal to zero on the cycle represented by K.

Note the following. We can assign to each cycle $c \in H_1(K, \mathbb{Z}_2)$ considered as a subgraph of K, the set of vertices of K which share exactly two non-opposite edges with c. Denote the set of the vertices by $d(c)$.

Let us show by induction on the number of vertices in $d(c)$ that c equals $\sum_i K_{v_i,1}$ up to an addition of the whole K, where v_i form the whole set $d(c)$. If $d(c)$ consists of one vertex, then the statement is evident: One of the two halves coincides with c. Let us suppose that the statement is true for all cycles c' for which the sets $d(c')$ contain the number of vertices less than n. Let c be an arbitrary cycle and the cardinality of $d(c)$ equals n. Let us choose an arbitrary vertex $v \in d(c)$ and consider the cycle c' obtained from c by changing the circuit at v (we move transversally at v, i.e. we move from a half-edge to the half-edge opposite to it). Then the cardinality of $d(c')$ is equal to $n-1$, and c is obtained from c' by adding a half corresponding to the vertex v. We get the validity of the statement.

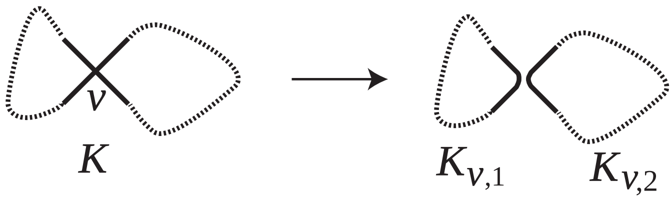


Fig. 8.7 The graphs $K_{v,1}$ and $K_{v,2}$.

Collecting the properties of this cohomology class h and recalling the parity axiomatics we see that:

(1) For every framed 4-graph K we have $h(K) = 0$.
(2) If a framed 4-graph K' is obtained from K by a first Reidemeister move adding a loop, then for every basis $\{\alpha_i\}$ of $H_1(K, \mathbb{Z}_2)$ there exist a basis of the group $H_1(K, \mathbb{Z}_2)$ consisting of one element β corresponding to the loop and a set of elements α'_i naturally corresponding to α_i. Then we have $h(\beta) = 0$ and $h(\alpha_i) = h(\alpha'_i)$ for all i.
(3) Let K' be obtained from K by a second increasing Reidemeister move. Then for every basis $\{\alpha_i\}$ of $H_1(K, \mathbb{Z}_2)$ there exists a basis in $H_1(K', \mathbb{Z}_2)$ consisting of one "bigon" K, the elements α'_i naturally corresponding to α_i and one additional element δ (see Fig. 8.8, left). The bigon equals the sum of two halves having the same parity in $H_1(K, \mathbb{Z}_2)$. Then the following holds: $h(\alpha_i) = h(\alpha'_i)$, $h(K) = 0$.
(4) Let a framed 4-graph K' be obtained from a framed 4-graph K by a third Reidemeister move. Then there exists a graph K'' with one

vertex of degree 6 and other vertices of degree 4. K'' is obtained from either of K or K' by contracting the "small" triangle to the point. This generates the mappings $i\colon H_1(K,\mathbb{Z}_2) \to H_1(K'',\mathbb{Z}_2)$ and $i'\colon H_1(K',\mathbb{Z}_2) \to H_1(K'',\mathbb{Z}_2)$ (see Fig. 8.8, right). Then the following holds: The cocycle h is equal to zero for small triangles, and if for $a \in H_1(K,\mathbb{Z}_2), a' \in H_1(K',\mathbb{Z}_2)$ we have $i(a) = i'(a')$, then $h(a) = h(a')$. Naturally, the equality of cocycle on small triangles to zero follows from the fact that each small triangle represents an element equal to (up to the graph K) the sum of the three halves in $H_1(K,\mathbb{Z}_2)$ from which the number of odd halves is equal to zero or two.

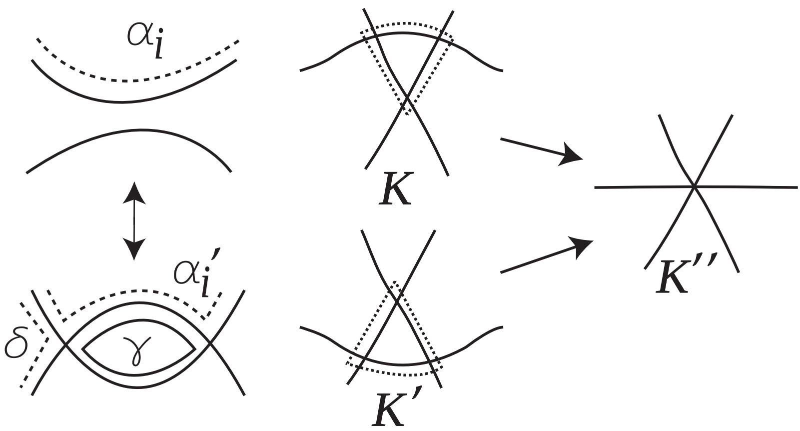


Fig. 8.8 The cohomology condition for Reidemeister moves.

Thus, every parity for free knots generates some "characteristic" $\mathbb{Z}_2$-cohomology class for all framed 4-graphs with one unicursal component, and this class behaves nicely under Reidemeister moves.

The converse is true as well.

Theorem 8.2. *Assume that we are given a certain "universal"* $\mathbb{Z}_2$*-cohomology class for all framed* 4*-graphs satisfying the conditions* (1)–(4) *described above. Then it originates from some parity.*

Indeed, it is sufficient to define the map p as follows. Let K be a framed 4-graph, and v be its vertex. We set $p_K(v)$ to equal the "universal" cohomology class of the corresponding half. The choice of a particular half does not matter, since the value of the cohomology class on the whole graph is zero. One can easily check that our map is a parity.

This point of view allows one to find parities for those knots lying in $\mathbb{Z}_2$-homologically non-trivial manifolds. For more details, see [188].

8.2.7 *The universal parity: The classification of parities for free knots*

In the present section, we shall classify parities for free knots. To do this, we shall introduce the notion of the universal parity for knots, i.e. any concrete parity on a given knot factors through the universal one. The main theorem of the present subsection is the theorem describing the universal parity for free knots.

Definition 8.8. A parity p_u with coefficients in A_u is called *universal* if for any parity p with coefficients in A there exists a unique homomorphism of groups $\rho\colon A_u \to A$ such that $p_K = \rho \circ (p_u)_K$ for any diagram K.

Let us describe a construction of the universal parity in the general case.

Let K be a knot diagram, and let v be its vertex. Denote by $1_{K,v}$ the generator of the direct summand in the group $\bigoplus_K \bigoplus_{v\in\mathcal{V}(K)} \mathbb{Z}$ corresponding to the vertex v of K.

Let A_u be the group

$$A_u = \left(\bigoplus_K \bigoplus_{v\in\mathcal{V}(K)} \mathbb{Z} \right) / \mathcal{R},$$

where $\mathcal{R}$ is the set of relations of four types:

(1) $1_{K',f_*(v)} = 1_{K,v}$ if $v \in \mathcal{V}(K)$ and there exists $f_*(v) \in \mathcal{V}(K')$;
(2) $1_{K,v_1} + 1_{K,v_2} = 0$ if f is a decreasing second Reidemeister move and v_1, v_2 are the disappearing crossings;
(3) $1_{K,v_1} + 1_{K,v_2} + 1_{K,v_3} = 0$ if f is a third Reidemeister move and v_1, v_2, v_3 are the crossings participating in this move.

The map $(p_u)_K$ for each diagram K is defined by the formula $(p_u)_K(v) = 1_{K,v},\ v \in \mathcal{V}(K)$.

If p is a parity with coefficients in a group A, one defines the map $\rho\colon A_u \to A$ in the following way:

$$\rho\left(\sum_{K,\ v\in\mathcal{V}(K)} \lambda_{K,v} 1_{K,v} \right) = \sum_{K,\ v\in\mathcal{V}(K)} \lambda_{K,v} p_K(v), \qquad \lambda_{K,v} \in \mathbb{Z}.$$

The main theorem of the present section is the following.

Theorem 8.3. *Let $\mathcal{K}$ be a free knot. Then the Gaussian parity* (*with coefficients in* $\mathbb{Z}_2$) *on diagrams of* $\mathcal{K}$ *is the universal parity.*

Remark 8.6. Theorem 8.3 means that for each free knot and for each parity on it either all vertices are even or they have the Gaussian parity.

This theorem will follow from Lemmas 8.6–8.8.

We encode free knots by Gauss diagrams with an ordered collection of distinct chords $\{a_1, \ldots, a_n\}$. Let us choose a point distinct from ends of chords on the core circle of a chord diagram. When going around the circle from the chosen point in counterclockwise, we shall meet each chord end. Denoting each end of a chord by the same letter as the chord, we shall get a word where each letter corresponds to a chord and occurs precisely twice.

Definition 8.9. Let D be a chord diagram. We shall say that an ordered collection of chords with numbers $i_1, \ldots, i_k$ of D forms a *polygon* (or k-*gon*) if a word corresponding to D, contains the following sequences of distinct letters $b_{2p-1}b_{2p}$, where $b_{2p-1}, b_{2p} \in \{a_{i_{\sigma(p)}}, a_{i_{\sigma(p-1)}}\}$, $p = 1, \ldots, k$, for some permutation $\sigma \in S_k$.

The pairs (b_{2p-1}, b_{2p}) of letters b_{2p-1}, b_{2p} from the definition of a polygon are said to be *sides* of polygon.

Example 8.2. Consider the chord diagrams depicted in Fig. 8.9. The chords denoted by a_2, a_4, a_5, a_6, a_8 form a convex pentagon (left) and a non-convex pentagon (right).

In Fig. 8.10 we depict a hexagon for a knot diagram. The knot diagram does not intersect the interior of the hexagon.

Lemma 8.6. *For every parity and any Gauss diagram, the sum of the parities of chords forming a polygon is equal to* 0.

Remark 8.7. The claim of Lemma 8.6 can be taken as a definition of a parity.

***Proof of Lemma* 8.6.** Let p be an arbitrary parity on diagrams of the free knot $\mathcal{K}$, and let D be a Gauss diagram representing $\mathcal{K}$. Let us prove the claim of the lemma by induction on the number of sides of a polygon.

The induction base. The validity of the claim for a loop, bigon, triangle follows from Lemma 8.2 and Definition 8.4, respectively.

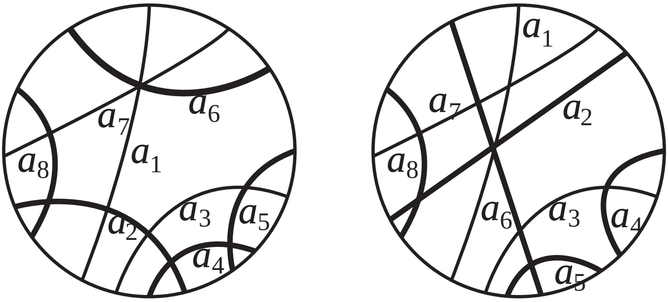


Fig. 8.9 Pentagons.

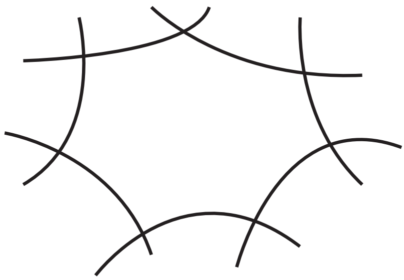


Fig. 8.10 A hexagon.

The induction step. Assume that the claim is true for $(k-1)$-gons. Let us consider an arbitrary k-gon $a_{i_1}a_{i_2}\dots a_{i_k}$.

Let us apply the second Reidemeister move to the Gauss diagram D by adding two chords b and c, see Fig. 8.11 (in Fig. 8.11 we have depicted the three possibilities of applying the second Reidemeister move depending on the ends of chords a_{i_1}, a_{i_2}, a_{i_3}, a_{i_k}).

As a result, we shall obtain the new chord diagram D' and the $(k-1)$-gon $c\,a_{i_3}a_{i_4}\dots a_{i_k}$ and the triangle $b\,a_{i_1}a_{i_2}$. By the induction hypothesis, we have

$$p_{D'}(c) + \sum_{j=3}^{k} p_{D'}(a_{i_j}) = 0, \qquad p_{D'}(b) + p_{D'}(a_{i_1}) + p_{D'}(a_{i_2}) = 0,$$

$$p_{D'}(b) + p_{D'}(c) = 0.$$

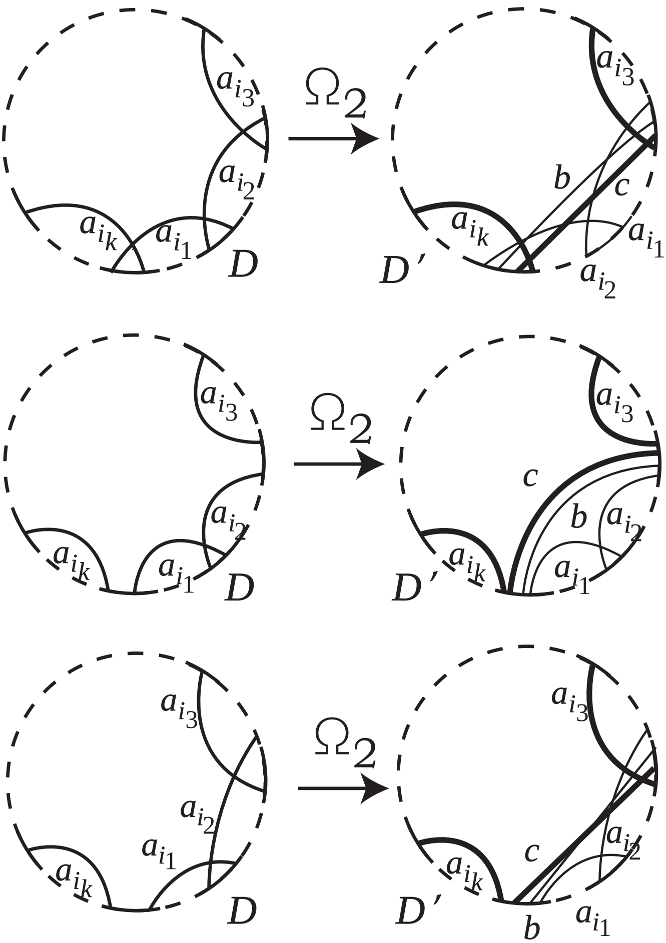


Fig. 8.11 The second Reidemeister move.

Therefore,

$$\sum_{j=1}^{k} p_{D'}(a_{i_j}) = \sum_{j=1}^{k} p_D(a_{i_j}) = 0.$$

□

Remark 8.8. If we work with knot diagrams, then the corresponding picture for Lemma 8.6 is shown in Fig. 8.12.

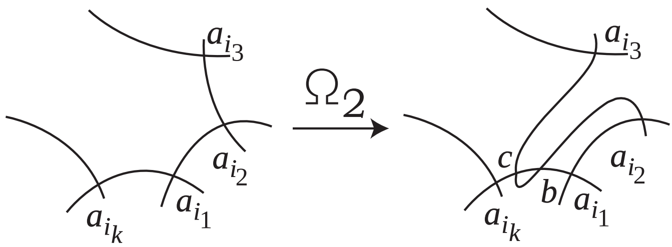


Fig. 8.12 The second Reidemeister move.

Lemma 8.7. *For a free knot with a diagram K and an arbitrary parity p we have $p_K(a) = 0$ if $\mathrm{gp}_K(a) = 0$.*

Proof. Let p be a parity, and let a be a chord of a chord diagram D with $\mathrm{gp}_D(a) = 0$. Let us consider the two halves of the core circle of D, which are obtained by removing the chord a from D. Since $\mathrm{gp}_D(a) = 0$, each half-circle corresponding to a contains an even number of ends of chords. Let us apply the induction on the number of ends of chords.

The induction base: If the number of ends on any half-circle is equal to 0, then $p_D(a) = 0$ because of the property of the first Reidemeister move.

The induction step: Assume that for any chord d of D with $\mathrm{gp}_D(d) = 0$ such that a half-circle contains less than $n = 2k$ ends of chords, we have $p_D(d) = 0$. Let us consider a chord a such that one of its half-circles, $D_{a,1}$, contains exactly n ends of chords and the other one, $D_{a,2}$, contains more than or equal to n chord ends.

Let us orient D in counterclockwise manner and consider the following two cases.

(1) The first two ends in $D_{a,1}$ belong to two distinct chords a_1, a_2, see Fig. 8.13. Apply the second increasing Reidemeister move by adding a pair of chords b, b' in such a way that the half-circle corresponding to b' contains the set of ends lying in $D_{a,1}$ minus the first ends of a_1, a_2 (see Fig. 8.14, the upper part). Let us show that $p_{D'}(a) + p_{D'}(b) = 0$ in the new chord diagram D'. Let us add the pair of chords c, c' to form the triangle a_1a_2c, see Fig. 8.14 (the lower part). Then $p_{D''}(a_1) + p_{D''}(a_2) + p_{D''}(c) = 0$ in D''. Moreover, we have the pentagon aa_1ca_2b and, therefore, the following equality holds (Lemma 8.6)

$$p_{D''}(a) + p_{D''}(a_1) + p_{D''}(c) + p_{D''}(a_2) + p_{D''}(b) = 0.$$

We get $p_{D''}(a) + p_{D''}(b) = 0$ and $p_{D'}(a) + p_{D'}(b) = 0$. In the half-circle corresponding to b', the number of ends is less than the number of ends in the half-circle corresponding to a. By the induction hypothesis, we get $p_{D'}(b) = p_{D'}(b') = 0$ and $p_D(a) = 0$.

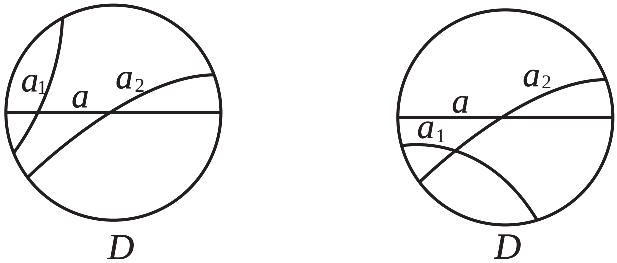


Fig. 8.13 The Gaussian parity zero.

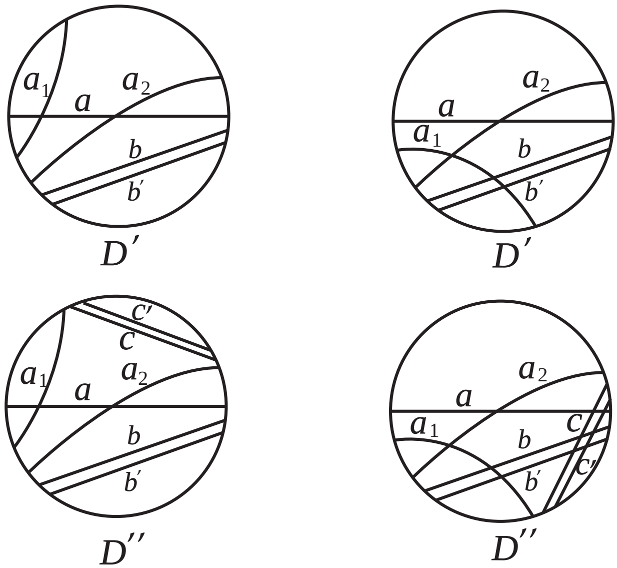


Fig. 8.14 The Gaussian parity zero.

(2) If the first two ends belong to the same chord c, then $p_D(c) = 0$ (the first Reidemeister move) and c forms the triangle in D' with the chords a and b. Therefore, $p_{D'}(a)+p_{D'}(b)+p_{D'}(c) = 0$. By the induction hypothesis, we get $p_{D'}(b) = p_{D'}(b') = 0$ and $p_D(a) = p_{D'}(b) = 0$. □

Lemma 8.8. *Let p be an arbitrary parity (with coefficients from a group A) on diagrams of the free knot represented by a Gauss diagram D. Then for any two chords a, b such that* $\mathrm{gp}_D(a) = \mathrm{gp}_D(b) = 1$ *we have* $p_D(a) = p_D(b) = x \in A$ *and* $2x = 0$.

Proof. Let $c_1, \ldots, c_k$ be ends of chords lying between the nearest ends of a and b.

Apply k times the second Reidemeister moves as it is shown in Fig. 8.15 (the center part). Let us show that $p_{D'}(d_l) = (-1)^l x$, where $x = p_{D'}(a)$. Apply the second Reidemeister move by adding two chords f, f' to form the triangle ad_1f. We have

$$\mathrm{gp}_{D''}(a) = \mathrm{gp}_{D''}(d_1) = 1 \Longrightarrow \mathrm{gp}_{D''}(f) = 0 \Longrightarrow p_{D''}(f) = 0$$
$$\Longrightarrow p_{D'}(d_1) = p_{D''}(d_1) = -x.$$

By the induction, we can prove that $p_{D'}(d_l) = (-1)^l x$ and $p_D(b) = (-1)^{k+1}x$.

Let us apply the third Reidemeister move to the triangle ad_1f. The parity p and the Gaussian parity of the chord a do not change but the parity of the number of ends of chords between a and b changes. Applying the previous trick, we get $p_D(b) = (-1)^k x$, i.e. $2x = 0$. □

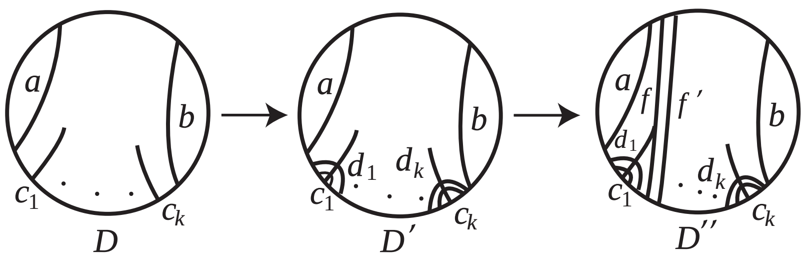


Fig. 8.15 The Gaussian parity one.

Using Lemmas 8.7 and 8.8 for any parity p (with coefficients from a group A) on diagrams of the free knot having a diagram K, we can construct the homomorphism $\rho\colon A \to \mathbb{Z}_2$ by taking $\rho(x) = 1$, where $p_K(a) = x$ and $\mathrm{gp}_K(a) = 1$. This concludes the proof of Theorem 8.3.

Remark 8.9. Let p be a parity for a free knot $\mathcal{K}$. It is not possible that there exist two diagrams K_1 and K_2 of $\mathcal{K}$, both having chords being odd in the Gaussian parity such that the parity p is trivial on all crossings of K_1, and p is the Gaussian parity for K_2. It follows from the fact that there is a sequence of Reidemeister moves transforming K_1 to K_2 such that any diagram in this sequence has chords being odd in the Gaussian parity.

In other words, we have an alternative: The parity for this knot is either Gaussian for all diagrams, or it is trivial for all diagrams.

Applying the previous arguments, let us prove the following.

Theorem 8.4. *For classical knots (a priori we do not suppose that this parity lifts to virtual knots), there exists a unique parity, the trivial parity.*

Before proving this theorem, we prove the following lemma.

Lemma 8.9. *Let p be a parity on diagrams of a knot $\mathcal{K}$. Then for any diagram K and any crossing v we have $2p_K(v) = 0$.*

Proof. By applying the second and third Reidemeister moves, we get diagrams K_1 and K_2 (see Fig. 8.16). We have the equality $p_{K_1}(v)+p_{K_1}(v_1) = 0$. Then $p_{K_2}(v)+p_{K_2}(v_1) = 0$. We also have $p_{K_2}(v)+p_{K_2}(v_2)+p_{K_2}(v_3) = 0$ and $p_{K_2}(v_1) + p_{K_2}(v_2) + p_{K_2}(v_3) = 0$. Hence, $p_{K_2}(v) = p_{K_2}(v_1)$ and $2p_{K_2}(v) = 0$. Then $2p_{K_1}(v) = 0$ and $2p_K(v) = 0$. □

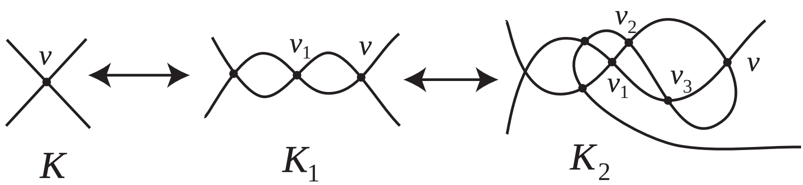


Fig. 8.16 The second and third Reidemeister moves.

***Proof of Theorem* 8.4.** It suffices to show that the sum of parities of crossings forming a bigon or a triangle in any parity is equal to 0.

Let K be a classical diagram, and let us consider only triangles. Assume vertices v_1, v_2, v_3 form a triangle. If one can apply the third Reidemeister move to the triangle, then the identity $p_K(v_1) + p_K(v_2) + p_K(v_3) = 0$ follows from the definition of parity. Otherwise the vertices constitute an alternating triangle. By applying three second and one third Reidemeister moves, we get the diagram K' (see Fig. 8.17), where the following equalities hold:

$$\begin{aligned} p_{K'}(v_2) + p_{K'}(v_3) + p_{K'}(v_4) &= 0, \\ p_{K'}(v_5) + p_{K'}(v_6) + p_{K'}(v_7) &= 0, \\ p_{K'}(v_1) + p_{K'}(v_6) + p_{K'}(v_7) &= 0, \\ p_{K'}(v_5) + p_{K'}(v_4) &= 0. \end{aligned}$$

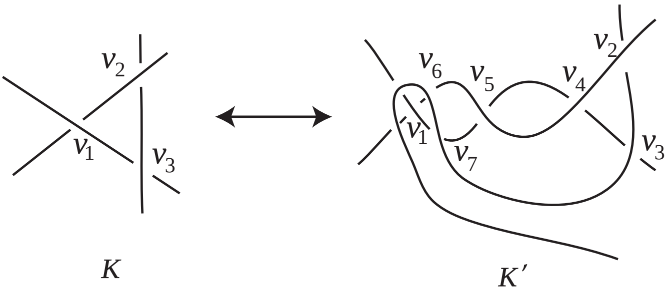


Fig. 8.17 An alternating triangle.

Then, we have $p_{K'}(v_1) = p_{K'}(v_5) = p_{K'}(v_4) = p_{K'}(v_2) + p_{K'}(v_3)$ (we do not need signs because of Lemma 8.9). Therefore, $p_{K'}(v_1) + p_{K'}(v_2) + p_{K'}(v_3) = 0$. □

Now, let us pass to flat knot theory and virtual knot theory. Let K be a (flat) virtual diagram. We have shown that a parity for K can be defined from the homology of the underlying framed 4-graph corresponding to K. Now we consider the homology of the underlying surface corresponding to K. Since bigons and triangles participating in the Reidemeister moves can be spanned by discs, we get the following corollary.

Corollary 8.1. *For every parity p and any (flat) virtual knot diagram, the sum of the parities $p_K(v_i)$ of the crossings v_i forming a polygon which is spanned by a disc in the underlying surface is equal to* 0.

Using virtualization moves, we can transform any polygon to a polygon which is spanned by a disc in the underlying surface. As a result we get the following corollary.

Corollary 8.2. *For every parity p and any pseudo-knot diagram K, we have*:

(1) *the sum of the parities $p_K(v_i)$ of the crossings v_i forming a polygon, is equal to* 0, *i.e. the writhe number has no influence on this property*;
(2) $p_K(v) = 0$ *if* $\mathrm{gp}_K(v) = 0$.

Taking into account Corollary 8.1, we get the following corollary.

Corollary 8.3. *Any parity for flat and virtual knots arises from the homology of the underlying surface. Thus, any parity on virtual knots gives the trivial parity on classical knots.*

8.3 A functorial mapping *f*

The mapping f given in the present section was first suggested by Turaev for the parity from Sec. 8.2.3. We construct this mapping for an arbitrary weakened parity.

8.3.1 *Construction*

Consider some knot theory represented by (flat or Gauss) diagrams and moves on them. Let K be a diagram from this theory.

Definition 8.10. A *subdiagram* of K is a diagram (possibly, with some further decorations) which is obtained from K by deleting several chords in the case of Gauss diagrams or replacing several classical crossings with virtual crossings in the case of flat diagrams.

Let us define the *closure* of knot theory to be the knot theory obtained from the initial knot theory as follows. Diagrams of the new theory are diagrams of the initial theory and their subdiagrams. Moves in the new theory are defined in the same manner as in the initial theory, i.e. if we have two equivalent diagrams K_1 and K_2 in the initial knot theory, then subdiagrams obtained from K_1 and K_2 by deleting the same chords (replacing the same crossings with virtual ones) not participating in the moves are equivalent in the new theory.

A knot theory is called *closed* if it coincides with its closure.

Theorem 8.5. *Given some knot theory and a weakened parity on diagrams of each knot. Let us construct the mapping f from the theory to its closure, which sets a diagram K to the diagram obtained from K as follows.*

Any even classical crossing of the diagram K remains classical, and any odd classical crossing is replaced with a virtual one (*i.e. when drawing a* *picture we substitute a virtual crossing for an odd classical crossing).*

Then the mapping f is well defined on equivalence classes of knots, i.e. this mapping takes equivalent diagrams to the equivalent diagrams.

Remark 8.10. In the case of free knots, i.e. when framed 4-graphs are considered as diagrams, the mapping f is the operation which deletes all odd crossings and joins two opposite half-edges to form one edge. When knot diagrams are depicted on the plane or a surface, deleted crossings are marked by a circle and considered as virtual crossings.

***Proof of Theorem* 8.5.** We need to check that if diagrams K_1 and K_2 are obtained from each other by applying a Reidemeister move, then the diagrams $f(K_1)$ and $f(K_2)$ either coincide (under drawing them on the plane they differ with a detour move), or differ by a Reidemeister move.

Indeed, in the case of the first Reidemeister move, after applying the mapping f, we get either a loop (if the crossing is even) or a virtual loop (if the crossing is odd). In the first case, two diagrams are connected by a detour move, and in the second case, they are related by the first Reidemeister move.

In the case of the second Reidemeister move, if the two crossings participating in the move are odd, then the two diagrams $f(K_1)$ and $f(K_2)$ differ by a detour move, and if the two crossings are even, then the two diagrams $f(K_1)$ and $f(K_2)$ differ by the second Reidemeister move.

Finally, in the case of the third Reidemeister move we can obtain either the third Reidemeister move for $f(K_1)$ and $f(K_2)$ if all the three crossings are even, or a detour move if the number of even crossings is one or zero.

Note that according to the first condition of Definition 8.4 all crossings not participating in Reidemeister moves remain fixed. The latter guarantees us that the diagram $f(K_1)$ does not change outside the neighborhoods of those crossings which take part in Reidemeister moves while we pass from $f(K_1)$ to the diagram $f(K_2)$. □

Let us describe the mapping f in concrete cases of knot theories.

8.3.2 *The mapping f in the case of the parity from Sec.* 8.2.3

Let K be a virtual diagram representing a free knot, $D_G(K)$ be the corresponding Gauss diagram, and let $\mathrm{At}(K)$ be the atom corresponding to the diagram K. Then the following theorem holds.

Theorem 8.6. *The atom* At(K) *is orientable if and only if all chords of the diagram* $D_G(K)$ *are even.*

In particular, the property of an atom to be orientable only depends on the free knot corresponding to K, *and does not depend on over/undercrossing information at classical crossings of* K.

Proof. It is known that an atom is orientable if and only if the frame of the atom admits a source–sink structure (see Sec. 7.10 and [232, 238]), i.e. we can orient all edges of its frame in such a way that at each vertex two opposite edges are *emanating*, and the other two opposite edges are *incoming*, see Fig. 8.18 in the center.

Recall that edges of the graph, the frame of an atom, correspond to arcs of the Gauss diagram. Besides, adjacent arcs of the Gauss diagram correspond to opposite half-edges. Therefore, a source–sink structure defines an orientation of the Gauss diagram, satisfying the following properties:

- arcs of the Gauss diagram alternate, i.e. each arc oriented in clockwise manner is followed by the arc oriented in counterclockwise manner;
- each chord has two emanating arcs in one of its ends, and two coming arcs in the other end.

From these conditions it easily follows that for each chord a half-circle of the core circle of the Gauss diagram contains an even number of chords' ends. Therefore, the chord is even. □

Example 8.3. Let us consider the Gauss diagram depicted in Fig. 8.18 (the left part). An orientation of arcs is given in it, this orientation gives rise to the source–sink structure (see Definition 5.7) for the frame of the corresponding atom. It is not difficult to check that one of the atoms corresponding to this diagram is spherical (i.e. the surface of the atom is the sphere), therefore, it is orientable, i.e. all atoms with the same frame are orientable.

We cannot define a source–sink structure for the chord diagram as depicted in Fig. 8.18 (the right part) since the chord a is odd (it is linked with one chord). Therefore, an alternating orientation of arrows along the core circle of the chord diagram leads to four incoming edges for the chord a.

Thus, the mapping f preserves those diagrams having orientable atoms (recall that we consider only diagrams of virtual knots, not links). If the atom At(K) corresponding to a diagram K is not orientable, then the diagram $f(K)$ has fewer classical crossings than the diagram K does.

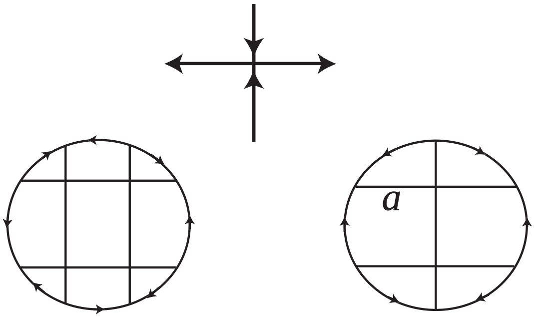


Fig. 8.18 Gauss diagrams and the source–sink structures.

In general, for some K we have $f^2(K) \neq f(K)$ since after the removal of odd chords of the chord diagram the formerly even chords may become odd. Thus, for instance, for the chord diagram with four chords a, b, c, d such that the linked pairs are (a,b), (a,c), (b,d), the removal of the odd chords c, d yields the diagram with the chords a, b being odd.

It is easy to see that for every diagram K of a virtual knot with n classical crossings there exists a number $m < n$ such that $f^m(K) = f^{m+1}(K)$, i.e. the atom corresponding to the diagram $f^m(K)$ is orientable

It is evident that the map killing odd crossings is a well-defined map on the set of *atoms*: It suffices to remove all vertices of the atom corresponding to odd crossings and to preserve the white–black structure in the neighborhoods of the remaining crossings.

This leads us to the following *filtration* on the set of atoms and virtual knots. Let At be an atom whose frame has one unicursal component. Then either At is orientable (in this case we say that At has *grading zero*), or the atom At is not orientable. In the second case there exists a unique natural number $n > 0$ such that $f^n(\mathrm{At})$ is an orientable atom, and $f^{n-1}(\mathrm{At})$ is not. In this case we say that the atom At is *of grading n*.

Analogously, one defines the grading on the set of virtual knots: We say that a virtual knot has grading 0 if it possesses a diagram with orientable atom; a knot K has grading $n > 0$ if the knot $f^n(K)$ has a diagram with orientable atom, whence the knot $f^{n-1}(K)$ has no such diagrams.

Thus we get a natural splitting of the set of virtual knots into subsets: $\mathcal{K}_0 \oplus \mathcal{K}_1 \oplus \mathcal{K}_2 \oplus \cdots \oplus \mathcal{K}_n \oplus \cdots$.

It is easy to construct examples showing that each of these sets $\mathcal{K}_n$ is non-empty. We construct these examples by induction on n. Let us fix a positive integer k and consider an irreducibly odd diagram on k chords. In the first step we add k pairwise unlinked chords, each linked precisely one chord of the initial diagram. In the lth step we add k pairwise unlinked chords, each of which is linked with exactly one chord added in the previous step. After performing the $n-1$ steps, we obtain a diagram of grading n.

Definition 8.11. We call a map which sets all virtual knots to knots with orientable atoms and takes all virtual knots with orientable atoms to themselves, a *projection*.

The approach above allows one to get a simple proof of the following theorem by Manturov–Viro (first proved in spring 2005 independently by V. O. Manturov and O. Ya. Viro and first published in [129]).

Theorem 8.7. *Let K, K' be two equivalent diagrams of virtual knots having orientable atoms. Then there exists a chain of Reidemeister moves*

$$K = K_0 \to K_1 \to \cdots \to K_n = K'$$

such that all atoms corresponding to diagrams K_i are orientable.

Proof. Let us consider a chain of diagrams $K = K_0 \to K_1 \to \cdots \to K_n = K'$ such that the diagrams K_i and K_{i+1} are obtained from each other by a classical Reidemeister move (plus, possibly, a detour move).

Denote the maximal number of odd crossings over all diagrams K_i by l.

Let us apply the map f to the chain of K_j's. We obtain a chain of diagrams

$$K = K_0 = K_0' \to K_1' \to \cdots \to K_n' = K_n = K',$$

where every two adjacent diagrams are obtained from each other by a classical Reidemeister move or coincide (i.e. they are obtained from each other by a detour move).

Let us apply the map f again (in total, we shall apply this map l times starting with the initial chain). Reiterating the process, we get a chain of diagrams

$$f^l(K) = K = K_0 \to f^l(K_1) \to \cdots \to f^l(K_n) = K_n = K',$$

where all atoms corresponding to all diagrams, are orientable, and every two adjacent diagrams either coincide (connected by a detour move) or differ by a Reidemeister move. □

Remark 8.11. The initial proof of the Manturov–Viro theorem in the general case of *multicomponent links* relies on the geometry of virtual knots and the Kuperberg theorem. Presently, the authors know the proof of this theorem in the general case, relying on the notion of *relative parity* [188].

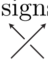
8.3.3 *The parity hierarchy on virtual knots*

It follows from the Manturov–Viro theorem that virtual knots with orientable atoms form a natural subclass of all virtual knots with the equivalence being defined only by using those Reidemeister moves which preserve the knot diagram within this subclass.

In particular, this class (let us denote it by $\mathfrak{V}^1$) includes all classical knots. We shall show that there exists a natural *filtration* on the set of virtual knots (unrelated to the grading n given above):

$$\mathfrak{V}^0 \supset \mathfrak{V}^1 \supset \mathfrak{V}^2 \supset \cdots \supset \mathfrak{V}^n \supset \cdots,$$

which starts with the set $\mathfrak{V}^0$ of all virtual knots and has as its limit "index zero knots" $\mathfrak{V}^\infty$, the set of knots containing all classical knots.

Thus, let K be a virtual diagram, and let $D_G(K)$ be the corresponding Gauss diagram. We endow the diagram $D_G(K)$ with signs and arrows in the usual way. The plus sign corresponds to a crossing , and the minus sign corresponds to a crossing . The arrow is pointing from the preimage of the overcrossing arc to the preimage of the undercrossing arc.

With each classical crossing we associate its *index*, which will be either a natural number or zero. Let v be a classical crossing of the diagram K, and let $c(v)$ be the corresponding (oriented) chord of the diagram $D_G(K)$. Consider all chords of $D_G(K)$ linked with $c(v)$. Let us count for them the sum of signs of those chords intersecting $c(v)$ from the left to the right and subtract the sum of signs of those chords intersecting the chords $c(v)$ from the right to the left. The absolute value of the obtained number will be called the *index* of the chord $c(v)$ (or the crossing v), and will be denoted by $\mathrm{ind}(v) \equiv \mathrm{ind}(c(v))$.

It is clear that if the atom corresponding to K is orientable, then the indices of the chords of $D_G(K)$ are all even.

Consequently, $\mathfrak{V}^1$ consists of those knots having all indices of all chords even.

Let us collect some facts whose proof follows from a simple check.

Statement 8.1. (1) *If a crossing takes place in the first Reidemeister move, then it has index zero.*
(2) *In the second Reidemeister move, the indices of the two crossings are equal.*
(3) *The index of a crossing does not change when the crossing is operated on by the third Reidemeister move*

$$\mathrm{ind}(v_1) = \mathrm{ind}(v_1'), \qquad \mathrm{ind}(v_2) = \mathrm{ind}(v_2'), \qquad \mathrm{ind}(v_3) = \mathrm{ind}(v_3'),$$

besides, if three crossings v_1, v_2, v_3 *participate in a third Reidemeister move, then* $\mathrm{ind}(v_1) \pm \mathrm{ind}(v_2) \pm \mathrm{ind}(v_3) = 0$, *see Fig. 8.19.*
(4) *The index of a crossing taking part in a certain Reidemeister move, does not change after this crossing undergoes the Reidemeister move.*
(5) *All crossings of a classical diagram have index zero.*

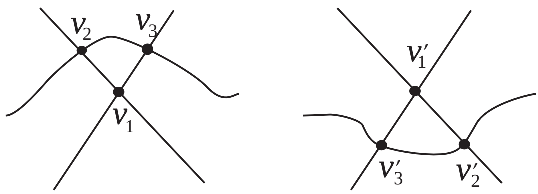


Fig. 8.19 The third Reidemeister move and corresponding crossings.

First, note that the property of the index to be equal to zero is a weakened parity. The same is true about the congruence to zero modulo some integer number. Thus, the map which eliminates all chords of non-zero indices is a well-defined map in a knot theory with an index.

Besides, the properties of the index described above show that the index can be used in order to define a parity (with coefficients from $\mathbb{Z}_2$).

Indeed, it follows from Statement 8.1 that one can introduce on diagrams of $\mathfrak{V}^1$ the following parity: Let K be a knot diagram from $\mathfrak{V}^1$; we decree those crossings of K having index divisible by four, to be even, and we decree the remaining ones to be odd.

From Statement 8.1 it follows that the parity defined in this way satisfies all parity axioms.

Let us apply to knots from $\mathfrak{V}^1$ the map which kills crossings of index not divisible by four. This operation may take us away from the class $\mathfrak{V}^1$. Nevertheless, if two diagrams K_1, K_2 from $\mathfrak{V}^1$ are equivalent, then so are $f(K_1)$ and $f(K_2)$ (even if they do not belong to $\mathfrak{V}^1$).

Arguing as above, we can define the sets $\mathfrak{V}^k$ of diagrams with all crossings having indices divisible by 2^k, and also the set $\mathfrak{V}^\infty$ as the set of diagrams with all crossings having index 0.

There is a set of maps $f_k\colon \mathfrak{V}^k \to \mathfrak{V}^0$; each map f_k eliminates all chords of index congruent to 2^k modulo 2^{k+1}. All these maps take equivalent diagrams to equivalent ones.

Then the following theorem holds.

Theorem 8.8. *Let K, K' be two diagrams of virtual knots from $\mathfrak{V}^k$ (where k is either a natural number or the symbol ∞) corresponding to equivalent virtual knots. Then there exists a chain*

$$K = K_0 \to K_1 \to \cdots \to K_n = K'$$

of diagrams from $\mathfrak{V}^k$, where every two adjacent diagrams are obtained from each other by a Reidemeister move or a detour move.

This theorem is proved in the same way as Theorem 8.7 by means of the functorial mapping eliminating all chords whose indices are not divisible by 2^k (in the case of finite k) or eliminating all chords of non-zero index (in the case $k = \infty$).

Note that the class $\mathfrak{V}^\infty$ is quite interesting: It is an "approximation" of classical knots by virtual knots, and all invariants defined on $\mathfrak{V}^\infty$, can be taken to virtual knots by means of the map f.

An example of a non-classical diagram from $\mathfrak{V}^\infty$ is shown in Fig. 8.20.

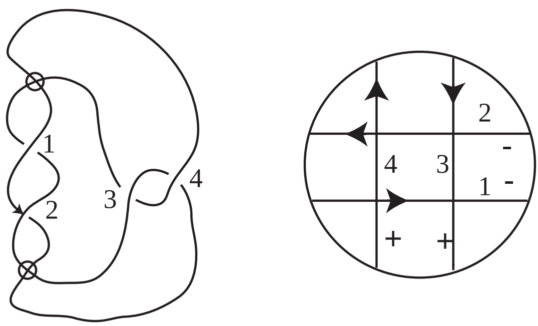


Fig. 8.20 A non-classical diagram from $\mathfrak{V}^\infty$.

In Fig. 8.20 we depict the Gauss diagram with arrows and signs. It is easy to see that in the Gauss diagram the signs corresponding to the crossings 1 and 2 have opposite directions, that guarantees that the indices of the chords 3 and 4 are both zeros. Analogously, the arrows, corresponding to crossings 3 and 4 are opposite.

The reader can easily check that the parity hierarchy gives rise to the flat hierarchy in the sense of Sec. 3.5.1: odd crossings will have label 0, those even crossings which become odd after one application of the map f, get label 1, and so on.

8.3.4 *The map f in the case of the parity from Sec.* 8.2.4

Let K be a diagram of a classical or virtual two-component link. Then the map f replaces all crossings formed by both components of the diagram K, with virtual ones. The geometric sense of the operation f is the following: With a link $K_1 \cup K_1$ it associates the split link formed by two components K_1 and K_2.

8.3.5 *The map f in the case of the parity from Sec.* 8.2.5

This map associates with knots in the thickened torus so-called virtual knots in thickened torus, i.e. equivalence classes of diagrams in $S^1 \times I^1$ with classical and virtual crossings modulo the Reidemeister moves.

The knots obtained in different ways as images of the map f can be further investigated by using different methods, e.g. either by considering them as usual virtual knots (by using the inclusion $S^1 \times I^1 \to \mathbb{R}^2$) or by investigating additional parities.

Analogously, one can consider the parity f for knots in a thickened surface which leads to the theory of "virtual knots in this thickened surface".

8.4 Invariants

It turns out that if some knot theory possesses a parity, then it allows one to construct invariants of knots from this theory valued in linear combinations of graphs. Such linear combinations arise from diagrams of the initial knot by means of *smoothing* and, therefore, they have many information about the knot in the large: about the number of crossings of its possible diagrams and their position with respect to each other.

In particular, for some knot theory with a parity, it is easy to prove theorems about minimal diagrams with respect to the number of crossings.

8.4.1 *Preliminaries: smoothings and linear spaces*

Let K be a diagram in some knot theory. Our main example will be the theory of free knots in which by a diagram we mean a framed 4-graph, therefore, we shall use two terms: a "diagram" and a "graph".

Let us define smoothing operations on diagrams (graphs). Later on, by a "smoothing" we shall call both a smoothing operation and the diagram obtained by applying a smoothing operation. Note that the idea of a "smoothing" of a 4-graph were studied by Bouchet, Nash-Williams and others, who called it a *detachment*, see, e.g. [40, 254].

Definition 8.12. Let v be a vertex (crossing) of the diagram K with four incident half-edges a, b, c, d, such that a is opposite to c and b is opposite to d at v. By a *smoothing* of K at v we mean any of the two framed 4-graphs obtained by removing v and repasting the edges as $a - b$, $c - d$ or as $a - d$, $b - c$, see Fig. 8.21.

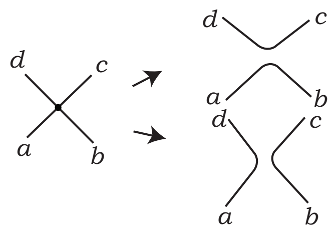


Fig. 8.21 Two smoothings of a vertex for a framed graph.

Herewith, the rest of the graph (together with all framings at vertices except v) remains unchanged.

We may then consider further smoothings of the diagram (graph) K at *several* vertices. Every smoothing is a result of consecutive smoothings at these vertices. In the case when a diagram in the knot theory contains an additional information at crossings (besides just framing), it is assumed that this information (for the corresponding crossings) is forgotten under a smoothing. For example, in the case of a virtual diagram the result of a smoothing is a framed 4-graph. We denote a result of a smoothing of a diagram K by K_s, where s is a *smoothing* (or a *state*) of K.

Let K be a diagram, and let $v_1, \ldots, v_n$ be all even crossings of K. We call a smoothing at all even crossings $v_1, \ldots, v_n$ an *even smoothing* of K. Thus, there are 2^n even smoothings for the diagram K.

We call an *even smoothing* of a framed 4-graph K (with respect to any parity) a 1-*even smoothing* if after the smoothing the resulting framed 4-graph has one unicursal component.

In the case of *braids* and *tangles* for initial objects we take graphs with vertices of degree one (called *terminal*) and degree four (each of which is framed), and in the case of tangles we also admit free components, i.e. separate circles. In this case, smoothings represent graphs with vertices of degrees 1 and 4 and free loops, and the set of vertices of degree 1 in every smoothing coincides with the set of vertices of degree 1 of the initial diagram.

For every framed 4-graph all of its smoothings are framed 4-graphs. For a virtual diagram K smoothings are virtual diagrams corresponding to smoothings of the corresponding framed 4-graph, obtained from K by forgetting extra information at crossings.

Later on, we shall be interested in the following sets of smoothings of a free graph (virtual diagram): the set S of all smoothings, the set S_{even} of all even smoothings, the set S_1 of all smoothings having one unicursal component and the set $S_{\text{even},1}$ of all *even* smoothings with one unicursal component. We shall denote elements of the corresponding sets (smoothings) by s, s_{even}, s_1, $s_{\text{even},1}$, respectively. To simplify the notation under taking the sum, we shall usually omit the set of smoothings over which the sum is taken; this set will be read from the variable indicating to it: s, s_{even}, s_1 or $s_{\text{even},1}$.

Framed 4-graphs running through the set of such smoothings will be later used for the construction of invariants of framed links and for some other theories with parities.

The theory of free knots (not links) is described by Gauss diagrams and the Reidemeister moves on them (for links one should construct Gauss diagrams on several circles). When passing from knots to free knots one should pass from Gauss diagrams with labeled edges and oriented edges to Gauss diagrams without labels and orientations.

Consider framed 4-graphs with one unicursal component modulo the equivalence relation generated by the second Reidemeister move. Let us define the linear space $\mathfrak{G}$ as the set of $\mathbb{Z}_2$-linear combinations of such equivalence classes.

Definition 8.13. The linear space $\widetilde{\mathfrak{G}}$ is the set of $\mathbb{Z}_2$-linear combinations of the following objects. One considers all framed 4-graphs modulo the following equivalence relations:

(1) the second Reidemeister move;
(2) $K \sqcup \bigcirc = 0$, i.e. a framed 4-graph having more than one component, and at least one trivial component, is assumed to be zero.

There is a natural map $g\colon \widetilde{\mathfrak{G}} \to \mathfrak{G}$ which takes to zero all equivalence classes of framed graphs having more than one unicursal component. Obviously, the map g is an epimorphism of groups.

8.4.2 *The invariants* $[\cdot]$, $\{\cdot\}$

The invariants given below will be explicitly defined for the case of free knots and links with respect to any parity with coefficients from $\mathbb{Z}_2$.

The definitions below can be directly extended to *free tangles*; these objects are encoded by graphs which, besides vertices of degree 4 (with the usual framing), have free ends of degree 1 (in particular, one can extend the definitions to free braids). We shall not give explicit definitions for the case of free tangles, leaving them for the reader as an exercise.

Since every virtual link generates the free link by "forgetting" the structure at classical crossings (the over/undercrossing structure and the local writhe number), but we remember the structure of opposite edges, then these invariants can be lifted to invariants of virtual knots and links for every parity with coefficients from $\mathbb{Z}_2$, which is well defined in the corresponding case.

Now we pass to the construction of the invariants $[\cdot]$ and $\{\cdot\}$ of free knots to be valued in $\mathfrak{G}$ and $\widetilde{\mathfrak{G}}$, respectively.

Let K be a framed 4-graph. Then the invariant $\{\cdot\}$ is given by the formula

$$\{K\} = \sum_{s_{\text{even}}} K_{s_{\text{even}}} \in \widetilde{\mathfrak{G}},$$

where it follows from the notation that the sum is taken over all even smoothings s_{even} of the framed 4-graph K, which are considered as elements from $\widetilde{\mathfrak{G}}$.

Theorem 8.9. *The bracket* $\{\cdot\}$ *is an invariant of free links.*

Proof. Let us check the invariance of the bracket $\{\cdot\}$ with respect to the Reidemeister moves. Here we shall use those properties of the parity, which are satisfied by those crossings of the diagram undergoing the Reidemeister moves.

Let diagrams K and K' differ from each other by a first Reidemeister Ω_1, such that the diagram K' contains one more vertex than the diagram K, and this vertex is denoted by v.

Note that the vertex v is even according to Lemma 8.2. One of the smoothings of the diagram K' at v leads us to a split component in such a way that every even state of the diagram K' where the vertex v is smoothed in the "wrong" way, will lead to a split trivial component. This will yield a trivial element from $\widetilde{\mathfrak{G}}$.

The smoothing of K' at v performed in the "right" way will lead us to the diagram K, which yields a one-to-one correspondence for even smoothings for K and K' with no split circles.

Thus we have proved the invariance of the bracket $\{\cdot\}$ under the first Reidemeister move.

For the second and third Reidemeister moves Ω_2 and Ω_3 we shall show that $\{K\} + \{K'\} \equiv 0 \pmod{\mathbb{Z}_2}$ by means of partitioning of all diagrams from $\{K\} + \{K'\}$ representing non-trivial elements from $\widetilde{\mathfrak{G}}$. Since $\widetilde{\mathfrak{G}}$ is a linear space over $\mathbb{Z}_2$, this will mean that $\{K\} = \{K'\}$.

Let K' be obtained from K by a second Reidemeister move adding two crossings v_1 and v_2. If both crossings are odd, then there is an obvious one-to-one correspondence between the set of even smoothings of the diagram K and the set of even smoothings of the diagram K'. The corresponding smoothings are obtained from each other by applying "the same" Reidemeister move to vertices v_1 and v_2. If both v_1 and v_2 are even, then there exist four smoothings of K' at these vertices:  $\to$ ,  $\to$ ,  $\to$ ,  $\to$ .

Note that the smoothing  has a split circle which will remain split for all subsequent smoothings at even crossings. Thus, such summands will have no impact in $\{K'\}$.

Furthermore, the smoothings  and  are in fact the same framed 4-graph (provided that the smoothings at the remaining vertices agree). Thus, these smoothings cancel in $\{K'\}$.

Even smoothings of type  of the diagram K' are naturally in one-to-one correspondence with even smoothings of the diagram K and give rise to framed 4-graphs.

Now assume that the diagram K is taken to a diagram K' by a third Reidemeister move Ω_3. Among the three crossings of K taking part in the Reidemeister move, either all three ones are even, or one crossing is even, and the other two ones are odd.

If the three crossings of K taking part in the Reidemeister move are even, then we have seven types of summands in the expansion of $\{K\}$ (and seven types for $\{K'\}$): At each of the three vertices we have two possible smoothings, here one of the eight possibilities leads to a split trivial circle for K and one of the eight possibilities for K' leads to a trivial circle (these two types of smoothings contribute to neither $\{K\}$ nor $\{K'\}$). Considering K (in Fig. 8.22 the summands corresponding to K are in the left-hand side and the summands corresponding to K' are in the right-hand side) we see that three of these seven types lead to coinciding sets of diagrams (these types are denoted by 1), thus in $\widetilde{\mathfrak{G}}$, the two sets are canceled with each other, so, only one set is left. Analogously, for the case of K' we have three "similar" types of smoothings (they are denoted by 2 in Fig. 8.22). Thus, in both $\{K\}$ and $\{K'\}$, five types of summands are left: 1, 2, 3, 4, 5.

In these five cases there is a one-to-one correspondence (see Fig. 8.22), which leads us to the equality $\{K\} = \{K'\}$.

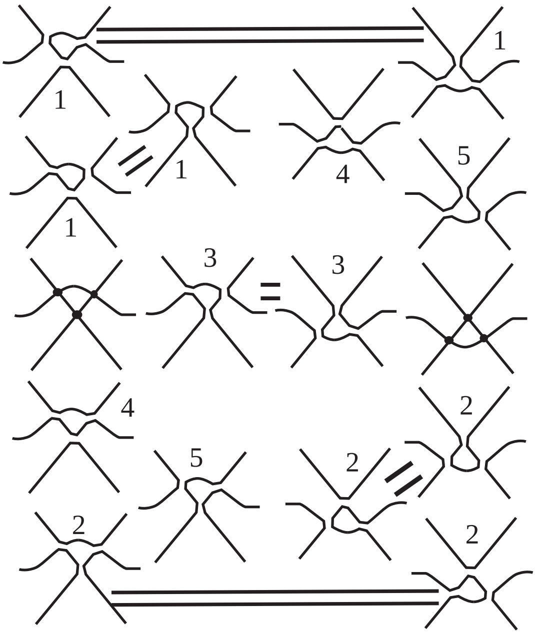


Fig. 8.22 The correspondence between smoothings for Ω_3 with three even vertices.

If among the vertices taking part in Ω_3 there is exactly one even vertex on the right-hand side and exactly one even vertex on the left-hand side (say, $v \to v'$), then we are in the situation shown in Fig. 8.23.

From the above figure we see that those smoothings where the vertex v (respectively, v') is smoothed *vertically*, give rise to coinciding summands for $\{K\}$ and $\{K'\}$, and those smoothings where v and v' are smoothed *horizontally* are in one-to-one correspondence for diagrams K and K', moreover, the corresponding graphs differ from each other by an application of two

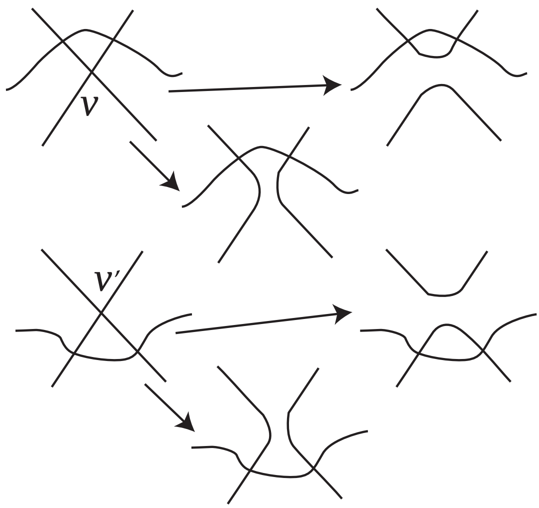


Fig. 8.23 The correspondence between smoothings for Ω_3 with one even vertex.

second Reidemeister moves. This proves that $\{G\} = \{G'\}$ in $\widetilde{\mathfrak{G}}$. □

The invariant $[\cdot]$ is given by the formula

$$[K] = \sum_{s_{1,\text{even}}} K_{s_{1,\text{even}}} \in \mathfrak{G}, \tag{8.1}$$

where the sum is taken over all even smoothings of the diagram K, which yield one unicursal component.

Obviously, $[K] = g(\{K\})$, which leads to the following theorem.

Theorem 8.10. *The bracket* $[\cdot]$ *is an invariant of free knots.*

Sometimes it will be convenient for us to use the bracket $[\cdot]$ just as defined by formula (8.1).

Remark 8.12. Note that the invariants defined above can be constructed for any parity with coefficients from $\mathbb{Z}_2$.

The invariants $\{\cdot\}$ and $[\cdot]$ take a certain equivalence class (of diagrams modulo the Reidemeister moves) to some linear combination of equivalence classes (of diagrams modulo the second Reidemeister move and some simple factorization of some class of diagrams). It turns out that the sets $\mathfrak{G}$ and $\widetilde{\mathfrak{G}}$ can be easily described algorithmically: Every element of each of these sets has a minimal representative which can be found by means of subsequent simplifications. By a simplification we mean the second Reidemeister move which decreases the number of crossings by two, and also (in case of elements from $\widetilde{\mathfrak{G}}$) the transformation which takes to zero any diagram with more than one unicursal component having one split trivial component.

More precisely, the above statement can be formulated as a lemma, which we shall formulate after some definition.

Definition 8.14. We say that a framed 4-graph is *simplifiable*, if it either contains a component without vertices or contains two vertices v_1 and v_2 connected by a couple of edges e_1, e_2 such that e_1, e_2 are adjacent in both v_1 and v_2.

The second case is just the situation when one can apply a second decreasing Reidemeister move to the graph.

Definition 8.15. Framed 4-graphs not admitting any simplifications will be called *minimal*. We say that a framed 4-graph $\widetilde{K}$ is obtained from a framed 4-graph K by a *subsequent simplification*, if there exists a chain of framed 4-graphs

$$K = K_n \to \cdots \to K_1 \to \widetilde{K},$$

where every subsequent graph is obtained from the previous one by an application of a second simplifying Reidemeister move.

A *minimal representative* of a graph K is a minimal graph which can be obtained from K by a subsequent simplification.

In the case of $\mathfrak{G}$ the notions of *minimal* and *non-simplifiable* graphs coincide.

Lemma 8.10. *If two framed 4-graphs K_0 and K_0' have one unicursal component and are obtained from some framed 4-graph K by a subsequent simplification, then K_0 is isomorphic to K_0'.*

The lemma states that every framed 4-graph with one unicursal component has a *unique* minimal representative.

From this lemma one gets the following lemma.

Lemma 8.11. *Framed 4-graphs K and K' with one unicursal component each are equivalent in $\mathfrak{G}$ if and only if their minimal representatives coincide.*

Proof. Let us deduce Lemma 8.11 from Lemma 8.10. The claim ($\Longleftarrow$) is evident. Assume that $K = K'$ is in $\mathfrak{G}$, and here minimal representatives of graphs K and K' are different. Let H and H' be two framed 4-graphs obtained from each other by a second Reidemeister move, in such a way that the graph H' has two crossings more than the graph H. Then by definition of the minimal representative and by Lemma 8.10 we conclude that the minimal representatives for H and H' coincide. Considering the chain $K = K_1 \to \cdots \to K_k = K'$ of second Reidemeister moves connecting K to K', we see that the minimal representatives of any two adjacent graphs in this chain coincide. Thus the minimal representatives of K and K' coincide as well, which yields the equivalence of these graphs in $\mathfrak{G}$. □

We preface Lemma 8.10 with two simple statements.

Statement 8.2. *Assume that a framed* 4*-graph* K *with one unicursal component has two non-isomorphic minimal representatives. Then there exists such a simplification* $\widetilde{K}$ *of the graph* K ($\widetilde{K}$ *can coincide with* K) *for which the following holds.*

Among minimal representatives of the graph $\widetilde{K}$ *there exist non-isomorphic framed* 4*-graphs* K_0 *and* K_0'*, and among elementary simplifications of the framed* 4*-graph* $\widetilde{K}$ *there exist framed* 4*-graphs* K_1 *and* K_1' *such that* K_0 *is one of minimal representatives for* K_1*, but not for* K_1'*.*

Proof. We choose two non-isomorphic minimal representatives K_0 and K_0' of the graph K and consider the chain of elementary transformations from K to K_0. In the initial moment the graph K has K_0' among its minimal representatives, and at the end of the chain, among minimal representatives of K_0 there is no graph K_0' (since the graph K_0 cannot be decreased). Consider the chain from K to K_0. Take the last graph of the chain having K_0 as a minimal representative and denote this graph by $\widetilde{K}$, we get the required statement. □

Statement 8.3. *If framed* 4*-graphs* K_1 *and* K_1' *are each obtained from a framed* 4*-graph* K *by one elementary simplification, then either* K_1 *is isomorphic to* K_1' *or there exists a framed* 4*-graph* K_2' *which can be obtained by one elementary simplification from each of* K_1*,* K_1'*.*

Proof. It suffices to consider the vertices $\{v_1, v_2\}$ of the graph K where the simplification $K \to K_1$ takes place, and the vertices $\{v_1', v_2'\}$ corresponding to the simplification $K \to K_1'$. If the set $\{v_1, v_2\} \cup \{v_1', v_2'\}$ consists of two or three elements, then it is evident that the graphs K_1 and K_1' are isomorphic. If this set consists of four elements, then it is easy to see that one can apply an elementary simplification to the graph K_1 at $\{v_1', v_2'\}$ so that the resulting graph is isomorphic to the graph obtained from K_1' by an elementary simplification at vertices $\{v_1, v_2\}$. □

Now we prove Lemma 8.10.

***Proof of Lemma* 8.10.** Consider the framed 4-graph K with one unicursal component and assume it has more than one minimal representative.

By virtue of Statement 8.2, there is a simplification $\widetilde{K}$ of K such that among graphs obtained from $\widetilde{K}$ by one elementary simplification, one can choose such a pair K_1 and K_1' for which one of the graphs K_0 or K_0' is a minimal representative for K_1 but not for K_1'. Without loss of generality we shall assume that this graph is K_0. According to Statement 8.3, there exists a graph K_2' which can be obtained by an elementary simplification from each of K_1 and K_1'.

Since by assumption K_1' does not have K_0 among minimal representatives, the graph K_2' does not have K_0 among minimal representatives. On the other hand, since K_0 is a minimal representative for K_1, then K_1 has at least two minimal representatives.

Changing the notation from K_1 to K and repeating the above argument, we see that one of the graphs (denote it by K_2) obtained by elementary transformation from K_1 has at least two non-isomorphic minimal representatives as well. Arguing as above, we shall get a chain $K \to K_1 \to K_2 \to \cdots$ of graphs, each of which is obtained from the previous one by an elementary simplification, and each K_i has at least two minimal representative.

This leads us to a contradiction for that graph K_i which is not simplifiable. □

Thus, we have completely described how to recognize the equivalence of framed 4-graphs as elements of $\mathfrak{G}$: One should take their minimal representatives and compare them. In the case of $\mathfrak{G}$, minimal representatives cannot be simplifiable because they (by definition of minimality) contain no bigons to be canceled by a second Reidemeister move, and the number of unicursal components is equal to one.

In the case of graphs from $\widetilde{\mathfrak{G}}$, a minimal representative can be simplifiable in the case it has a split component. In this case the graph is equivalent to zero in $\widetilde{\mathfrak{G}}$.

Analogously to Lemma 8.11 for $\widetilde{\mathfrak{G}}$, one proves the following lemma.

Lemma 8.12. *Framed* 4*-graphs* K *and* K' *from* $\widetilde{\mathfrak{G}}$ *are equivalent if their minimal representatives* $\widetilde{K}$ *and* $\widetilde{K}'$ *either both contain split trivial components, or do not contain split trivial components and are isomorphic.*

Using the parity constructed in Sec. 8.2.5 and the invariant $[\cdot]$ defined for this parity, one can prove the following theorem.

Theorem 8.11. *Let* K *be a framed* 4*-graph, whose Gauss diagram is irreducibly odd. Then for every framed* 4*-graph* K' *representing the same free knot as* K *there exists a smoothing which is isomorphic to the framed* 4*-graph* K (*as a framed* 4*-graph*).

Evidently, this theorem yields Theorem 8.1.

Example 8.4. A framed 4-graph, given by a chord diagram shown in Fig. 8.24, is irreducibly odd.

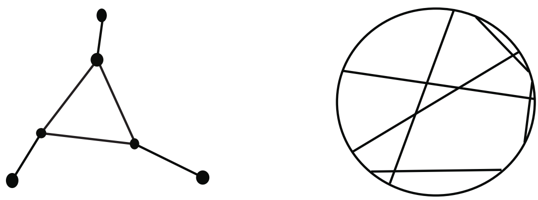


Fig. 8.24 Irreducibly odd chord diagram and its intersection graph.

It is easy to see that the set of *irreducibly odd Gauss diagrams* is infinite (see also [37]), thus, the set of different minimal framed 4-graphs is infinite as well.

Theorem 8.11 is a corollary from the following fundamental statement which is easy to prove.

Statement 8.4. *For an irreducibly odd framed* 4*-graph* K *with one unicursal component, the following equality takes place*:

$$[K] = K. \tag{8.2}$$

On the left-hand side of (8.2), the graph K is considered as a representative of a *free knot*, and on the right-hand side it is considered as an element from $\mathfrak{G}$.

The proof of Statement 8.4 is obvious: Since all crossings of the framed 4-graph K are *odd*, then in the formula for $[K]$ one has to smooth only the empty set of crossings. Thus, we get the only summand which is the 4-graph K itself.

Let us prove Theorem 8.11.

***Proof of Theorem* 8.11.** Let us deduce Theorem 8.11 from Statement 8.4. Let a framed 4-graph K' be equivalent to a framed 4-graph K as a free knot. Then $[K'] = [K] = K$. Consequently, the graph K' has at least one smoothing $\widetilde{K}$, which is equivalent to the graph K as an element from $\mathfrak{G}$. Thus K is a minimal representative for $\widetilde{K}$. Furthermore, note that if a framed 4-graph H' is obtained from a framed 4-graph H by one elementary simplification, then H' is obtained from H by a smoothing in two vertices (just those vertices where the elementary simplification was performed). Consequently, the graph $\widetilde{K}$ is obtained by smoothings of some vertices of the graph K', and the graph K is obtained by smoothing of $\widetilde{K}$. Thus, K is obtained by smoothing from $\widetilde{K}$. □

Analogous results about minimality can be obtained for links with arbitrarily many components. To that end, instead of the bracket $[\cdot]$ and the parity from Sec. 8.2.3 one can use, for example, the bracket $\{\cdot\}$ and the parity from Sec. 8.2.4. Namely, analogously to Theorem 8.11 one can prove the following theorem.

Theorem 8.12. *Let K be a diagram of a free two-component link with all crossings belonging to both components, such that no second decreasing Reidemeister move is applicable to it. Then for every framed 4-graph K' generating the same free link as K, there exists a smoothing which is isomorphic to K as a framed 4-graph.*

Proof. By definition of the invariant $\{\cdot\}$ for the parity from Sec. 8.2.4, we have $\{K\} = K$ (note that the invariant $[\cdot]$ for the parity from Sec. 8.2.3 does not work, since after smoothing the empty set of crossings of the diagram K having more than one unicursal component, we get K itself, consequently, $[K] = 0$).

Since the graph K cannot be decreased by a second Reidemeister move, we see in $\widetilde{\mathfrak{G}}$ that $K \neq 0$. In addition, for every diagram K' which gives the same link as K one has $\{K'\} = K$. Thus, at least one of the smoothings of K' represents an element equivalent to K in $\widetilde{\mathfrak{G}}$, which, in turn, yields that the framed 4-graph K is a smoothing of K'. □

As an example we give the following statement.

Statement 8.5. *A free link diagram given in Fig.* 8.25 *is minimal* (*with respect to the number of crossings*), *and the corresponding atom is orientable.*

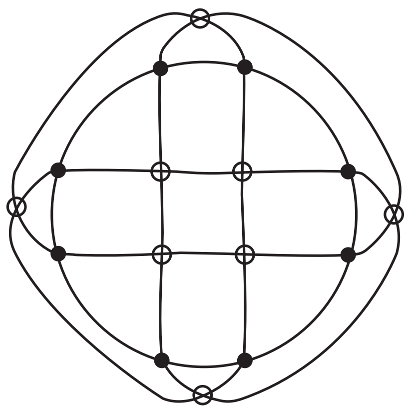


Fig. 8.25 A minimal representative of a two-component link.

Proof. The orientability of the corresponding atom follows from a straightforward check of the source–sink condition. The minimality follows from Theorem 8.12. □

Note that examples of minimal diagrams of framed links in the case of non-orientable atom can be obtained in a much easier way. One may take the simplest two-component framed link with one vertex belonging to both components.

However, the atoms corresponding to this free link as a frame are *non-orientable* since the frame has no source–sink orientation.

It turns out that the methods given above allow one to prove the minimality for diagrams of free knots with orientable atoms as well, i.e. such knot diagrams given by framed 4-graphs having all vertices being even.

8.4.3 *Non-invertibility of free links*

There is a natural operation of orientation reversal on the set of free knots. Let us introduce the notation: For an oriented free knot K (framed 4-graph) by $\overline{K}$ we shall denote the oriented free knot (framed 4-graph) obtained from K by inverting the orientation. Analogously, for an oriented free link L by $\overline{L}$

we shall denote the oriented free link obtained by inverting the orientation of all components of L.

In this section we shall show the existence of non-invertible free links: $L \neq \overline{L}$. To this end, we shall modify the bracket $\{\cdot\}$ in a way such that those graphs which appear as summands of it, carry the information about the orientation of the initial link. By $|K|$, $|L|$ we shall denote the *unoriented* knots and links obtained from K and L by *forgetting the orientation.*

We shall prove the following theorem.

Theorem 8.13. *Let $L = K_1 \cup K_2$ be a framed 4-graph representing a diagram of an oriented free two-component link with components K_1, K_2, and assume the following conditions hold.*

(1) *All crossings of L are formed by both components K_1 and K_2.*
(2) *There is no room to apply any second decreasing Reidemeister move to L.*
(3) *The total number of crossings of L is odd.*
(4) *The framed oriented 4-graph L is not isomorphic to any of the graphs $\overline{K}_1 \cup K_2$, $\overline{K}_1 \cup \overline{K}_2$ with respect to the orientation. In other words, the orientation change of the component K_1 does change the 4-graph regardless of the orientation of the component K_2.*
(5) *There is no isomorphism of the framed 4-graph $|L|$ onto itself taking $|K_1|$ to $|K_2|$ and taking $|K_2|$ to $|K_1|$.*

Then the diagram L is not equivalent to $\overline{L}$.

The first two conditions of the theorem guarantee the minimality of the diagram L according to Theorem 8.12.

It is clear that the diagram $\overline{L}$ is also minimal by the same theorem. Nevertheless, the bracket $\{\cdot\}$ does not allow to distinguish between L and $\overline{L}$ because all summands in the bracket are equivalence classes of *unoriented* framed 4-graphs.

Let us now modify the bracket $\{\cdot\}$. Note that for two-component links the *oddness* of the number of crossings formed by both components is an invariant property.

Let $\mathfrak{H}$ be the set of equivalence classes of $\mathbb{Z}_2$-linear combinations of framed 4-graphs with two unicursal components, where one of which is oriented, by the following two equivalence relations:

(1) the second Reidemeister move (taking into account the orientation of the oriented component);

(2) the relation $K \sqcup \bigcirc = 0$, where diagrams having a split trivial component are equated to zero.

For the set of two-component free links with oddly many crossings between components and with *one oriented component*, we shall construct an invariant $L \mapsto \{L\}_2 \in \mathfrak{H}$.

If both components of the link L are oriented and the number of crossings between the components is odd, then there will be two invariants $\{L\}_{2,K_1}$, $\{L\}_{2,K_2}$ corresponding to L and depending on the choice of an oriented component.

Now, let us construct the invariant $\{\cdot\}_2$.

For the two-component link diagram L, an *odd* crossing is the crossing formed by both components. Let us consider even smoothings of the diagram L. Each of them will be a framed 4-graph representing a link of at least two components: Since we smooth even crossings only, we shall get some set of components originating from K_1 and some set of components originating from K_2.

We shall select only those summands where the number of components is equal to two; besides, we shall endow the component coming from K_1 with an orientation.

The orientation is chosen according to the following rule. For each diagram $L_s = (K_s)_1 \cup (K_s)_2$ obtained from L by a smoothing of all even crossings, at each odd crossing we have the orientation of the component $(K_s)_1$, corresponding to the orientation of the component K_1 at the same crossing.

The number of such crossings is odd, and in each of them, K_1 generates one of the two orientations of the component $(K_s)_1$. For the orientation of $(K_s)_1$, we choose that one orientation (of two) which occurs oddly many times.

More precisely, we have

$$\{L\}_2 = \sum_{s_{\text{even, 2 comp}}} ((K_s)_{1,\text{odd or.}}, (K_s)_2) \in \mathfrak{H}.$$

Then the following theorem holds.

Theorem 8.14. *The bracket $\{L\}_2$ is an invariant of two-component free links with one fixed components and the odd number of mixed crossings between the two components.*

Proof. Let us repeat the proof of Theorem 8.9 paying more attention to the orientation and parity arguments.

First, note that the "non-orientable" version $\{\overline{L}\}_2$ of the invariant $\{L\}_2$ is obtained by a natural projection of the invariant $\{L\}$ to the linear space of framed 4-valent graphs generating two-component links.

Thus, it suffices for us to take care of the behavior of orientations of components under the Reidemeister moves for those links which appear in $\{L\}$.

Under the first Reidemeister move, the check is evident: The crossing participating in the move is smoothed in a way compatible with the orientation and it does not affect the resulting orientation of the summand in the bracket.

The same happens in the case of the second Reidemeister move with two even crossings and in the case of the third Reidemeister move with three even crossings: The corresponding diagrams before and after the Reidemeister move have *the same set of odd crossings with the same orientation of the component K_1 in each of them*.

In the case of the second increasing Reidemeister move $L \mapsto L'$, applied to two odd crossings, the summands in $\{L\}_2$ and in $\{L'\}_2$ are in one-to-one correspondence with each other and are obtained from each other by a second Reidemeister move. It remains to show that in every summands of $\{L'\}_2$ both new crossings generate the same orientation of the oriented component. Thus, the existence of these two crossings does not affect the rule for defining the orientation of the oriented component.

Now, let us consider the case of the third Reidemeister move where both components take place. If two of the three branches belong to the non-oriented component K_2 and one branch belongs to the oriented component K_1, then both in the diagram L before applying the Reidemeister move and in the diagram L' after the Reidemeister move, we have two consecutive crossings of the same orientation. Consequently, the orientations of the corresponding summands in $\{L\}_2$ in $\mathfrak{H}$ coincide.

We are left with the most difficult case, when the oriented component forms two branches, and the non-oriented component forms one branch, moreover, the only even crossing taking part in the third Reidemeister move lies on the self-intersection of the oriented component.

The two versions of this move are shown in Fig. 8.26.

In the upper part of Fig. 8.26 we see that the summands from the first pair on the right and the summands on the left have two crossings each, moreover, the orientations of the component K_1 in these crossings are *opposite* in both cases, consequently, the resulting orientation for the

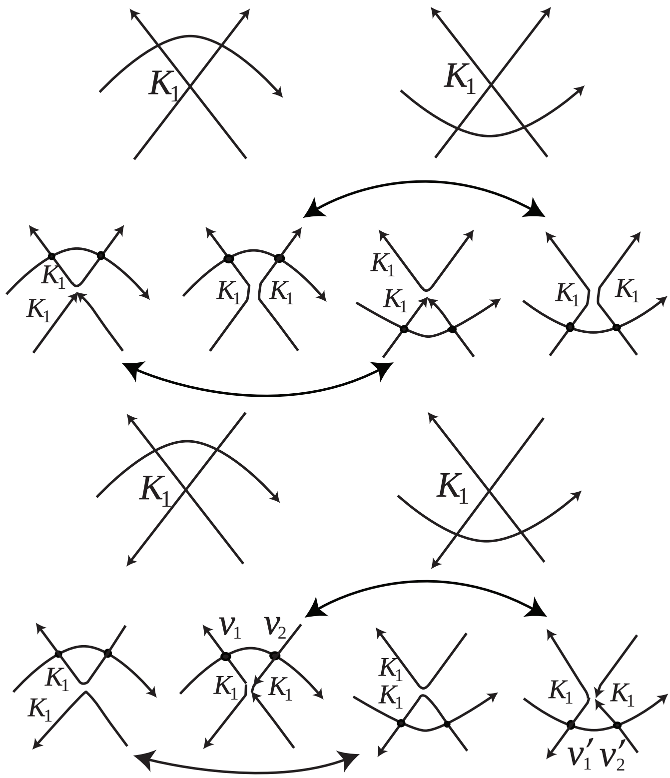


Fig. 8.26 The behavior of orientations under the third Reidemeister move.

summand as an element of $\mathfrak{H}$ is the same (up to second Reidemeister moves).

As for the second summand in the upper part of Fig. 8.26, the two intersection points (left and right) of the component K_1 with the component K_2 generate the same orientations as the two intersection points in the right part, thus, in this case the rule for orienting the component for $\{L\}_2$ in the right part is the same as the rule for the left part.

In the lower picture, the first summand in the left part has two crossings with similar orientation, thus, their common contribution to the definition of orientation of K_1 cancels. The same happens in the first summand in the right part.

In the lower picture, the second summand in the left part has two crossings v_2, v_2 between the components K_1, K_2, and the second summand in the right part has two crossings v_1', v_2' between the corresponding components. Note that the orientation of the component K_1 at v_1 is opposite to the orientation at v_1', and the orientation at v_2 is opposite to the orientation at v_2'. If the orientations of the component K_1 at v_1 and v_2 agree in the left-hand side, then the corresponding orientations agree in the right-hand side. If the orientations in v_1 and v_2 in the left-hand side are different, then in both the right-hand side and the left-hand side we have a couple of crossings which give both possible orientations for the components K_1. Summarizing the above statements, we conclude that in all these cases the orientation of the corresponding summand in $\{L\}_2$ will be the same.

Thus we have proved that the bracket $\{L\}_2$ is invariant under Reidemeister moves as an element of $\mathfrak{H}$. □

From the definition one easily gets the following statement.

Statement 8.6. *If for a link $L = K_1 \cup |K_2|$ all crossings belong to both components, and the number of components is odd, then $\{K_1 \cup |K_2|\}_2 = K_1 \cup |K_2|$ in $\mathfrak{H}$.*

Let us prove Theorem 8.13.

***Proof of Theorem* 8.13.** Assume that the link L satisfies the conditions of the theorem. Also assume that the link L is equivalent to the link $\overline{K}_1 \cup \overline{K}_2$. Applying the bracket $\{\cdot\}$ to $|L|$, we see that the Reidemeister moves cannot take the link $|L|$ to itself and switch the components K_1 and K_2 (the fifth condition of our theorem).

Furthermore, $K_1 \cup |K_2|$ is not equivalent to $\overline{K}_1 \cup |K_2|$ as a two-component free link with a selected oriented component. Passing to $\{\cdot\}_2$, we see that $K_1 \cup |K_2| \neq \overline{K}_1 \cup |K_2|$ in $\mathfrak{H}$.

Thus the links L and $\overline{L}$ are not equivalent. For the coordinated enumeration of components, this non-equivalence follows from the properties of the new invariant $\{\cdot\}_2$, and for the uncoordinated orientation the equivalence is forbidden by the hypothesis of the theorem. □

As an example for Theorem 8.13 we take the free link shown in Fig. 8.27. All conditions of Theorem 8.13 follow from a straightforward check.

8.5 Goldman's bracket and Turaev's cobracket

It turns out that on the set of pairs (a closed oriented 2-surface, a curve on this surface) there are interesting operations which are invariant under the

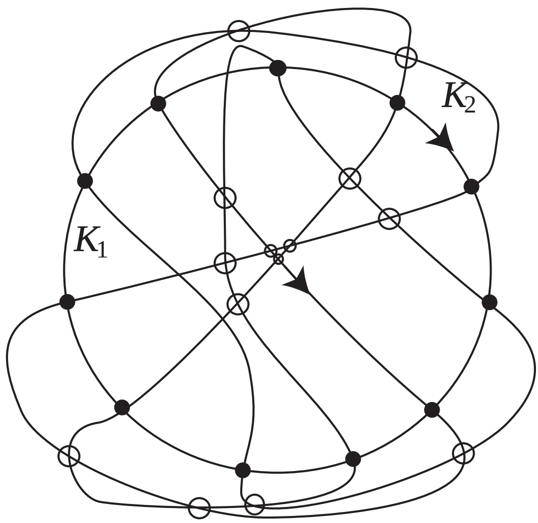


Fig. 8.27 A non-invertible free link.

Reidemeister moves. Namely, let S be a fixed 2-surface (we assume this surface to be oriented; otherwise all arguments below work for the case of coefficients from $\mathbb{Z}_2$).

We think of all curves to be generically immersed. Such a curve represents an embedding of a framed 4-graph. Two immersed curves in general position are *homotopic* if and only if one of them is obtained from the other by a composition of the Reidemeister moves (certainly, in the case when graphs are embedded in a surface, the Reidemeister moves respect the structure of this surface, unlike the Reidemeister moves for free graphs).

Let Γ_S be a set of all linear combinations of homotopy classes (note that homotopy may violate the smoothness condition whenever a first Reidemeister move is applied) of curves on S with coefficients from a ground field F (the field can be arbitrary in the case of orientable surfaces; in the non-orientable case it should be the field of characteristic 2). All curves are assumed to be oriented.

Furthermore, let Γ_S^2 be the set of F-linear combinations of homotopy classes for ordered pairs of oriented curves on S, and let $\Gamma_{S,0}^2$ be the quotient space of the space Γ_S^2 modulo the following relation: $K \sqcup \bigcirc = 0$, i.e. we take to zero all those links having a diagram on S, with two connected components, one of which is homotopy trivial.

Now let us pass to the construction of invariant maps.

8.5.1 *The map* $m\colon \Gamma_S^2 \to \Gamma_S$

Let γ_1, γ_2 be a pair of oriented curves generically immersed in S, and let $v_1, \ldots, v_n$ be crossings from $\gamma_1 \cap \gamma_2$ (since the curves are in general position, they are transverse, and there are finitely many of crossings). Let $m(\gamma_1, \gamma_2)_k$ be the oriented curve obtained by smoothing $\gamma_1 \cup \gamma_2$ at v_k in the way compatible with the orientation. We set

$$m(\gamma_1, \gamma_2) = \sum_{k=1}^{n} \mathrm{sign}_k(1,2) m(\gamma_1, \gamma_2)_k \in \Gamma_S, \tag{8.3}$$

where the sum is taken over all numbers of crossings $k = 1, \ldots, n$, and $\mathrm{sign}_k(1,2)$ at a crossing v_k denotes the sign of the crossing v_k, i.e. it is equal to 1 if the basis formed by tangent vectors $(\dot\gamma_1, \dot\gamma_2)$ is positive, and -1, otherwise.

This sum is considered as an element from Γ_S. A direct check shows that the following theorem holds.

Theorem 8.15 (Goldman [110]). *The map* $m\colon \Gamma_S^2 \to \Gamma_S$ *is well defined, i.e. if the pair* (γ_1, γ_2) *is equivalent to a pair* (γ_1', γ_2'), *then* $m(\gamma_1, \gamma_2) = m(\gamma_1', \gamma_2')$.

Remark 8.13. In the case of framed links, the operation (8.3) is defined only over the field of characteristic 2, because we can define the sign, $\mathrm{sign}_k(1,2)$, only when the curves locally lie on an oriented surface; this is the case, say, for flat virtual knots but not for free knots. Moreover, without such a surface, we are unable to define the map m for any *pair of curves*; it is possible to do that only for a two-component free link (without any surface we do not have any intersection points for two curves defined abstractly).

8.5.2 *Goldman's Lie algebra*

The map m can be treated in another way. Having a pair of curves γ_1, γ_2 (in fact, we were talking about a pair of homotopy classes of curves on S), we may think that there is a well-defined map

$$[\cdot,\cdot]\colon \gamma_1, \gamma_2 \mapsto [\gamma_1, \gamma_2] = \sum_k \mathrm{sign}_k(1,2) m(\gamma_1, \gamma_2)_k,$$

here the permutation of γ_1 and γ_2 leads to the sign change: $[\gamma_1, \gamma_2] = -[\gamma_2, \gamma_1]$.

It can also be easily checked that the operation $[\cdot, \cdot]$ satisfies the Jacobi identity:

$$[[\gamma_1, \gamma_2], \gamma_3] + [[\gamma_2, \gamma_3], \gamma_1] + [[\gamma_3, \gamma_1], \gamma_2] = 0$$

for any triple of curves $\gamma_1, \gamma_2, \gamma_3$.

Thus, the set Γ_S possesses the structure of a Lie algebra.

8.5.3 *The maps* $\Gamma_S \to \Gamma^2_{S,0}$ *and* $\Gamma_S \to \Gamma_S \otimes \Gamma_S/\langle triv.\rangle$: *Turaev's cobracket*

Let γ be an oriented curve in general position on an oriented surface S, and let $v_1, \dots, v_n$ be the intersection points of the curve in S. Then, as a result of smoothing γ at v_i in a way compatible with the orientation, we get two curves, one of which, $\gamma_{i,L}$, can be naturally called *the left one*, and the other one ($\gamma_{i,R}$) is called *the right one*. Thus, we can define a map

$$\gamma \mapsto \sum_{i=1}^{n} (\gamma_{i,L} \otimes \gamma_{i,R} - \gamma_{i,R} \otimes \gamma_{i,L}). \tag{8.4}$$

This map is skew-symmetrical, however, it is not quite well defined. If we apply the first increasing Reidemeister move to the curve γ, then in the right part of the equality (8.4) we shall have additional summands of the form $\gamma_0 \otimes \gamma' - \gamma' \otimes \gamma_0$, where γ_0 is a contractible curve, and γ' is a curve homotopic to γ.

Thus, in order to have a well-defined cobracket, we have to take the quotient of the set of curves by the relation taking contractible curves to zero (in the tensor product, we set $0 \otimes a = a \otimes 0 = 0$). The same arguments allow one to construct an invariant map

(a curve $\to$ a linear combination of pairs of curves),

which does not lead to a coalgebra structure, but it suffices to consider a map given by the formula

$$\Delta\colon \gamma \to \sum_{i=1}^{n} \gamma_{v_i} \in \Gamma^2_{S,0},$$

where the sum is taken over $\mathbb{Z}_2$, and γ_{v_i} means a pair of curves on the surface S obtained by smoothing the curve γ at the crossing v_i coordinated with the orientation, and the result of the map Δ is considered as an element from $\Gamma^2_{S,0}$.

Thus we get a map $\Delta\colon \Gamma_S \to \Gamma^2_{S,0}$, to be called *Turaev's delta* in the sequel.

Let Γ_{Fr} be the set of all linear combinations of all free knots with coefficients from $\mathbb{Z}_2$, and let $\Gamma^2_{\mathrm{Fr},0}$ be the set of $\mathbb{Z}_2$-linear combinations of two-component free links modulo the following relation: $K \sqcup \bigcirc = 0$, i.e. we take to zero all those free links having a diagram with two components one of which is a split unknot.

Analogously, let us define Turaev's delta from Γ_{Fr} to $\Gamma^2_{\mathrm{Fr},0}$: With each framed 4-graph K with one unicursal component we associate a linear combination $\sum_i K_i$ (over $\mathbb{Z}_2$) of framed 4-graphs with two unicursal components each, obtained by smoothing at corresponding vertices, and the resulting sum will be considered as an element from $\Gamma^2_{\mathrm{Fr},0}$.

8.6 Applications of Turaev's Delta

In Sec. 8.4.2 we have shown that the parity (in the sense of Sec. 8.2.3 or in the sense of Sec. 8.2.4) can be used for proving minimality of free knot diagrams. However, the very first example (the non-triviality of an irreducibly odd knot) deals only with free knots whose Gauss diagrams have all odd chords. We have used the invariant $[\cdot]$ for proving its minimality.

Then we considered an example of a two-component free link whose atom is orientable. By means of the invariant $\{\cdot\}$, applied for the parity from Sec. 8.2.4, we proved minimality of this diagram L *in the strong sense*: We proved that every framed 4-graph L' realizing the same link as L admitted a smoothing which represented a graph isomorphic to L as a framed 4-graph.

Now, we are going to give an example of a *free knot* with *orientable* atoms for which one can prove minimality of one of its diagrams by using the above methods.

Statement 8.7. *The diagram K_1 of a free knot, shown in Fig.* 8.28, *is minimal.*

Proof. The orientability of atoms corresponding to the diagram K_1 can be easily checked by a straightforward construction of a source–sink orientation.

Let us consider $\Delta(K_1) \in \Gamma^2_{\mathrm{Fr},0}$.

By construction, $\Delta(K_1)$ consists of nine summands (since K_1 has nine

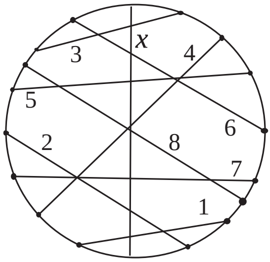


Fig. 8.28 A minimal Gauss diagram.

vertices), and every summand is a two-component link. These summands are obtained as smoothings of K_1 at crossings. One of these summands (obtained by smoothing along the chord x) is the two-component link L, shown in Fig. 8.25. Denote the remaining summands by L_i, $i = 1, \ldots, 8$.

Thus,

$$\Delta(K_1) = L + \sum_i L_i = L$$

(in the last equality we used the central symmetry of the chord diagram; for example, the summand L_1 coincides with L_3, etc.).

Considering $\{\Delta(K_1)\}$ and taking into account the invariance of the map $\{\cdot\}$, we see that every diagram of the free knot has at least one smoothing representing L as a framed 4-graph, i.e. every diagram contains at least nine crossings. □

8.6.1 *Non-invertibility of free knots*

Using non-triviality of free links, one can prove non-triviality of free knots. Analogously, non-invertibility of free links yields non-invertibility of free knots. To this end, it suffices to note that Turaev's Δ is orientation sensitive. Moreover, the following theorem holds.

Theorem 8.16. *The free knot K_2 shown in Fig.* 8.29 *is not invertible.*

Proof. One can easily see that $\Delta(K_2) = L + \sum_i L_i$, where L is the free link shown in Fig. 8.27, and L_i are two-component free links, each of which has at least one crossing formed by two branches of the same component. Indeed, for the chord diagram shown in Fig. 8.29, there is exactly one chord x which is linked with all the other chords.

From this we easily see that $\{\Delta(K_2)\}_2 \neq \{\Delta(\overline{K}_2)\}_2$. □

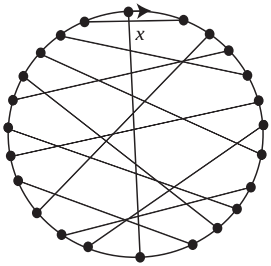


Fig. 8.29 An example of a non-invertible free knot.

8.6.2 *Even and odd analogues of Goldman's bracket and Turaev's cobracket*

Assume some knot theory having a parity with coefficients from $\mathbb{Z}_2$ is given (usually we shall deal with free knots and the parity from Sec. 8.2.3).

Consider the sets $\Gamma_{\rm Fr}$ and $\Gamma^2_{{\rm Fr},0}$. For them, let us define the maps $\Delta_{\rm even}$ and $\Delta_{\rm odd}\colon \Gamma_{\rm Fr} \to \Gamma^2_{{\rm Fr},0}$.

Let K be a framed 4-graph with a unique unicursal component. We set

$$\Delta_{\rm even}(K) = \sum_{S_{\rm even}} K_s, \qquad \Delta_{\rm odd}(K) = \sum_{S_{\rm odd}} K_s.$$

Here in the first case we take the sum of all framed 4-graphs obtained by a smoothing of one even vertex of K (the sum is taken over all even vertices) in a way compatible with the orientation. In the second case the sum is taken over all odd vertices, and again we take those smoothings of one vertex of the graph K compatible with the orientation.

When applying any of the maps $\Delta_{\rm even}$ and $\Delta_{\rm odd}$ to a concrete framed 4-graph K we obtain a sum which can be considered as an element from $\Gamma^2_{{\rm Fr},0}$.

Note that the term "the way compatible with the orientation" makes sense for unoriented framed 4-graphs as well: Among the two smoothings at the vertex one has to choose the one which leads to a two-component link.

Then the following theorem holds.

Theorem 8.17. *The maps $\Delta_{\rm even}$ and $\Delta_{\rm odd}$ are well defined as maps from $\Gamma_{\rm Fr}$ to $\Gamma^2_{{\rm Fr},0}$.*

Proof. To prove the claim, note the following. If two framed 4-graphs K and K' differ from each other by a first Reidemeister move, then the following equality $\Delta_{\text{odd}}(K) = \Delta_{\text{odd}}(K')$ holds termwise: the number of odd crossings of K coincides with the number of odd crossings of K', consequently, there is a one-to-one correspondence between the summands in the expansions for $\Delta_{\text{odd}}(K')$ and for $\Delta_{\text{odd}}(K)$. The corresponding summands are obtained from each other by an application of one first Reidemeister move. In the case of Δ_{even}, in the expansion for K', the number of summands is one more than the number of summands in the expansion for K. This "additional" summand in K' is a two-component link with a split trivial component. Thus, this summand is trivial in $\Gamma^2_{\text{Fr},0}$. The remaining summands in $\Delta_{\text{even}}(K')$ are in one-to-one correspondence with summands from $\Delta_{\text{even}}(K)$, moreover, these corresponding summands are isomorphic as framed 4-graphs.

Now, let K' be obtained from K by a second Reidemeister move in such a way that the number of crossings of K' is two more than the number of crossings of K. These two "extra" crossings have the same parity. If both crossings are odd, then in the expansions of $\Delta_{\text{even}}(K)$ and $\Delta_{\text{even}}(K')$ the number of summands is the same, and all the corresponding summands are obtained from each other by means of a second Reidemeister move. If both crossings are even, then in $\Delta_{\text{even}}(K')$ we have three extra summands in comparison with $\Delta_{\text{even}}(K)$, but two of these three summands coincide identically, and one summand has a split trivial circle. Thus, all new summands contribute zero. All the remaining terms in $\Delta_{\text{even}}(K)$ are in one-to-one correspondence with the terms from $\Delta_{\text{even}}(K')$, and all corresponding terms are isomorphic as framed 4-graphs.

Analogously, if we consider the case Δ_{odd}, then in the case of two odd crossings we get two "extra" summands in $\Delta_{\text{odd}}(K')$ in comparison with those in $\Delta_{\text{odd}}(K)$; these two new summands will cancel; in the case of two even crossings we get a one-to-one correspondence between the terms in the expansion. Thus, $\Delta_{\text{odd}}(K) = \Delta_{\text{odd}}(K')$.

If K' is obtained from K by a third Reidemeister move, then there is a one-to-one correspondence between crossings of the framed 4-graph K and crossings of the framed 4-graph K'. Moreover, even crossings correspond to even crossings, and odd crossings correspond to odd crossings.

If we smooth the diagrams K and K' at v (and the corresponding crossing, to be denoted by the same letter v), and the crossing v does not take part in the third Reidemeister move, then it is evident that the corresponding framed two-component free links will coincide, since the representing graphs are obtained from each other by "the same" third Reidemeister move, which transforms K into K'.

If this crossing v is one of the three crossings taking part in the third Reidemeister move, then it is easy to see that the smoothing of the diagram K at the crossing v either coincides with the smoothing of the diagram K' at the crossing v' (corresponding to the crossing v) or differs from it by an application of two second Reidemeister move, one of which decreases the number of crossings by two, and the other one increases the number by two. □

Analogously to the "comultiplying maps" Δ_{even}, Δ_{odd} one can construct the "multiplying maps" m_{even}, m_{odd}, herewith for the target space we can take a larger space than $\Gamma^2_{\text{Fr},0}$: it is not necessary to equate to zero those diagrams having a split trivial component in the case of "multiplication" maps.

Remark 8.14. It would be interesting to consider the compositions of maps Δ, Δ_{even} and Δ_{odd} (and their reasonable modifications) taken in different orders and analyze which links (free links) can be obtained from some concrete knot at concrete steps for given parities.

Remark 8.15. In order to define the map Δ we do not need the over/undercrossing structure.

8.7 An analogue of the Kauffman bracket

In this section, we are going to construct a refinement of the Kauffman bracket $\langle\cdot\rangle$ for knot theories with a parity, this refinement generalizes the usual Kauffman bracket in the case of classical knots. There are many refinements of the Kauffman bracket for the case of virtual knots, see, e.g. [42, 77, 79, 163–165, 233, 247] and references therein.

We shall explicitly write down the formulae in the case of virtual knots. The generalization of the Kauffman bracket given below works in other cases as well, namely, it works in the cases when there is a natural rule to decree one way of smoothing to be A, and the other rule of smoothing to be B, herewith the Reidemeister moves are "in a natural way compatible with these rules". Note that these generalizations work, for example, for the theory of graph-links, see, e.g. Chap. 9 and [129, 130].

Consider the free module $\mathcal{F}$ over the ring $\mathbb{Z}[a, a^{-1}]$ generated by all framed 4-graphs.

Let $\widetilde{\mathcal{F}}$ be the module obtained by factoring the module $\mathcal{F}$ by the following two relations:

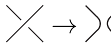
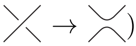
(1) the second Reidemeister move,
(2) the relation $K \sqcup \bigcirc = (-a^2 - a^{-2})K$, where K denotes any arbitrary framed 4-graph, and $K \sqcup \bigcirc$ is the disjoint sum of K with a split circle.

The algorithmic recognizability of elements from $\widetilde{\mathcal{F}}$ (the existence of a unique minimal representative) is proved analogously to the case of $\widetilde{\mathfrak{G}}$.

Virtual knot theory possesses the Gaussian parity (in the sense of Sec. 8.2.3); for every knot diagram at every classical crossing we shall use the fact that there are smoothings of types A and B (but we are not going to dwell on the *axioms*, these crossings, participating in Reidemeister moves, of types A and B have to satisfy).

Now, let us construct the *even Kauffman bracket* valued in the module $\widetilde{\mathcal{F}}$, in the following way:

$$\langle K \rangle_{\text{even}} = \sum_{s_{\text{even}}} a^{\alpha(s)-\beta(s)} K_s,$$

where $\alpha(s)$ (respectively, $\beta(s)$) is the number of positive  (respectively, negative ) smoothings in the state s, and K_s is the free link diagram obtained by smoothing the diagram K according to the state s, considered as an element from $\widetilde{\mathcal{F}}$.

Remark 8.16. There is a natural map $\widetilde{\mathcal{F}} \to \widetilde{\mathfrak{G}}$, obtained by factoring the quotient field: $\mathbb{Z}[a, a^{-1}] \to \mathbb{Z}_2$, where $a \mapsto 1$, $2 \mapsto 0$.

Thus, the invariant $\{\cdot\}$ represents a simplification of the invariant $\langle\cdot\rangle_{\text{even}}$.

Theorem 8.18. *The bracket $\langle\cdot\rangle_{\text{even}}$ is an invariant of knots with respect to the Reidemeister moves Ω_2, Ω_3. When applying the move Ω_1, the value $\langle\cdot\rangle_{\text{even}}$ gets multiplied by $(-a)^{\pm 3}$, and the following normalization for $\langle\cdot\rangle_{\text{even}}$ is invariant under all Reidemeister moves*:

$$X_{\text{even}}(K) = (-a)^{-3w(K)} \langle K \rangle_{\text{even}},$$

where $w(K)$ stays for the writhe number of the oriented diagram K.

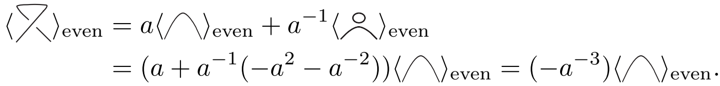
Definition 8.16. We shall call $X_{\text{even}}(K)$ the *even Jones polynomial* (cf. [141]) for the knot generated by K.

Remark 8.17. The even Jones polynomial is a generalization of the invariant $\{\cdot\}$ for every knot theory with a parity and the rules A and B for smoothings. To get the bracket from the Jones polynomial one takes the map $\mathbb{Z}[a, a^{-1}] \to \mathbb{Z}_2 : a \mapsto 1,\ 2 \mapsto 0$.

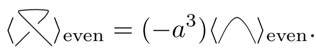
Besides, for the parity in the sense of Sec. 8.2.3 the polynomial $X_{\text{even}}(K)$ is a generalization of the usual Jones polynomial for the case of classical knots and for those knots with corresponding orientable atoms. In this case, in the definition of the bracket $\langle \cdot \rangle_{\text{even}}$, all elements K_s are trivial links which in the module $\widetilde{\mathcal{F}}$ are equal to the multiples of the unknot with coefficients equal to some powers of the polynomial $(-a^2 - a^{-2})$. Taking the generator of the module $\mathcal{F}$, generated by the unknot, to be 1, we get the standard Jones polynomial.

***Proof of Theorem* 8.18.** This proof is completely analogous to the invariance proof for the bracket $\{\cdot\}$. Let K' be obtained from K by one Reidemeister move.

In the case of Ω_1 we have (in $\widetilde{\mathcal{F}}$):



Analogously, for the other case of the first Reidemeister move we have



When applying a second Reidemeister move we have to distinguish between the following two cases. If both crossings taking part in the second Reidemeister move are odd, then there is a one-to-one correspondence between crossings of the diagrams K and K', which yields a one-to-one correspondence between summands of $\langle K \rangle_{\text{even}}$ and $\langle K' \rangle_{\text{even}}$. Herewith the corresponding summands are obtained from each other by applying a second Reidemeister move, thus they represent equal elements from $\widetilde{\mathcal{F}}$.

In the case of two even crossings taking part in the second Reidemeister move, the standard way of proving the invariance for the usual Kauffman bracket under the second Reidemeister move works. As in the standard Kauffman bracket, the three summands in $\widetilde{\mathcal{F}}$ are canceled because of the coefficients a^2, a^{-2} and $(-a^2 - a^{-2})$.

In the case of the third Reidemeister move, one has to distinguish between the cases when the number of even crossings taking part in this move, is equal to one or is equal to three.

If the number of even crossings of K (or K') taking part in the third Reidemeister move is equal to one, then after smoothing the diagrams K

and K' at the corresponding crossings v and v' we get the following sums

$$\langle K\rangle_{\text{even}} = a\langle K\rangle_{\text{even},+} + a^{-1}\langle K\rangle_{\text{even},-},$$
$$\langle K'\rangle_{\text{even}} = a\langle K'\rangle_{\text{even},+} + a^{-1}\langle K'\rangle_{\text{even},-},$$

where the indices plus and minus are responsible for positive (negative) smoothings at the crossings v and v'. It is easy to see that the corresponding summands for K and K' in one of the two smoothings ($+$ or $-$) coincide, and in the other smoothing they differ by an application of the second Reidemeister move to the corresponding graphs, i.e. they are equal in $\widetilde{\mathcal{F}}$.

In the case when all three crossings of the diagram K taking part in the third Reidemeister move $K \to K'$ are even, the proof repeats verbatim the invariance proof for the Kauffman bracket under the third Reidemeister move in the case of classical and virtual knots. □

8.8 Virtual crossing numbers for virtual knots

The main result of this section is Theorem 8.19 showing that the minimal number of virtual crossings for some families of virtual knots grows quadratically with respect to the minimal number of classical crossings.

The main idea of the section is to use the parity arguments, which allow one to reduce some problems about virtual knots to analogous problems about their diagrams (representatives).

Thus, we have to find a certain family of 4-graphs for which the crossing number (minimal number of *additional crossings* (prototypes of virtual crossings) for an immersion in $\mathbb{R}^2$) is quadratic with respect to the number of *vertices* (prototypes of classical crossings).

In the case of graphs, such families having quadratic growth for the number of additional crossings with respect to the number of the crossings themselves are quite well known to graph theorists: Even for trivalent graphs the generic crossing number grows quadratically with respect to the number of vertices, see, e.g. [264].

Notational remark. For a graph Γ, we shall use the standard terminology: the number of vertices, $\mathrm{v}(\Gamma)$, and the crossing number, $\mathrm{cr}(\Gamma)$, the latter referring to the minimal number of additional crossings for a generic immersion, see ahead. For virtual knots, we shall use the notations: vi and cl for minimal virtual crossing number and minimal classical crossing number over all diagrams of a given knot.

Definition 8.17. The *classical* (respectively, *virtual*) *crossing number* cl($\mathcal{K}$) (respectively, vi($\mathcal{K}$)) of a virtual knot $\mathcal{K}$ is the minimum of the numbers of classical (respectively, virtual) crossings over all diagrams of $\mathcal{K}$.

For estimates of virtual crossing numbers for virtual knots see, [1, 3, 4, 23, 77, 274].

In the last years, some attempts to compare the classical and virtual crossing numbers were undertaken, e.g. Satoh and Tomiyama [274] proved that for any two positive numbers $m < n$ there was a virtual knot $\mathcal{K}$ with minimal virtual crossing number $\text{vi}(\mathcal{K}) = m$ and minimal classical crossing number $\text{cl}(\mathcal{K}) = n$.

However, no results were found in the opposite direction: For all known virtual knots the number of classical crossings was greater than or equal to the number of virtual crossings (see tables due to Green [115]).

In this section, we prove that the minimal number of virtual crossings grows quadratically with respect to the minimal number of classical crossings by reducing the problem *from knots to graphs*: We take some family of graphs for which cr grows quadratically with respect to the number of vertices, transform them into 4-graphs (which can correspond to diagrams of virtual knots with vertices corresponding to classical crossings), turn these graphs into a good shape (irreducibly odd) by some minor transformations which increase the complexity a little, and then use the fact that for irreducibly odd graphs the crossing number is equal to the virtual crossing number of the underlying knots.

Given a graph Γ; analogously to the case of 4-graphs, by a *generic* immersion of Γ in $\mathbb{R}^2$ we mean an immersion $\Gamma \to \mathbb{R}^2$ such that

(1) the number of points with more than one preimage is finite;
(2) each such point has exactly two preimages;
(3) these two preimages are interior points of edges of the graph, and the intersection of the images of edges at such a point is transverse.

Definition 8.18. By the *crossing number* $\text{cr}(\Gamma)$ of a graph Γ we mean the minimal number of crossing points over all generic immersions $\Gamma \to \mathbb{R}^2$.

When we deal with framed 4-graphs, we restrict ourselves for such immersions for which at the image of every vertex the images of any two formally opposite edges turn out to be opposite on the plane (in other words, the opposite edge structure is preserved).

Example 8.5. Consider the 4-graph with one vertex v and two edges e_1, e_2 connecting v to v. There are two possible framings for this graph; one of these framings (where one half-edge of e_1 is formally opposite to the other half-edge of e_2) leads to a framed 4-graph with two unicursal components. Such a graph is certainly non-planar, and its crossing number is equal to one. The other framing (where a half-edge of the edge e_1 is opposite to a half-edge of the edge e_2) is planar, so, for that framing the crossing number is zero.

Now, let us present some examples of families of graphs where the crossing number grows quadratically. Let $p > 2$ be a prime number. Consider the chord diagram with $(p-3)/2$ chords obtained as follows. We take the standard circle $x^2 + y^2 = 1$ on the plane, take all residue classes modulo p except $0, p-1, 1$, and put the residue class on the standard (core) circle in the following way. The vertex corresponding to the residue class r will be located at $\left(\cos\frac{2\pi r}{p}, \sin\frac{2\pi r}{p}\right)$. Now, every vertex r is coupled with the vertex s where $rs \equiv 1 \pmod{p}$.

It is known that for such chord diagrams the crossing number grows quadratically in p as $p \to \infty$.

Other examples of families of trivalent graphs with quadratic growth can be constructed by using an *expander family*; for more about expanders, see, e.g. [103, 264]. The idea is as follows. For a graph Γ and a set $V = V(\Gamma)$ of vertices of it, we define the neighborhood $N(V)$ to be the set of vertices of Γ not from V which are connected to at least one vertex from V by an edge. It is natural to study the ratio $\frac{|N(V)|}{|V|}$. A family F_n of graphs is called an *ε-expander family* for some positive constant ε if this ratio exceeds ε for all graphs F_n with sufficiently large n and for all sets V_n of vertices smaller than the half of all vertices of F_n.

We are now ready to state and to prove our main result.

Theorem 8.19. *For some infinite set of positive integers i, there is a family $\mathcal{K}_i$ of virtual knots such that the virtual crossing number of $\mathcal{K}_i$ grows quadratically with respect to the classical crossing number of $\mathcal{K}_i$ as i tends to the infinity.*

The proof of this theorem relies upon the following two lemmas.

Lemma 8.13. *Let K be a framed 4-graph. Let K' be a graph obtained from K by smoothings at some vertices. Then* $\mathrm{cr}(K') \leqslant \mathrm{cr}(K)$.

Proof. Indeed, consider an immersion of K in $\mathbb{R}^2$ preserving the framing and realizing the crossing number $\mathrm{cr}(K)$. Now, take those vertices of K

where the smoothing $K \to K'$ takes place and perform this smoothing just on the plane. The further proof is obvious. □

Lemma 8.14. *Let H_n be a family of trivalent graphs such that the crossing number* $\mathrm{cr}(H_n)$ *grows quadratically with respect to the number of vertices* $\mathrm{v}(H_n)$ *as n tends to the infinity. Then there are two families of framed 4-graphs K_n and K'_n such that*

(1) *the graphs K_n are all irreducibly odd;*
(2) *the number of vertices of K_n does not exceed three times the number of vertices of H_n;*
(3) *K'_n is obtained from K_n by smoothing of some vertices; both K_n and K'_n are graphs with one unicursal component;*
(4) *H_n is a subgraph of K'_n obtained by removing some edges.*

Proof. Let H_n be a connected trivalent graph. Obviously, the number of vertices of H_n is even. Let us arbitrarily partition all the vertices of H_n into $n/2$ disjoint sets consisting of two vertices each. Connect vertices from one set by an edge. We get a 4-graph. We shall denote it by K'_n; to complete the construction of K'_n, we have to find a framing for it in order to get a diagram of a free knot (with one unicursal component).

To do this, we shall use Euler's theorem asserting that for every connected graph with all vertices of even degree there exists a circuit which passes once through every edge. Let us choose an Euler circuit. Define the framing for K'_n in such a way that the Euler circuit is the unicursal circuit for K'_n (two consecutive edges at every vertex are decreed to be formally opposite).

Consider the chord diagram corresponding to K'_n. This diagram might have even and odd chords. Our goal is to construct the chord diagram of K_n by adding some chords to K'_n. Namely, for every chord l of K_n we shall either add one small chord at one end of l (linked only with l) or add two small chords at both ends of l. Our goal is to show that we obtain an irreducibly odd chord diagram such that the framed 4-graph K'_n is obtained from the framed 4-graph of K_n by smoothing at some vertices.

Note that whenever a chord diagram D is obtained from a chord diagram D' by adding one chord linked precisely with one chord of D', then the corresponding 4-graph can be obtained from the framed 4-graph corresponding to D' by smoothing the vertex corresponding to the added chord. Indeed, view Fig. 8.30.

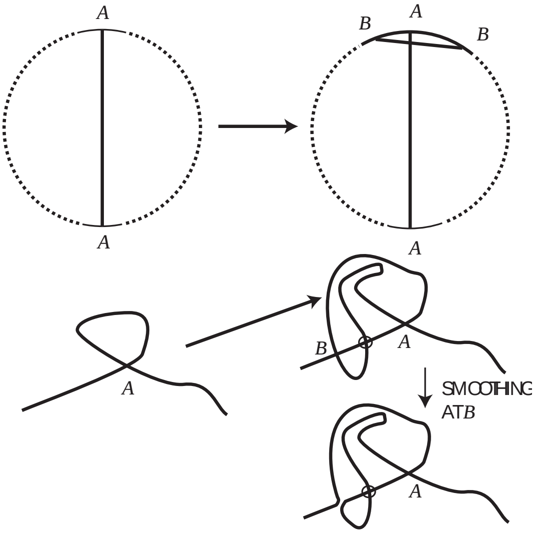


Fig. 8.30 Addition of a chord and the inverse operation.

We shall consider only graphs K'_n such that their chord diagrams have no *solitary* chords (chords not linked with any other chord).

Now, to every even chord of K'_n we add one small chord on one end of it. To every odd chord of K'_n we add two chords on both flanks. This will guarantee that the resulting chord diagram is odd (all "small" chords are odd since each of them is linked with exactly one chord). Besides, this guarantees that the resulting chord diagram (or framed 4-graph) is irreducible.

We shall distinguish between *former* chords (belonging to K'_n) and *new* chords ("small" added chords).

Now, no two former chords (for K'_n) can be operated on by the second decreasing Reidemeister move: for each two chords a, b of such sort there is at least one chord c distinct from a, b which is linked with a and not with b (it suffices to take some "small" chord linked with a).

A former chord cannot participate in the second decreasing Reidemeister move together with a new chord because every former chord is linked with at least one former chord and with at least one new chord, and every new chord is linked with exactly one chord.

If two new chords a and b are linked with different former chords, they cannot participate in the second decreasing Reidemeister move. Neither can they even if they are linked with the same former chord: In this case, since the former chord is not solitary, there is at least one former chord c lying "between" a and b, so, the endpoints of a and b cannot be neighboring.

Now, an obvious estimate shows that the number of chords of K_n does not exceed $3n$. □

Let us prove Theorem 8.19.

***Proof of Theorem* 8.19.** Let us take a family of trivalent graphs H_n with quadratical growths of the crossing number. Denote their numbers of vertices by v_n and denote their crossing numbers by cr_n.

Apply Lemma 8.14. Consider the families of framed 4-graphs K_n and K'_n. Consider an arbitrary immersion of K_n in $\mathbb{R}^2$. Endow all vertices of this immersion with any classical crossing structure; denote the obtained virtual diagram by L_n.

We claim that the classical crossing number $\mathrm{cl}(\mathcal{K}_n)$ of the knot $\mathcal{K}_n$ given by L_n grows linearly with respect to v_n, whence the virtual crossing number $\mathrm{vi}(\mathcal{K}_n)$ grows quadratically with respect to v_n.

The first claim follows from the construction: the number of classical crossings of L_n does not exceed three times the number of vertices of H_n, so, the minimal classical crossing number over all diagrams representing the knot $\mathcal{K}_n$ can only be smaller.

Now, consider $\mathrm{vi}(\mathcal{K}_n)$. Let $\widetilde{L}_n$ be a diagram of the knot $\mathcal{K}_n$. So, if we consider the framed 4-graphs corresponding to diagrams L_n and $\widetilde{L}_n$, they will represent the same free knot. By definition, the framed 4-graph K_n corresponds to L_n. Denote the framed 4-graph corresponding to $\widetilde{L}_n$ by $\widetilde{K}_n$. We see that $\widetilde{K}_n$ represent the same free knot as K_n.

Now, apply Theorem 8.11 to the free knot generated by K_n. By construction, K_n is irreducibly odd. Thus, we see that K_n can be obtained from $\widetilde{K}_n$ by means of a smoothing at some vertices. So, by Lemma 8.13, we have $\mathrm{cr}(K_n) \leqslant \mathrm{cr}(\widetilde{K}_n)$. Therefore, by construction, we get $\mathrm{vi}(\mathcal{K}_n) \geqslant \mathrm{cr}(K_n)$. By Lemma 8.13, we have $\mathrm{cr}(K_n) \geqslant \mathrm{cr}(K'_n)$. The following inequality $\mathrm{cr}(K'_n) \geqslant \mathrm{cr}(H_n) = \mathrm{cr}_n$ holds because H_n is a subgraph of K'_n. As a result, we get $\mathrm{vi}(\mathcal{K}_n) \geqslant \mathrm{cr}_n$ and cr_n grows quadratically with respect to v_n.

This completes the proof of the theorem. □

Remark 8.18. In this direction, one can prove slightly more than stated in Theorem 8.19: The number of virtual crossings grows quadratically with respect to the number of classical crossings not only for virtual knots, but also for virtual knots considered modulo virtualization.

8.9 Cobordisms of free knots

8.9.1 *Introduction*

A curve immersed in a surface admits a natural notion of *null-cobordance* or *sliceness*.

Definition 8.19. One says that an immersed curve $\gamma \subset S_g$ in an oriented closed 2-surface S_g of genus g is *null-cobordant* or *slice* if there exists an oriented 3-manifold M, $\partial M = S_g$, and a smooth map $f\colon D \to M$ of a disc D such that $f(\partial D) = f(D) \cap \partial M = \gamma$.

Definition 8.20. Analogously, one says that the *slice genus* of $\gamma \subset S_g$ does not exceed h if in Definition 8.19 one uses a surface D_h of genus h with one boundary component instead of the disc D.

The first obstructions for curves to be null-cobordant were found by Carter [49]; after that, the theory was also studied by Turaev [301], Orr, and others.

Remark 8.19. In the sequel, we deal only with *generic immersions* of curves in 2-surfaces, unless otherwise specified. By a generic immersion for a curve we mean an immersion such that all whose singularities consist of a finite number of self-intersection points and each self-intersection point is a transverse double point.

It can be easily proved that if two curves are homotopic and one of them is null-cobordant, then the other one is also null-cobordant. Thus one can talk about *null-cobordant classes of homotopy classes of curves.* Moreover, if a curve γ in a surface S_{g+1} does not share a point with a meridian of a handle of the surface, then one can consider the curve γ to lie in the surface S_g obtained from S_{g+1} by cutting S_{g+1} along the curve (meridian) and pasting resulting components of the boundary with discs. It is evident that the pairs (S_g, γ) and (S_{g+1}, γ) are simultaneously either null-cobordant or not.

Therefore, one can talk about *null-cobordant classes of flat virtual knots* [298], which represent equivalence classes of pairs (a circle immersed in an oriented 2-surface, and the surface itself) up to isotopy of curves on our surface and stabilization/destabilization.

The paper [233] (full version is published in [237]) pioneered the overall study of *free knots.* One can naturally define a notion of null-cobordant (or slice) for free knots (see below) in such a way that, if a flat knot is null-cobordant, then the free knot corresponding to it is also null-cobordant.

In this section, we extend the notion of parity from one-dimensional objects (curves with self-intersections) to two-dimensional ones (discs with self-intersections), it allows us to construct invariants of sliceness for free knots. This question (about existence of non-sliced free knots) became actual after the first examples of non-trivial free knots appeared, see accurate definitions of *null-cobordant* free knots and *slice genus* for free knots below, Definition 8.27.

The obstructions for curves to be null-cobordant (and the corresponding construction of cobordism invariants) suggested by Carter, Orr, Turaev cannot be straightforwardly defined for the case of free knots, since they use some homological data of the surface, which free knots do not possess. In some sense, parity can replace homological/homotopy information, when no "genuine" homology is present, see Sec. 8.2.6.

The concept of parity has some other applications in the cobordism theory for free knots and immersed curves. In particular, if a free knot represented by a framed 4-graph Γ is slice, then so is the free knot represented by a framed 4-graph obtained from Γ by "killing odd crossings", see Theorem 8.16.

The aim of this section is to construct one simple (in fact, integer-valued) invariant of free knots which gives an obstruction for a free knot to be slice (null-cobordant).

In [131] we introduced an equivalence relation related to the notion of cobordisms which we called *combinatorial cobordism*: Instead of a topological definition using the notion of a *spanning disc* we dealt with a formal *combinatorial* definition following Turaev's paper [301]. According to this definition two free knots are (*combinatorially*) *cobordant* if one of them can be transformed to the other one by a finite sequence of moves from a given list. These moves include all the Reidemeister moves, and each of them corresponds to a "real" (topological) cobordism.

The section is organized as follows. First we consider a notion of combinatorially cobordant free knots. We construct an invariant and example showing that free knots are non-trivial in the wide sense.

Further, we construct an invariant for topologically cobordant free knots. We prove its invariance under the Reidemeister moves. Then, to show that the invariant is well behaved under cobordisms, we have to extend the notion of parity from one free knot to the spanning disc of the cobordism. This is done by marking double lines of the spanning disc as "even" and "odd". We use homological approach to the definition of parity of a double line which agrees with the homological definition of parity of crossings of a free knot.

After that we give basic definitions of Morse theory for cobordisms of immersed curves, and outline the proof of the main theorem. Taking a Morse function on a spanning disc and having the homological definition of parity for double lines, one extends the invariant to all level lines of this function. Non-triviality of the initial invariant coupled with simple Morse theoretic arguments and the "additivity property" of the invariant leads to a contradiction. The key point in the proof is the way to extend the notion of parity from self-intersection points of a curve to double lines of surfaces.

Note that a free knot combinatorially null-cobordant is topologically null-cobordant. But the validity of the inverse statement is still an open problem.

8.9.2 *Combinatorial cobordism of free knots*

Let D be a chord diagram.

Definition 8.21. By an *even symmetric configuration* C on a chord diagram D we mean a set of pairwise disjoint arcs C_i on the core cycle of the chord diagram which possesses the following properties:

(1) The ends of the arcs do not coincide with chord ends, and the number of endpoints of chords inside any arc is even.
(2) Every chord having one endpoint in C has the other endpoint in C.
(3) Consider the involution i of the core cycle which fixes all points outside the arcs C_i and reflects all arcs along the radii connecting the center of the core circle with the middle of the arc. Connecting by a chord the images of two points which formed a chord in the initial chord diagram we get the chord diagram $i(D)$. We require that the configuration C be symmetric, i.e. the chord diagrams D and $i(D)$ are equal.

Definition 8.22. *Elementary cobordism* means a transformation of a chord diagram deleting all chords belonging to an *even* symmetric configuration, as well as the inverse transformation.

We say that two Gauss diagrams are *cobordant* if one can be obtained from the other by a sequence of elementary cobordisms and third Reidemeister moves.

Remark 8.20. The parity of chords does not change under elementary cobordisms.

The definition of cobordisms given above agrees with the definition of *word cobordism* (nanoword cobordism) [301], since there is a natural map from the cobordism classes of (nano)word to the cobordism classes of free knots.

Note that the first two Reidemeister moves are particular cases of elementary cobordisms, unlike the third Reidemeister move. Therefore, it makes sense to talk about *cobordism classes of free knots.*

The main result of this subsection is a proof of existence of free knots being not combinatorially cobordant to the trivial free knot. To solve this problem we shall construct a combinatorial cobordism invariant of free knots.

Let K be a framed 4-graph. Each unicursal component K_i of K can be treated as a framed 4-graph with the Gauss diagram D_i. Thus, some vertices of the graph K can be represented by chords of one of D_i's (namely, those vertices lying on one unicursal component). Among these, let us choose *even* vertices (in the sense of the Gauss diagram D_i), and at each even vertex v of K, we consider the smoothing K_v (one of the two possible) for which the number of unicursal components is greater than that of K by 1.

Now let $\mathcal{R}$ be the set of $\mathbb{Z}_2$-linear combinations of equivalence classes of framed 4-graphs modulo the second and third Reidemeister moves. Set

$$\Delta(K) = \sum_v K_v \in \mathcal{R},$$

where the sum is taken over all even crossings v.

Remark 8.21. In this subsection, we use the same notation Δ as well as for Turaev's delta, since these two operations, in some sense, are similar. In Turaev's delta a linear combination of ordered pairs of curves is assigned to each curve, and, here, we assign a linear combination of equivalence classes of framed 4-graphs with $k+1$ unicursal components to each framed 4-graph with k unicursal components.

Statement 8.8. *The map Δ is a well-defined map from $\mathcal{R}$ to $\mathcal{R}$.*

Proof. Indeed, assume that K is obtained from K' by a third Reidemeister move. Consider the three crossings v_1, v_2, v_3 of K involved in this move, and the corresponding crossings v'_1, v'_2, v'_3 of K'. By construction, the vertex v_i lies in one unicursal component of K if and only if the vertex v'_i lies in one unicursal component of K'. Moreover, v_i is even if and only if v'_i is even. It is now easy to see that whenever v_i is even, the smoothing K_{v_i} gives the same impact to $\mathcal{R}$ as that of $K'_{v'_i}$ (the corresponding framed 4-graphs are either isomorphic or differ by a second Reidemeister move).

Now, let K and K' differ by a second Reidemeister move, and let K' have two more crossings v_1, v_2 in comparison with K. If both vertices v_1, v_2 are odd, then the summands in $\Delta(K)$ are in one-to-one correspondence with those in $\Delta(K')$ and the corresponding diagrams in each pair differ by a second Reidemeister move. If both v_1 and v_2 are even, then it is obvious that the smoothings at these crossings give equal impact to K', and since we are working over $\mathbb{Z}_2$, they cancel each other. □

Therefore, for any $l \in \mathbb{N}$ the map Δ^l, the iteration of l times of the map Δ, is a well-defined map. So, if K and K' are two framed 4-graphs which are obtained from each other by a third Reidemeister move, then for every positive integer l we have $\Delta^l(K') = \Delta^l(K)$.

Let K be a framed 4-graph with l unicursal components. With K we associate a graph $\Gamma(K)$ (not necessarily 4-valent, but without loops and multiple edges) and a number $j(K)$ according to the following rule. The graph $\Gamma(K)$ will have l vertices which are in one-to-one correspondence with unicursal components of K. Two vertices are connected by an edge if and only if the corresponding components share an odd number of points. The following statement is evident.

Statement 8.9. *If two framed* 4*-graphs K and K' are equivalent, then $\Gamma(K)$ and $\Gamma(K')$ are isomorphic.*

Define the number $j(K)$ *from the graph* $\Gamma(K)$ in the following way. If $\Gamma(K)$ is disconnected we set $j(K) = 0$, otherwise $j(K)$ is set to be the number of edges of $\Gamma(K)$.

Fix a natural number n. Let $\mathcal{N}$ be a linear space generated over $\mathbb{Z}_2$ by formal vectors $\{\mathbf{a}_i, i \in \mathbb{N}\}$. For a framed 4-graph K, we set $\mathcal{J}(K) = \mathbf{a}_{j(K)}$ if $j(K) > 0$ and $\mathcal{J}(K) = 0$ otherwise. We extend this map to $\mathbb{Z}_2$-linear combinations of framed 4-graphs by linearity.

Now, set $\mathrm{I}^{(n)}(K) = \mathcal{J}(\Delta^n(K))$.

Theorem 8.20. *If K and K' are combinatorially cobordant, then* $\mathrm{I}^{(n)}(K) = \mathrm{I}^{(n)}(K')$.

The proof of this theorem follows from two statements, first of them, Statement 8.9, is evident, and the second one, Statement 8.10, is a central. By virtue of Statement 8.9, the mapping $\mathrm{I}^{(n)}(\cdot)$ is invariant with respect to the third Reidemeister move (since so is Δ^n). Moreover, the following holds.

Statement 8.10. *If K_2 is obtained from K_1 by elementary cobordisms, then* $\mathrm{I}^{(n)}(K_1) = \mathrm{I}^{(n)}(K_2)$.

Having proved Statement 8.10, we shall get Theorem 8.20.

We give a sketch of the proof of Statement 8.10.

***Proof of Statement* 8.10.** Instead of framed 4-graphs we shall consider Gauss diagrams.

Let D_2 be the Gauss diagram obtained from a Gauss diagram D_1 by deleting an even symmetric configuration C.

The chords of D_1 belong to three sets:

(1) The set of those chords corresponding to chords of D_2; we denote them for both D_1 and D_2 by γ_j's.
(2) The set of those chords β_j which are fixed under the involution i on C.
(3) The set of pairs of chords α_k and $\bar{\alpha}_k = i(\alpha_k)$ which are obtained from each other by the involution i (here $\bar{\bar{\alpha}}_k = \alpha_k$).

Recall that for every chord diagram D, $\Delta^n(D)$ is a sum of some consecutive smoothings $\Delta^{(p_1 \cdots p_k)}(D)$ along chords $p_1, \ldots, p_k$, where all p_i's are chords of D and p_i occurs to be even after smoothing all $p_1, \ldots, p_{i-1}$.

$\Delta^n(D_1)$ naturally splits into three types of summands:

(1) Those summands where all chords p_i are some γ_j. These smoothings are in one-to-one correspondence with smoothings of D_2. We claim that the corresponding elements $I(\Delta^{(p_1 \cdots p_k)}(D_1))$ and $I(\Delta^{(p_1 \cdots p_k)}(D_2))$ are equal.
(2) Those summands where at least one of p_i is β_j, and neither α's nor $\bar{\alpha}$'s occur among p_i. We claim that each of these summands $I(\Delta^{(p_1 \cdots p_k)}(D_1))$ is zero.
(3) Those summands where at least one of p_i's is α_j or $\bar{\alpha}_j$. These summands are naturally paired: the elements $I(\Delta^{(p_1 \cdots p_k)}(D_1))$ and $I(\Delta^{(\bar{p}_1 \cdots \bar{p}_k)}(D_1))$ are equal.

Let us consider the first case. Since arcs of our even symmetric configurations have no common points with chords γ_j, we see that after smoothing along any of γ's, every arc will completely belong to one circle. This means that the corresponding graphs $\Gamma(\Delta^{(p_1\cdots p_k)}(D_1))$ and $\Gamma(\Delta^{(p_1\cdots p_k)}(D_2))$ are isomorphic.

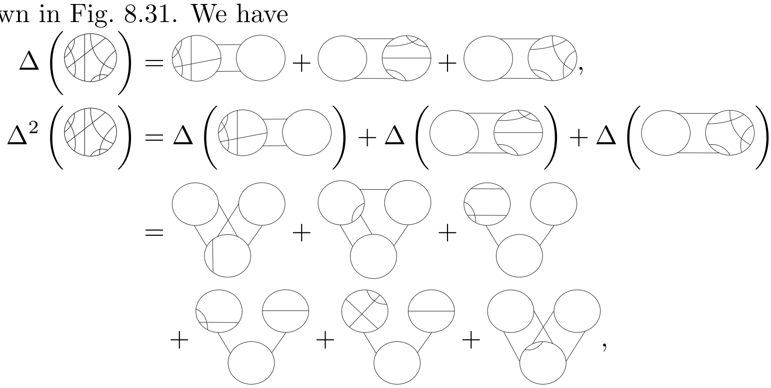
Indeed, chords β_j do not change the graph $\Gamma(\Delta^{(p_1\cdots p_k)}(D_1))$ at all, since they always lie on one unicursal component. Chords α_i and $\bar{\alpha}_i$ belong to either one unicursal component or the same pair of unicursal components. Therefore, they either do not take part or their impacts to the construction of the graph $\Gamma(\Delta^{(p_1\cdots p_k)}(D_1))$ cancels.

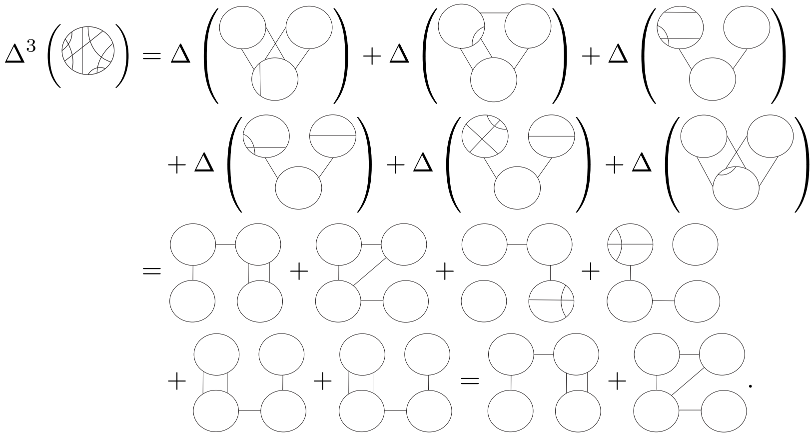
Let us prove the second case. Let us take one chord $p_j = \beta_k$, and consider the arc C_i of D_1 where β_k lies. Without loss of generality, we may assume that β_k is the innermost chord in C_i among those chords β_l we use for smoothings.

Now, our summand looks like $\Delta^{\cdots\beta_k\cdots}$. Smoothing the corresponding diagram (obtained as a result of smoothings along the chords being before the chord β_k) along β_k cuts the free knot (the unicursal component) which contains the arc C_i. It is obvious that this unicursal component will be split in the sense of the graph Γ. It will give a new vertex corresponding to the unicursal component which shares chords of only types α and $\bar{\alpha}$ with other unicursal components. Since we do not smooth those chords, this unicursal component will always share an even number of chords with each unicursal component, i.e. the graph is not connected.

The third case is proved with the help of Theorem 7.8. Namely, using Theorem 7.8 we can always realize whether two ends of a chord belong to either two different circles or the same circle, and whether two chords ends of which lie on distinct circles connect either two distinct circles or the same. □

Example 8.6. Consider the free knot K represented by the Gauss diagram shown in Fig. 8.31. We have





Thus $\mathrm{I}^{(3)}(K) = \mathbf{a}_4$.

Thus, by Theorem 8.20, the cobordism class of K is non-trivial.

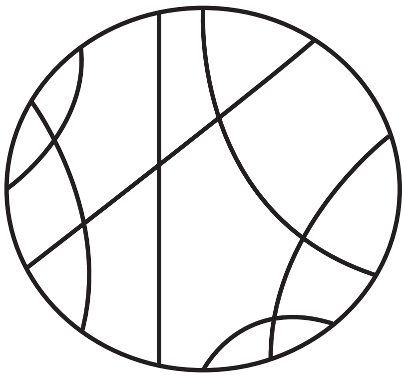


Fig. 8.31 A Gauss diagram of the free knot not cobordant to the unknot.

8.9.3 *An invariant of free knots*

In this subsection, we shall construct an invariant of free knots constructed from the parity and justified parity, and prove its invariance. We shall later prove that this invariant delivers a sliceness obstruction for a free knot. Within the present subsection, by *parity* and *justified parity* we mean the Gaussian parity and the Gaussian justified parity.

An extension of the invariant to be presented below is constructed in [202]; for our purposes (sliceness obstruction) the version given here will suffice. Nevertheless, it is important to investigate the invariant from [202] (in [202] its invariance under the Reidemeister moves for free knots was proved) from the point of view of sliceness.

To construct the invariant we need to introduce the notion of *justified parity* which is analogous to a parity.

Definition 8.23. By a *justified parity* of crossings we mean a parity with coefficients from $\mathbb{Z}_2$ where each odd crossing is marked by a letter b or b' (in these cases we call a crossing *an odd crossing of the first type* or *an odd crossing of the second type*, respectively), so that the following propositions hold:

(1) If a second Reidemeister move is applied to two odd crossings, then they are of the same type (either both are b or both are b').
(2) If in a third Reidemeister move we have two odd crossings, then each of them changes its type after the Reidemeister move is applied (the crossing marked by b before the Reidemeister move is applied, should correspond to the crossing marked by b' after the Reidemeister move is applied).
(3) Moreover, odd crossings not taking part in the Reidemeister move do not change their types.

We define the *Gaussian justified parity* on framed 4-graphs with one unicursal component, as follows.

Definition 8.24. Let D be a chord diagram with the Gaussian parity, i.e. a chord of D is *even* if and only if the number of chords linked with it, is even, and *odd*, otherwise. Furthermore, an odd chord is *of the first type* (after Gauss) if it is linked with an even number of even chords; otherwise an odd chord is said to be *of the second type* (after Gauss).

For the framed 4-graph corresponding to a chord diagram D the Gaussian parity and justified parity are defined as those of the corresponding chord diagram.

It can be checked easily that the Gaussian parity and the Gaussian justified parity satisfy the axioms of parity and justified parity.

Definition 8.25. A section of a cobordism, i.e. a section (level line) of the spanning disc, is called *singular* if it contains a critical point, which is neither Morse critical point nor Reidemeister singularity. Otherwise, a section is called *regular*.

Later in this section, for cobordism purposes we shall then extend the notion of the Gaussian parity and justified Gaussian parity for another situation. First, we shall define the Gaussian parity and justified Gaussian parity for double lines of a 2-disc with generic intersections, and then for every regular section of this disc (which will be a framed 4-graph representing a free link) we shall define the parity and justified parity for crossings to be the parity and justified parity of double lines it comes from.

However, for the first goal (the construction of an invariant of free knots) it would be sufficient for us to have a well-defined parity for just framed 4-graphs with one unicursal component.

Let us consider the group

$$G = \langle a,\, b,\, b' \,|\, a^2 = b^2 = b'^2 = e,\, ab = b'a\rangle$$

with the unit e. Note that G is isomorphic to the infinite dihedral group. For a word γ in the alphabet consisting of a, b, b' we shall denote by $[\gamma]$ the element of G corresponding to γ.

Our first goal is to construct an invariant of free long knots (respectively, of compact free knots) valued in G (in the set of conjugacy classes of the group G).

Let D be an oriented chord diagram, with a marked point X on the core circle C distinct from any chord end. Later, we shall see how one can get rid of the orientation of D.

We distinguish between *even* and *odd* chords of D; moreover, we distinguish between *two types of odd chords* of D.

With a marked oriented chord diagram (D, X) we associate a word in the alphabet $\{a, b, b'\}$ as follows. Let us walk along the core circle C starting from X. Every time we meet a chord end, we write down a letter a, b or b' depending on whether the chord whose endpoint we met, is even, first type odd, or second type odd. Having returned to the point X, we obtain a word $\gamma(D, X)$; this word determines an element of G; by abuse of notation we shall denote this element just by $\gamma(D, X)$. Moreover, sometimes we shall omit X from the notation when it is clear from the context which initial point we have chosen.

Theorem 8.21. *If two marked chord diagrams* (D, X) *and* (D', X') *generate equivalent free knots, then* $[\gamma(D, X)] = [\gamma(D', X')]$ *in* G.

Proof. Indeed, if D and D' differ by a first Reidemeister move (say, D' has one extra chord with respect to D), then the word $\gamma(D')$ is obtained from $\gamma(D)$ by an addition of two consecutive letters $a \cdot a$; thus, the corresponding elements from G coincide.

Analogously, if D' obtained from D by an increasing second Reidemeister move, then the two new chords of D' are of the same parity (and, if they are odd, of the same type); denote the letter corresponding to each of these two chords (a, b or b'), by u. Thus, the word $\gamma(D', X')$ is obtained from $\gamma(D, X)$ by addition of $u \cdot u$ in two places. As in the first case, it does not change the corresponding element of G.

The third Reidemeister move $D \to D'$ may be of one of the two types. In the first case, all three chords participating in the third Reidemeister move, are even.

In this case the words $\gamma(D)$ and $\gamma(D')$ coincide identically.

In the second case, two of the three chords taking part in the Reidemeister move are odd, and one chord is even. Recall that under the third Reidemeister move each of odd chords participating in the move changes its type.

Consider those three segments of the words $\gamma(D)$ and $\gamma(D')$ where the ends of the three moving chords are located. For those segments containing an end of the odd chord, we get one of the two substitutions $ab \longleftrightarrow b'a$ or $ba \longleftrightarrow ab'$. Both changes correspond to some relations in G.

Now consider the segment of the diagram containing the two ends of the odd chords. If these two odd chords are of the same type in D, then on D' they are of the same type as well. Consequently, when passing from $\gamma(D)$ to $\gamma(D')$ we replace $b \cdot b$ by $b' \cdot b'$ or vice versa. Since both subwords correspond to the trivial element of G, we have $\gamma(D) = \gamma(D')$ in G.

Finally, if the two chords participating in the third Reidemeister move are of different types on the diagram D, then when passing from $\gamma(D)$ to $\gamma(D')$ the fragment of the corresponding word *stays the same*. Indeed, the adjacent letters b and b' change their position twice because the chords change their type and chords' ends change their positions.

Thus, no Reidemeister move changes the element of G corresponding to the oriented chord diagram with a marked point. □

This theorem immediately yields the following corollary.

Corollary 8.4. *The conjugacy class of the element* $[\gamma(D, X)]$ *in* G *is an invariant of free knots given by the diagram* D*, i.e. it does not depend on* *the marked point X.*

Indeed, moving the marked point through a chord end corresponds to a cyclic permutation of the letters, which, in turn, generates a conjugation in G.

8.9.3.1 *The Cayley graph of G*

Its Cayley graph looks like a vertical strip on a squared paper between $x = 0$ and $x = 1$: We choose the point $(0, 0)$ to be the unit in the group; the multiplication by a on the right is chosen to one step in a horizontal direction (to the right if the first coordinate of the point is equal to zero, and to the left if this first coordinate is equal to one), the multiplication by b is one step upwards if the sum of coordinates is even and one step downwards if this sum is odd, and the multiplication by b' is one step downwards if the sum of coordinates is even and one step upwards if the sum of coordinates is odd, see Fig. 8.32.

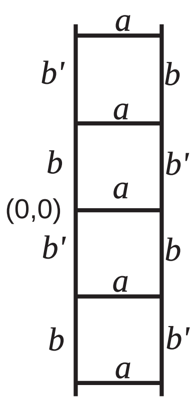


Fig. 8.32 The Cayley graph of the group G.

With each pointed chord diagram (D, X) one associates an element from G. Let us show that each element has coordinates $(0, 4m)$.

Consider a chord of type b'. Since by definition there are odd number of ends of even chords and even number of ends of odd chords between the ends of the chord, then before meeting the first end of the chord we have the coordinate (k, l) and before meeting the second end of the chord we have the coordinate (p, q), where k and p are of different parities, and l and q

are of the same parity. Hence, when we pass through the ends of the chord, the total coordinate shift we have is $(0, 2)$ (if l and q are odd) or $(0, -2)$ (if l and q are even). Analogously, we can show that total coordinate shift at the ends of a chord of type b is $(0, \pm 2)$. Since the number of chords of type b and b' is even, the word in G corresponding to a chord diagram has coordinates $(0, 4m)$.

Moreover, the conjugacy class of the element $(0, 4m)$ for $m \neq 0$ consists of the two elements: $(0, 4m)$ and $(0, -4m)$. Thus, for each *long free knot* one gets an integer-valued invariant, equal to $l = 4m$; we shall denote this invariant for a knot K by $l(K)$; each compact free knot has, in turn, the invariant equal to the absolute value $|l|$; we shall denote the latter by $L(K)$.

It is obvious that if we invert the orientation of the chord diagram, we shall reverse the order of letters in the word γ; this leads to the switch $(0, 4m) \to (0, -4m)$. So, the invariant $L(K)$ can be defined for *unoriented* free knots. The last fact immediately yields the following two corollaries.

Corollary 8.5. *If for an oriented free knot K we have $l(K) \neq 0$, then K is non-invertible.*

Corollary 8.6. *$L(K)$ is an invariant of* unoriented *free knots.*

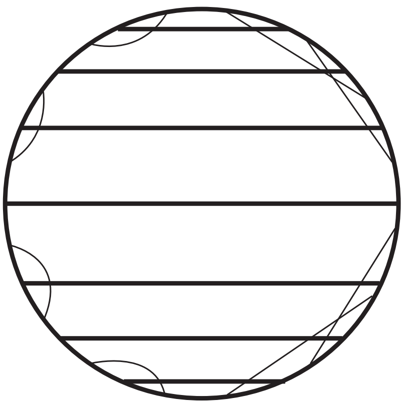


Fig. 8.33 A non-slice free knot.

In Fig. 8.33 we have a free knot K_1 for which $L(K_1) = 16$.

We use *bold* lines for describing even chords. The corresponding word in G (with an appropriate choice of the marked point) looks like $(b'a)^7 b'b(ab)^7 = (b'b)^{16}$.

8.9.3.2 *Remarks on the definition of the invariant L for links*

Note that Theorem 8.21 works for *any parity*, not only for the Gaussian one, and the proof in the case of any parity follows line-by-line the proof of Theorem 8.21.

For cobordism purposes, we shall be able to understand the behavior of the invariant L not only under the Reidemeister moves, but also under Morse bifurcations. Here we shall only use the Gaussian justified parity.

Moreover, first we have to *define* this invariant for links with many components, since after a Morse bifurcation we can obtain a link from a knot.

Our further strategy is as follows. Assuming we have a cobordism (see Definitions 8.26 and 8.27) $\mathcal{D} \to D$ spanning a framed 4-graph K, we shall define the *parity and justified parity* for this cobordism, i.e. we say which of *double lines* (i.e. those lines on D having preimages consisting of two connected components $\mathcal{D}$) are *even*, and which ones are *odd*, and, furthermore, each odd double line will be divided into segments: *odd of the first type* and *odd of the second type*. This parity will be defined in such a way that the parity (the justified parity) for a double point on K will coincide with the parity (justified parity) for the double line this point belongs to. Besides, this approach will allow us to define the parity and the justified parity for any *generic section* of a cobordism; such a section is a framed 4-graph representing a multicomponent link. With these parity and justified parity, we shall be able to extend our invariant L to sections of L with respect to a Morse function on D and then understand the behavior of this invariant under Morse bifurcations. We shall see that the values of this invariant (sets of numbers) will behave nicely under Morse bifurcations, and as a result, we get their invariance.

For a single component free link (i.e. a free knot), the value of the invariant γ can be expressed by one non-negative integer L. For a cobordism, every section is a multicomponent free link, so, we have to define the invariant γ on any component to be a collection of conjugacy classes of elements of G (one for each component), and we may require that these elements of G (or the corresponding conjugacy classes) are expressed by one integer (respectively, non-negative integer) each. In this case we shall be able to associate an integer (respectively, non-negative integer) with each component of a link appearing in a section of our cobordism.

To this end, we shall need that:

(1) The parity and justified parity are well defined for sections and well

behaved under Morse bifurcations.

(2) The number of intersection points of each unicursal component of a section with each double line is even; moreover, the number of intersection points of this unicursal component of the section with odd double lines is even; this condition is necessary in order that the element of G, corresponding to this unicursal component, has in its representation an even number of the letter a and can be described by one coordinate on the Cayley graph.

(3) The value of the first coordinate of the element of G (which is the same as L) behaves well with respect to Morse bifurcations (we shall describe the exact meaning of this below).

In particular, every component of a non-singular level link has an even number of intersection points with double lines: it is necessary for the parity to be well defined. Indeed, in order to define the Gaussian parity of some crossing, one has to take some "half" of the circle corresponding to this crossing and count the number of intersection points belonging to this half. In the case of a free knot the parity of this number of points does not depend on the "half" one chooses because it equals the parity of the number of chords linked with the chord in question.

When we have a two-component link, and we take a crossing formed by a single component, the two parities corresponding to the two halves will be different if the total number of crossings between components is odd.

So, for those two-component links having an odd number of intersection points between components, there is no immediate way to extend the Gaussian parity.

As we shall see further, all these conditions necessary for naturally extending the Gaussian parity will be automatically satisfied for those multicomponent link which are *sections* of a disc cobordism.

In order to do this we need to give a more topological definition of the Gaussian parity.

8.9.4 *Slice genus and cobordisms of free knots*

Now we are ready to give the topological definition of cobordism.

Definition 8.26. Let K be a framed 4-graph with one unicursal component. We say that K has *slice genus at most* g if there exist a surface $\mathcal{D}_g$ of genus g with one boundary component (circle) S, a 2-complex $D_g \supset K$ containing K as a subcomplex, and a continuous map $\nu\colon \mathcal{D}_g \to D_g$ such that:

(1) $\nu(\partial\mathcal{D}_g) = K \subset D_g$; for every vertex v of K we have $\nu^{-1}(v) = \{v_1, v_2\}$, and a small neighborhood $U(v_i) \subset S$ is mapped to a pair of opposite edges of K at v;
(2) the map ν is one-to-one everywhere except a union of intervals: $\Sigma = \{x \in D_g \,|\, \text{card}(\nu^{-1}(x)) > 1\}$;
(3) the set $\Sigma_3 = \{x \in D_g \,|\, \text{card}(\nu^{-1}(x)) > 2\} \subset \Sigma$ is a finite subset of the zero-dimensional skeleton of the complex D_g and consists only of those points having exactly three preimages; moreover, $\Sigma_3 \cap \partial D_g = \emptyset$;
(4) "local three-dimensionality": in the complex D_g neighborhoods of double points, triple points and cusps have the following structure:

   (a) a neighborhood of a cusp (points in which a map is not immersion) is homeomorphic to the Whitney umbrella;
   (b) a neighborhood of a double point from $\Sigma \setminus \Sigma_3$ being not cusp is homeomorphic to a neighborhood of the point $(0, 0, 0)$ in the set $\{(x, y, z) \,|\, xy = 0\} \subset \mathbb{R}^3$;
   (c) a neighborhood of a triple point is homeomorphic to a neighborhood of the point $(0, 0, 0)$ in the set $\{(x, y, z) \,|\, xyz = 0\} \subset \mathbb{R}^3$;
   (d) a neighborhood of a vertex of K in D_g is homeomorphic to a neighborhood of the point $(0, 0, 0)$ in the set $\{(x, y, z) \,|\, xy = 0,\ z \geqslant 0\} \subset \mathbb{R}^3$.

The surface $\mathcal{D}_g$ will be called *the spanning surface* of genus g or the *cobordism of genus* g for K.

In other words, in the definition we require that a free knot (a framed 4-graph K represented by an image of a circle S) is spanned by the 2-complex D_g: an image of the 2-surface $\mathcal{D}_g$, the boundary of which is the circle S, and the singularities of the map $\nu\colon \mathcal{D}_g \to D_g$ are generic singularities (i.e. a neighborhood of each singularity is embedded in $\mathbb{R}^3$).

Analogously, one defines the slice genus for framed 4-graphs with many unicursal components, the cobordism of genus g for free knots (in this case a spanning surface has several boundary components), and the equivalence relation $\sim$ for free knots to be cobordant. Two free knots K_1 and K_2 are *cobordant* if the free link $K_1 \sqcup K_2$ is cobordant to the trivial link (with a cobordism of genus 0).

Remark 8.22. Further, saying a word "cobordism" we always mean a cobordism of genus 0 unless otherwise stated.

The closure $\overline{\Sigma}$ except for points from Σ, also contains *cusps*, i.e. such points $x \in D_g$ for which $\text{card}(\nu^{-1}(x)) = 1$, and, moreover, for any small neighborhood $U(x)$ of x the intersection $U(x) \cap \Sigma$ represents a punctured interval. Denote by Σ_2 the complement $\Sigma \setminus \Sigma_3$.

The intersection $\nu(S) \cap D_g$ is a framed 4-graph in D_g. This graph is obtained from $S = \partial \mathcal{D}_g$ by *pasting double points* in S. The framing (the structure of opposite edges) for this graph is obtained from S. Namely, for a point x in $\nu(S) \cap \Sigma$ the preimage $\nu^{-1}(U(x) \cap \nu(S))$ consists of two branches of S. The images of these two branches under ν will generate two pairs of opposite edges.

Definition 8.27. A free knot is called *null-cobordant* or *slice* if it has slice genus zero.

If a free knot K admits any cobordism of genus g and does not admit a cobordism of genus $g-1$, we say that K has *slice genus* g. Notation: $sg(K) = g$.

The following lemma follows from the definition of a free knot.

Lemma 8.15. *If framed* 4*-graphs* K, K' *represent the same free knot, then* K *and* K' *are cobordant and, therefore,* $\text{sg}(K) = \text{sg}(K')$.

Indeed, in Fig. 8.34 we demonstrate that an equivalence under each Reidemeister move leads to them to be cobordant; since to be cobordant is an equivalence relation, we get the necessary. The first Reidemeister move corresponds to a cusp point, the second Reidemeister move corresponds to a passage through a tangency point, and the third Reidemeister move corresponds to a triple point.

Thus, it makes sense to speak about *the slice genus* of free knots, but not only for framed 4-graphs.

Remark 8.23. Let K be a flat knot and $|K|$ be the underlying free knot. Then it follows from the definition that the slice genus of K is greater than or equal to the slice genus of $|K|$. In particular, if K is slice, then so is $|K|$.

Example 8.7. The first example of a non-slice flat virtual knot was constructed by Carter [49], it is shown in Fig. 8.35.

This knot is embedded in an oriented surface of genus 2. Let us orient this surface. In Fig. 8.35, the arrows indicate the *clockwise direction of branches*. Namely, orient the core circle of the chord diagram counterclockwise and orient the immersed curve accordingly. If two oriented branches

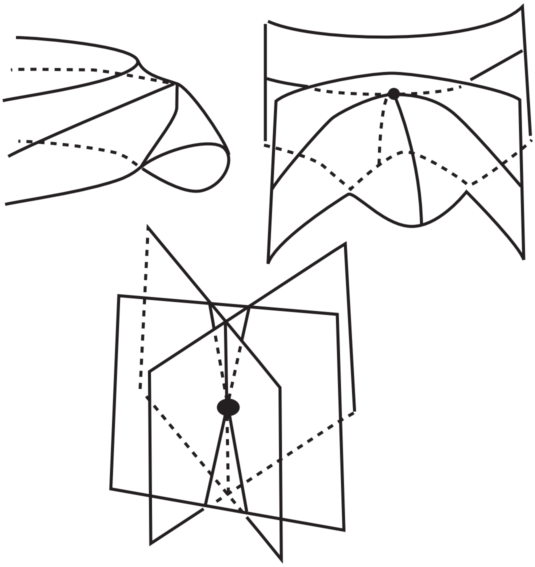


Fig. 8.34 Cobordisms corresponding Reidemeister moves.

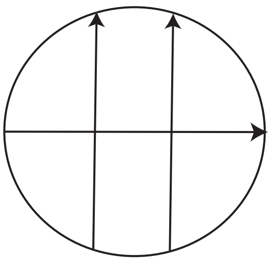


Fig. 8.35 Carter's non-slice flat virtual knot.

(a, b) of the curve have an intersection at a double point v and the tangent vectors $\tau_{v,a}$, $\tau_{v,b}$ form a positively oriented basis, then the arrow is directed from a to b.

In this notation, for flat virtual knots, two chords participating in a second Reidemeister move should have opposite orientations. The flat knot K in Fig. 8.35 is non-trivial as a *flat virtual knot*, and, moreover, it is non-slice. Nevertheless, when we forget about the arrows, we can first cancel the two vertical arrows (by the second Reidemeister move) and then cancel the horizontal arrow (by the first Reidemeister move). So, the corresponding free knot $|K|$ is trivial, and hence *slice*.

So, if a free knot is *non-slice*, then so is *every* underlying virtual knot.

The problem of finding *non-slice* free knots is rather complicated.

In Sec. 8.9.2 we introduced the notion of combinatorial cobordism. Our combinatorial transformations consist of the third Reidemeister move and addition/removal of an "even symmetric configuration". The latter transformations include the first and second Reidemeister moves as particular cases, and each of them represents a (topological) cobordism of genus zero. Thus, if two knots are combinatorially cobordant, then they are also topologically cobordant. Interconnection between topological and combinatorial cobordant relations will be considered in another paper.

In the work by Carter [49] and Turaev [301], topological sliceness obstructions for immersed curves (which come from flat knots) were studied. For each double point v of an immersed curve Γ, one considers the homology class of the halves $\Gamma_{v,1}$, and takes the *homological pairing* of these halves in the surface. These pairings form an integer matrix, an obstruction for a cobordism is formulated in terms of properties of the obtained matrix. This approach cannot be applied to free knots because a framed 4-graph is not assumed to be embedded in *any* 2-surface. Moreover, embeddings into different 2-surfaces may crucially change the intersection form for "halves" even with $\mathbb{Z}_2$-coefficients.

8.9.5 *Parity of curves in 2-surfaces*

Let us now pay more attention to the structure of cobordisms of free knots. Assume there is a cobordism $\nu\colon \mathcal{D} \to D$ (of genus zero) spanning the free knot (framed 4-graph) $K = \nu(\partial\mathcal{D})$.

Set $\Psi = \nu^{-1}(\overline{\Sigma})$. Then Ψ has a natural stratification containing strata of dimensions zero and one. The strata of dimension zero are *double points on the boundary*, *cusps*, and *triple points*; all other points form strata of dimension one. By a *double line* we mean a minimal (with respect to inclusion) collection of one-dimensional strata possessing the following properties:

(1) Two 1-strata attaching the same cusp from opposite sides belong to the same double.
(2) Two 1-strata attaching the same triple point from *opposite* sides belong to the same double line, see Fig. 8.36.

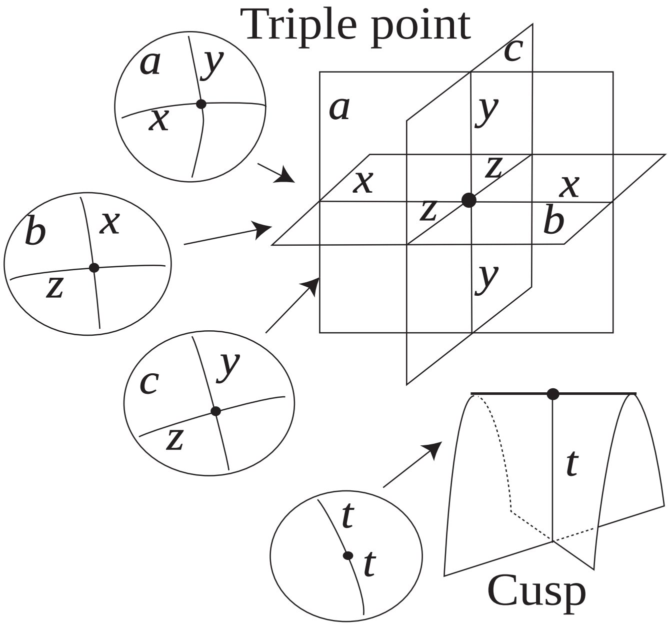


Fig. 8.36 Double lines a, x, y, z, cusps and triple points.

Let $v \in K \cap \Sigma$ be a double point on the boundary of ∂D. Assume $\nu^{-1}(x) = \{x_1, x_2\}$.

Recall the definition of the Gaussian parity for framed 4-graphs. Let us consider a double point x. We take an arc (a half of the core circle) ι of K connecting x to x, being the image of an arc $\tilde{\iota}$ on S, and count the parity of the double point number on ι. Note that here we take a half of the core circle, but not just a curve on K connecting the point x to itself. The fact of the matter is that if we take a curve on the circle $\partial\mathcal{D}$ connecting two preimages x_1 and x_2 of x, but locally (in neighborhoods of these preimages) pointed towards *different* "halves", then the number of double points on such curve is greater (or smaller) by one, since one of the preimages of x is added.

Now, let us consider the preimage $\tilde{\iota} \subset S$ connecting x_1 to x_2. Then the definition of parity $p(x)$ can be reformulated as the parity of card$(\tilde{\iota} \cap \Sigma)$. Note that this line $\tilde{\iota}$ belongs to the disc $\mathcal{D}$.

If we want to generalize the notion of parity on double lines and make it more topological, then instead of $\tilde{\iota} \subset \partial\mathcal{D}$ we may take an arbitrary path

$\tilde{\eta} \subset \mathcal{D}$ in generic position connecting x_1 to x_2 (see Fig. 8.37). Generic position means that the path intersects $\overline{\Sigma}$ only in Σ_2, and all intersections are transverse. Moreover, we have to reformulate the condition of right-orientation of the path in neighborhoods of the starting point and ending point (see below). Having done this, we immediately obtain a well-defined definition of parity, since such two paths are homotopic *with respect to the boundary*, and the parity of the number of intersection points with the set Σ will not change.

We impose the following condition concerning the behavior of the curve in neighborhoods of x_1 and x_2. When we take $\tilde{\eta} \subset \mathcal{D}$, neighborhoods $\tilde{\eta} \cap U(x_1)$ and $\tilde{\eta} \cap U(x_2)$ of x_1 and x_2 belong to the same half of the circle S. In terms of $\mathcal{D}$ and D, this can be reformulated as follows.

Let $\zeta \subset D$ be the 1-stratum in D attaching the point x. Orient ζ arbitrarily, and orient the two preimages $\zeta_1 \cap U(x_1)$ and $\zeta_2 \cap U(x_2)$ accordingly. Consider two vectors v_1 and v_2 tangent to $\tilde{\eta}$ at x_1 and x_2, respectively (see Fig. 8.37). We require that the bases $(\dot{\zeta}_1, v_1)$ and $(\dot{\zeta}_2, v_2)$ generate two *different* orientations of $\mathcal{D}$. If we change the direction of both x_1 and x_2 it will not change the parity of $\tilde{\eta} \cap \Sigma_2$.

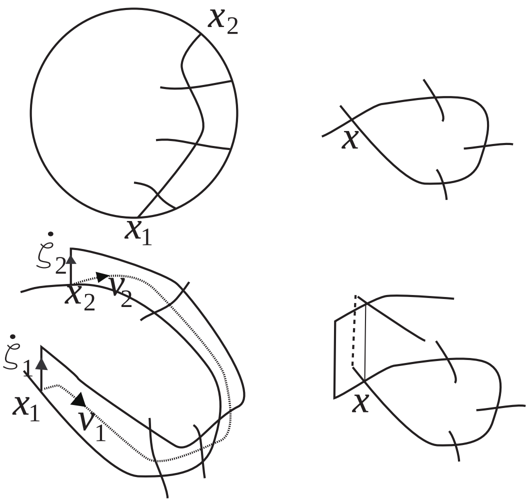


Fig. 8.37 The geometric way for defining parity.

So, this gives another way to define a parity for x. Consider it as the definition of parity for *any* double point from Σ_2.

Actually, let x be a point on a double line from Σ_2. Having counted the number of intersection points of a curve connecting x_1 to x_2 by the way mentioned above, we get the number $p(x)$. We call it *the Gaussian parity* of the point x.

The following statement easily follows from the definition.

Statement 8.11. *The Gaussian parity is constant along double lines.*

Proof. This is evident for two points belonging to the same 1-stratum and for points on two 1-strata attaching the same cusp. When passing through a triple point, the parity does not change, see Fig. 8.38. We see that the curve connecting the two preimages of A is "parallel" to the curve connecting the two preimages of B everywhere except for the two small domains; inside these two domains, we have two intersections with double lines p and q which cancel each other. Thus, the parity of points A and B coincides. □

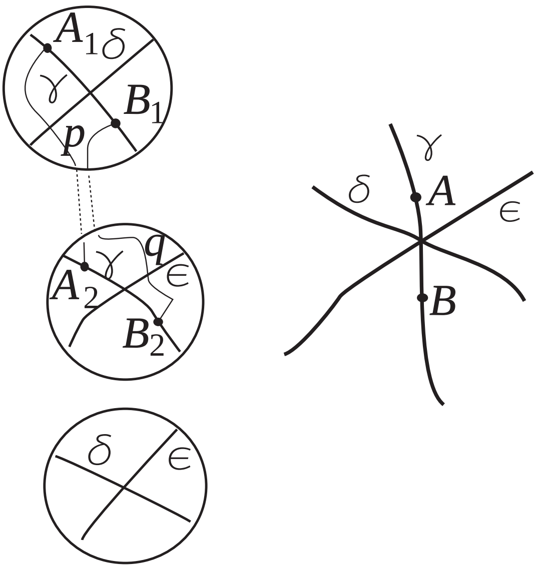


Fig. 8.38 The behavior of parity and justified parity when passing through a triple point.

This leads to the definition of parity for a double line.

Definition 8.28. Let γ be a double line on a cobordism. Take an arbitrary x on $\gamma \cap \Sigma_2$, and consider the two preimages x_1 and x_2 of it on $\mathcal{D}$. Connect x_1 to x_2 by a generic path $\tilde{\eta}$, such that its behavior in neighborhoods $U(x_1)$ and $U(x_2)$ is coordinated (as in the definition of the Gaussian parity). Now, the *Gaussian parity* of the double line containing x is the parity of the number of intersection points between $\tilde{\eta}$ and Ψ.

This allows us to define the set Ψ_{even} (respectively, Ψ_{odd}) to consist of the closure of all those points of Ψ belonging to *even* (respectively, *odd*) double lines.

From the definition of the Gaussian parity for double lines it is easy to get the following statement.

Statement 8.12. *Amongst three double lines intersecting at the same point the number of odd lines is even* (*equals zero or two*).

Now, to define the *justified Gaussian parity*, we should take an *odd double line* γ, an arbitrary generic point x on it, and consider the two preimages x_1 and x_2 of x. Then we connect x_1 to x_2 by a generic path $\delta \in \mathcal{D}$ (of course, its behavior in neighborhoods $U(x_1)$ and $U(x_2)$ is coordinated) and count the number of intersections between δ and Ψ_{even}. If this number is even, we say that the 1-stratum containing x is of the *first type*; otherwise we say that this 1-stratum is of the *second type*.

Note that here we do not require any coordination (coorientation) of the first and last segments of the path: under small perturbation of the path in a neighborhood of its start (end) the number of intersection points only with *odd double line* can change, but we are interested in intersections with even double lines.

From the definition we have the following statement.

Statement 8.13. *The Gaussian justified parity is constant on* 1*-strata belonging to* Ψ*. It does not change when passing through a cusp point* (*from one stratum to another stratum of the same double line*), *and it changes from* b *to* b' *or from* b' *to* b *when passing through a triple point formed by two odd double lines and one even double line.*

Proof. The statement that the type does not change along 1-strata is evident. Consider now Fig. 8.38. Assume the double lines γ and δ are odd and the double line ϵ is even. Then the two lines connecting the preimages of A and the preimages of B are "parallel" except for two domains where one of them passes through a double line at p, and the other one passes through a double line at q. Since we disregard intersection with *odd* double lines, we see that q counts and p does not. Therefore, we have proved that the type changes by one when we pass through a triple point. □

Having defined the *Gaussian parity* for double lines and the *Gaussian justified parity* for strata, we will be able to construct the invariant L for any *section* of a cobordism D. The parity properties proved above for the parity of double lines and the justified parity of strata guarantee that this invariant is well defined and behaves nice under the Reidemeister moves.

8.9.6 *Sliceness of free knots*

It turns out that the invariant L of free knots is an *obstruction to sliceness.*

Before proving the main theorem, we shall make several observations concerning sliceness. From Lemma 8.15, it follows that the slice genus is well defined on the set of free knots.

The following statement is trivial (see Fig. 8.34).

Statement 8.14. *If a framed 4-graph K' is equivalent to a framed 4-graph K (by Reidemeister moves), then the slice genus of K' is equal to that of K. In particular, if K is slice then so is K'.*

Thus, it makes sense to speak about *cobordisms* of free knots, but not only for framed 4-graphs.

As a corollary, we get the following statement.

Statement 8.15. *If a framed 4-graph K is embeddable in S^2 or T^2, then K is slice.*

Indeed, every framed 4-graph on S^2 is equivalent (even as a flat virtual knot) to the (free) unknot; every framed 4-graph on the torus T^2 is homotopic either to the trivial loop on the torus (with no crossings) or to a knot lying in a cylinder (a complement to a simple non-contractible curve in the torus). In the latter case this free knot is equivalent to a knot lying in a sphere (as cylinder is a submanifold of a sphere) and, therefore, it is trivial.

Statement 8.16. *Let K be a free knot, and let $f(K)$ be a free knot obtained from K by deleting* odd *crossings. If K is slice, then so is $f(K)$.*

Proof. Indeed, any cobordism (of genus zero) for K generates a cobordism of genus zero for $f(K)$ obtained by separating all *odd double lines*: Two points from Σ_2 will be pasted together in the new cobordism if and only if they lie on an even double line in the first cobordism. Note that the rule of pasting is agreed with triple points, since if we have two even double lines in a triple point, then the third double line is also even. □

Denote the obtained cobordism for $f(K)$ by $f(D)$ where D is the cobordism for K.

The main result of this section is the following theorem.

Theorem 8.22. *If a free knot K has $L(K) \neq 0$, then K is not slice.*

In particular, this yields the following corollary.

Corollary 8.7. *Let γ be a curve immersed in an oriented closed 2-surface S_g. Then if for a free knot K corresponding to γ one has $L(K) \neq 0$, then the flat virtual knot corresponding to γ is not slice.*

Indeed, a disc immersed in a 3-manifold is a spanning disc. The converse statement is generally not true.

The problem of finding obstructions for a surface S_g with a curve γ to span a disc immersed in a 3-manifold M with boundary S_g was studied by Carter [49], Turaev [301] etc. Some topological obstructions based on homology of S_g were constructed.

In this section, we consider a more complicated problem: Instead of curves in 2-surfaces we consider framed 4-graphs, and instead of spanning 2-discs in 3-manifolds we consider "abstract" spanning 2-discs. In this case we cannot define a "homology group", since a framed 4-graph can be embedded in different surfaces, which have different homology groups.

From this point of view the notion of parity plays in some sense the role of a "substitute for homology of S_g".

8.9.6.1 *Constructing the Morse function and the Reeb graph*

The proof of the main theorem will consist of several steps.

First, let us adopt the following notation: by ν we shall denote a map $\mathcal{D} \to D$ corresponding to the cobordism, and by μ we shall denote as a Morse function $\mu\colon D \to \mathbb{R}$ on the disc with self-intersections (see the definition below) as the composition $\mu \circ \nu\colon \mathcal{D} \to \mathbb{R}$ representing a Morse function on the disc $\mathcal{D}$.

Assume a free knot K (represented by a framed 4-graph) admits a cobordism $\nu\colon \mathcal{D} \to D$ (of genus zero).

Definition 8.29. By a *Morse function* on D we mean a Morse function $\mu\colon \mathcal{D} \to [0,\infty)$ such that if $\nu(x) = \nu(y)$, then $\mu(x) = \mu(y)$, all triple points and cusp points on $\mathcal{D}$ lie on non-critical levels of μ, and $\mu^{-1}(0) = K$, $\mu^{-1}(1) = \emptyset$. By abuse of notation we shall denote the function on $\mathcal{D}$ and the function on D by the same letter f.

By a *non-singular* value of the function μ we mean a non-critical value c of the function μ such that $\mu^{-1}(c) \subset \mathcal{D}$ contains no cusps and no triple points; a singular (respectively, non-singular) level is the preimage of a singular (respectively, non-singular) value. A Morse function on D will be called *simple* if every singular level contains either exactly one critical point, or exactly one triple point, or exactly one cusp point.

From now on, we require that the Morse function on D is simple and the level 0 is non-singular. It is clear that such Morse functions are *everywhere dense in the class of all functions.* Every Morse function has singular levels of two types: those corresponding to Morse bifurcations (saddles, minima, and maxima) and those corresponding to Reidemeister moves. Denote singular levels of the Morse function μ by $c_1 < \cdots < c_k$ and choose non-singular levels a_i: $0 = a_0 < c_1 < a_1 < c_2 < \cdots < a_k < c_k < a_{k+1} = 1$.

Let us construct the *Reeb graph* Γ_μ (*molecule*) of the Morse function μ as follows. All vertices of this graph have degree either 1 or 3. The univalent vertices of the Reeb graph (except one) will correspond to minima and maxima of the function μ; the vertices of degree three will correspond to saddle points (note that each saddle consists of a transformation of one circle to two circles or two circles to one, since the surface-disc is oriented); edges will connect critical points; every edge will correspond to a cylinder $S^1 \times I \subset D$ which is continuously mapped by μ to a closed interval (this edge) between some two critical points (vertices); this cylinder has no critical Morse points inside (but may contain inside critical points corresponding to Reidemeister moves). One edge will emanate from the point 0 (a non-critical point) corresponding to the circle $S = \partial\mathcal{D}$, see Fig. 8.39.

Since this graph is the Reeb graph of a Morse function on the disc $\mathcal{D}$, the graph Γ_μ is a *tree*.

Our next goal is to endow each edge of the Reeb graph with a non-negative integer *label*. The label of the edge emanating from 0 will coincide with $L(K)$.

For every non-singular level c of μ the preimage $K_c = \mu^{-1}(c) \subset D$ is a framed 4-graph representing a free link; when passing through a Reidemeister singular point, it is operated on by the corresponding Reidemeister

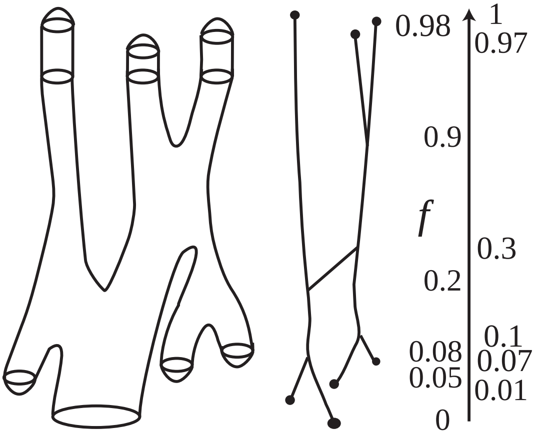


Fig. 8.39 The Reeb graph and circle bifurcations.

move; when passing through a Morse critical point it gets operated on by a Morse-type bifurcation. Every crossing of K_c belongs to some double line. Define the *parity* of a crossing to be the Gaussian parity of the double line, it belongs to. Analogously, define the *justified parity* of a crossing to be that of the 1-stratum, it belongs to.

One can easily see that if a section of the Morse function is a free knot (a framed 4-graph with one unicursal component), then the parity and justified parity coincide with the Gaussian parity and justified parity defined directly via Gauss diagrams.

Choose a non-singular level c, and consider the free link K_c and orient its component arbitrarily (as we shall see further, the orientation will be immaterial); for every unicursal component $K_{c,j}$ of the free link K_c we may define the conjugacy class $\delta(K_{c,j})$ of $\gamma(K_{c,j})$ in G just as it is done for free knots with the Gaussian parity and justified parity. Let $\delta(K_c)$ be the unordered collection of all $\delta(K_{c,j})$ for all j (with repetitions).

From Statements 8.11 and 8.13 we get the following lemma.

Lemma 8.16. *The parity and justified parity defined on the set of all non-singular levels K_c satisfy the parity and justified parity axioms under those Reidemeister moves which happen under passing from one non-singular level to another one within the cobordism D.*

Thus, we get the following lemma.

Lemma 8.17. *When changing a parameter c along an interval $[a, b] \subset [0, 1]$ on which the level K_c is operated on by Reidemeister moves but not Morse bifurcations, and the levels corresponding to a and b contain no Reidemeister move, we have $\delta(K_a) = \delta(K_b)$ if the orientations of components of the links K_a and K_b are agreed with each other: these sets represent conjugacy classes of elements from G.*

Moreover, $\delta(K_0)$ is the conjugacy class of $\gamma(K)$.

The proof literally repeats the proof of Theorem 8.21.

Now, we would like to treat δ as a collection of *non-negative integers* with multiplicities and to forget about orientations of components of K_c. To this end, we prove the following lemma.

Lemma 8.18. *Let c be a non-singular level of μ, and let $K_{c,1}, \ldots, K_{c,n}$ be unicursal components of the free link $K_c = \mu^{-1}(c) \subset D$. Then for every $i = 1, \ldots, n$ the following properties hold:*

(1) *The total number of intersection points between $K_{c,i}$ and $K_{c,j}, j \neq i$, is even.*
(2) *The number of odd intersection points between $K_{c,i}$ and $K_{c,j}$, $j \neq i$ (i.e. intersection points lying on odd double lines), is even.*

Proof. The proof follows from the fact that the preimage of the framed 4-graph $K_{c,i}$ in $\mathcal{D}$ is a circle, and the intersection of a closed curve with the set Ψ (or Ψ_{even}) in $\mathcal{D}$ consists of an even number of points. □

Lemma 8.18 immediately means that every $\gamma(K_{c,i})$ for a non-singular value c is represented by an element $(0, 2k)$ on the Cayley graph of G (for some integer k).

Let $L_c = \{l_{c,1}, \ldots, l_{c,m}\}$ be the unordered collection of integers (with repetitions) obtained from $\delta(K, c)$ by replacing conjugacy classes of elements from G with absolute values of their second coordinates. Since the reversing of the orientation changes the coordinate of an element of the group G from $(0, 2k)$ to $(0, -2k)$, then under reversing the orientation the unordered collection of modules of numbers (which is talked about in Lemma 8.18) corresponding to components of links does not change.

Each $l_{c,i} \in \mathbb{N} \cup \{0\}$ corresponds to a component of the free link K_c and does not change under Reidemeister moves when changing c without passing through Morse critical points. Associate it with the corresponding edge of the graph Γ_μ.

Now, let us analyze the behavior of these labels $l_{c,i}$ at vertices of the graph Γ_μ.

Lemma 8.19. *Assume $K_{c-\varepsilon}$ and $K_{c+\varepsilon}$ differ by one Morse bifurcation at the level c. Then:*

(1) *If this bifurcation corresponds to a birth of a circle, then $L_{c+\varepsilon}$ is obtained from $L_{c-\varepsilon}$ by an addition of 0.*
(2) *If it corresponds to a removal of a circle, then $L_{c+\varepsilon}$ is obtained from $L_{c-\varepsilon}$ by a removal of 0.*
(3) *In the case of fusion of two circles into one the set $L_{c+\varepsilon}$ is obtained from $L_{c-\varepsilon}$ by applying the following operation: All elements except two (being equal to m and n) remain the same, and the elements m and n turn into some $k = \pm n \pm n$ to form an element.*
(4) *The fission operation is the inverse to the fusion: Instead of one element k one gets a pair of elements m, n such that $\pm m \pm n = k$.*

Proof. The first two assertions are obvious: The trivial circle has no double points, thus the corresponding element of G is the unit of G, its coordinate is $(0, 0)$ and the corresponding label is equal to 0.

The last two assertions follow from the following observation. If a circle with a marked point X splits into two circles by a Morse bifurcation connecting X to some point Y, then the corresponding word $w \in G$ splits into the product $w = w_{XY} w_{YX}$.

The rest of the proof follows from the multiplication rule in G: For elements u, $v \in G$ having coordinates (u_1, u_2) and (v_1, v_2), respectively, the product $u \cdot v$ has coordinates $(\pm u_1 \pm v_1, \pm u_2 \pm v_2)$. □

The proved lemma leads to the following way of proving Theorem 8.22. The graph Γ_μ has all vertices of degree one except possibly one (corresponding to the initial knot K), having label 0. At each vertex of degree 3 the three labels with signs $\pm$ sum up to give zero. Thus, taking into account that the Reeb graph is a tree, we get $L(K) = 0$. The contradiction completes the proof of Theorem 8.22.

Example 8.8. Consider the free knot K_1 shown in Fig. 8.33. By Theorem 8.22, it is not-slice. Thus, all flat virtual knots with the underlying free knot K_1, are not slice, either.

8.9.7 Cobordisms of higher genus

The methods used for proving the fact that the invariant L gives an obstruction to the sliceness are not immediately generalized for obtaining lower estimates on the slice genus of free knots. Two reasons are as follows. First, when we define a *parity* and *justified parity* for double lines on $\mathcal{D}$, we chose an arbitrary curve connecting the two preimages of a point on the double curve. We assert that any two curves connecting these two preimages (and behaving correctly in neighborhoods of the ends) are homotopic. It is true in the case of cobordism of genus zero, but in the case of a surface of an arbitrary genus h it is, indeed, not true.

Thus, in order to define a parity for double lines we have to impose some restrictions on the spanning surface. We have to require that the cohomology class dual to the graph Ψ was $\mathbb{Z}_2$-homologically trivial. The significance of the property for even-valent graphs to be $\mathbb{Z}_2$-homologically trivial is closely connected with *atoms* (for details see [234]).

Another problem is that the Reeb graph of an arbitrary Morse function (not necessarily corresponding to the disc) is not necessarily a disc, see Fig. 8.40.

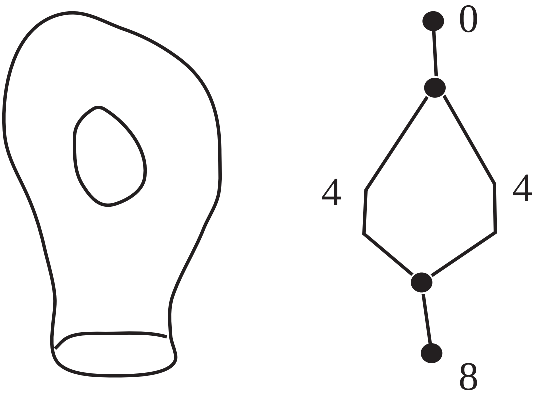


Fig. 8.40 The Reeb graph corresponding a cobordism of genus one.

Thus, starting from a free knot K for which, we say, $L(K) = 8$, we (in principle) can turn it by a Morse bifurcation into free two-component link consisting of two free knots K_1 and K_2, for which $L(K_i) = 4$, and then by another Morse bifurcation we can reconstruct this trivial link into the unknot. The invariant L is not an obstruction to this, since the sum of 4 and -4 is zero.

In some cases, we can overcome these two difficulties for cobordisms (of arbitrary genus).

Let $\mathcal{D}_g$ be a surface with boundary S^1. Obviously, the collection of double lines of $\mathcal{D}_g$ defines a relative $\mathbb{Z}_2$-homology class $\kappa \in H_1(\mathcal{D}_g, S^1; \mathbb{Z}_2)$. This homology class is an obstruction for the surface to be checkerboard-colorable; also, this is an obstruction for well-definedness of even/odd double lines.

Namely, if we look at the definition of an even/odd double line: We see that there is an ambiguity in the choice of path connecting two preimages of a generic point on the double line. For the case of a disc cobordism, the parity of double lines is well defined, because all such curves are homotopic. For $\mathcal{D}_g$ the unique obstruction to this well-definedness is the class κ.

We call a cobordism of genus g *checkerboard* (or *atomic*) if the corresponding class κ vanishes.

The next task (after detecting which 1-stratum is even and which one is odd) is to distinguish between b and b'. To this end, one should do the same for preimages of points lying on odd 1-strata, connect them by a generic curve, and count the intersection with *even* double lines. So, we see that the only obstruction is the relative $\mathbb{Z}_2$-homology class $\kappa' \in H_1(\mathcal{D}_g, S^1; \mathbb{Z}_2)$ generated by even double lines.

We say that a checkerboard cobordism is *two-atomic* if κ' vanishes.

It turns our that Theorem 8.22 is true for 2-atomic cobordisms. Namely, the following theorem holds.

Theorem 8.23. *Assume for a one-component framed 4-graph K we have $L(K) \neq 0$. Then there is no two-atomic cobordism spanning the knot K of any genus.*

Proof. Let us revisit the proof of Theorem 8.22. Assume there is a two-atomic cobordism $\mathcal{D}_g \to D_g$ of some genus g spanning the knot K. Fix a generic Morse function f on $\mathcal{D}_g$ (the corresponding Morse function on D_g will be denoted by the same letter f).

Following the lines of the proof of Theorem 8.22, we see that:

(1) at each non-critical level c of f we have a free link K_c of some number of components, and with each component we associate a natural number coming from a conjugacy class in G;
(2) these numbers behave nicely under Morse bifurcations, i.e. a birth/death of a circle corresponds to an addition/removal of an oc-

currence of 0;

(3) for a saddle point the three numbers k, l, m corresponding to the adjacent edges satisfy $\pm k \pm l \pm m = 0$.

Note that every saddle point merges two circles into one or splits one circle into two circles (the Möbius bifurcation is impossible because $\mathcal{D}_g$ is orientable).

Recall that each of these numbers k, l, m at this point was defined only up to sign (equivalently, we had an absolute value of these three numbers), and this is not sufficient to prove the theorem in the case of cobordism of arbitrary genus.

Now, let us be more specific and study these numbers on edges in more detail. Let t be a non-critical level of f, and let $K_1, \dots, K_n$ be the free knots composing the corresponding free link K_t. For each of these knots K_j, we have defined the integer by taking an initial point of K_j. This number $2l$ comes from an element of G of the form $x = (bb')^l$, and we see that a conjugation of x by any of a, b, b' takes x to $(b'b)^l = x^{-1}$. So, a conjugation by a word of *even length* does not change the number $2l$, whence the conjugation by a word of an *odd length* takes it to $-2l$.

Let us take a checkerboard coloring of $\mathcal{D}_g$ with respect to the cell decomposition generated by double lines.

Whenever we take an initial point of any section, this initial point does not belong to any double line, so, it has some color, black or white. We see that a conjugation by a word of even length does not change the color of the initial point.

This means that the numbers $l_b(K_j)$ and $l_w(K_j)$ are well defined and $l_b(K_j) = -l_w(K_j)$, where subscripts b and w correspond to the color choice of the initial point on the circle, for every component K_j.

Fix the color black once and forever. Then every edge of the Reeb graph acquires an integer number l_b, and for every level when we merge/split circles, the sum of these l_b's does not change.

So, the sum l_b remains invariant for every non-critical level of the Morse function. Since it is non-zero at $t = 0$, it will remain non-zero for every t. This completes the proof of the theorem. □

Chapter 9

Theory of Graph-Links

9.1 Introduction

It is well known that classical and virtual knots [158] can be represented by Gauss diagrams, and the whole information about the knot and its invariants can be read out of any Gauss diagram encoding it, see Fig. 9.1. If a chord diagram is not a classical knot Gauss diagram, i.e. the Gauss code is not planar, see Fig. 9.2, then we obtain a virtual knot.

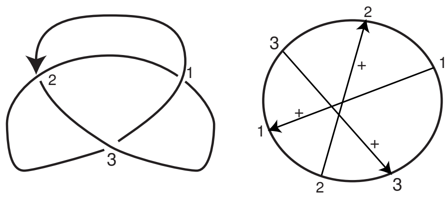


Fig. 9.1 The right trefoil and its Gauss diagram.

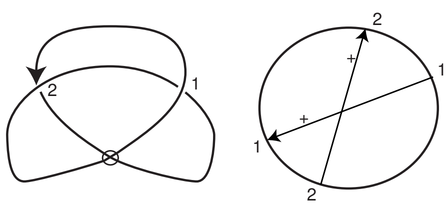


Fig. 9.2 The virtual trefoil and its Gauss diagram.

It turns out that some information about the knot can be obtained from a more combinatorial data: the intersection graph of a Gauss diagram, see Definition 7.13 and Fig. 9.3. Vertices of the intersection graph are endowed with the local writhe number of the crossing. However, sometimes a Gauss diagram can be obtained from the intersection graph in a non-unique way, see Fig. 9.4, and some graphs (shown in Fig. 9.5) cannot be represented by chord diagrams at all [39] (we call these graphs *non-realizable*). Other questions related to non-realizable graphs were considered, e.g. in [38, 41].

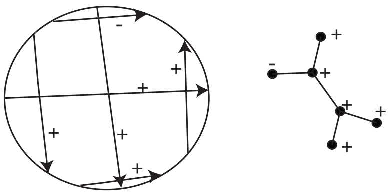


Fig. 9.3 A Gauss diagram and its labeled intersection graph.

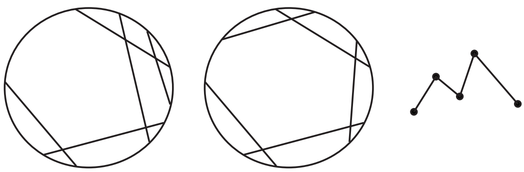


Fig. 9.4 A graph not uniquely represented by chord diagrams.

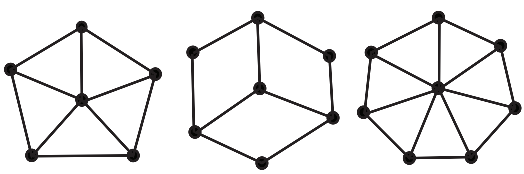


Fig. 9.5 Non-realizable Bouchet graphs.

When passing to the intersection graph, we remember the writhe number information, but forget the information about the cyclic order of half-edges at each vertex encoded by the arrows. In principle, it is possible to describe analogous objects when all information is saved in the intersection graph; however, already the writhe number information is sufficient to recover a lot of data, as we shall see. Even more, if we forget about the writhe number information and only have the structure of opposite edges we shall get non-trivial objects (modulo the Reidemeister moves), see Chap. 8.

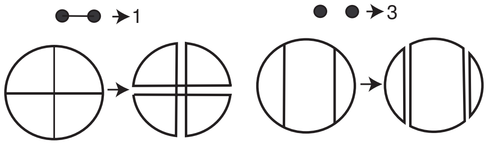


Fig. 9.6 Resmoothing along two chords yields one or three circles.

Probably, the simplest evidence that one can get some information out of the intersection graph is Theorem 7.8 allowing one to count the number of circles in Kauffman's states out of the intersection graph, see Fig. 9.6 (intersection graphs and chord diagrams are depicted in the figure). In particular, this means that graphs not necessarily corresponding to knots admit a way of generalizing the Kauffman bracket polynomial, which coincides with the usual Kauffman bracket when the graph is realizable by a chord diagram. This was the initial point of investigation for Traldi and Zulli [293] (*looped interlacement graphs*): They constructed a self-contained theory of "non-realizable graphs" possessing lots of interesting knot theoretic properties. These objects are equivalence classes of (decorated) graphs modulo "Reidemeister moves" (translated into the language of intersection graphs). A significant disadvantage of this approach was that it had applications only to knots, *not links*: In order to encode a link, one has to use a more complicated object rather than just a Gauss diagram, a Gauss diagram *on many circles*. This approach was further developed in Traldi's works [290–292], and it allowed to encode not only knots but also links with any number of components by decorated graphs. The first question arises here: Whether or not every simple graph is Reidemeister equivalent to the looped interlacement graph of a virtual knot diagram? The negative answer to this question was obtained in [233], i.e. not every simple graph is equivalent to a graph realizable by a chord diagram.

We suggested another way of looking at knots and links and generalizing them: Whence a Gauss diagram corresponds to a transverse passage along a knot, one may consider a rotating circuit. Moreover, one can also encode the type of smoothing (Kauffman's A-smoothing or Kauffman's B-smoothing) corresponding to the crossing where the circuit turns right or left and never goes straight, see Fig. 9.7. We note that each vertex has a label depending on the orientation of opposite edges (framing 0 or framing 1).

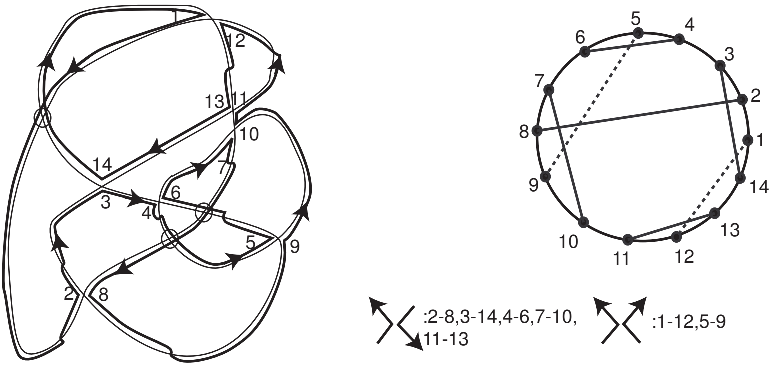


Fig. 9.7 Rotating circuit shown by a thick line; chord diagram.

The second important question which arises here is whether there is an equivalence between the set of *homotopy classes of looped interlacement graphs* introduced by Traldi and Zulli [293] and the set of *graph-knots* constructed in [129, 130]. According to Sec. 7.8.4 for virtual knots there is an explicit formula connecting the adjacency matrices of the intersection graphs of the Gauss diagram and a chord diagram constructed by using a rotating circuit, and vice versa. This formula allows one to prove the equivalence between two theories [127, 128], and, moreover, the homotopy class of looped interlacement graphs and the graph-knot both constructed from a given virtual knot diagram are related by this equivalence. This construction will be described in detail in Sec. 9.2. In general, one can consider an arbitrary Euler tour on links and construct the chord diagram, and then the intersection graph [291]. In this book we consider only rotating circuits and Gauss circuits.

It is obvious that whenever an intersection graph is realizable in the sense of Gauss diagrams, then the corresponding "rotating" intersection graph is realizable in the sense of rotating diagrams and vice versa: Just because if one of these two graphs is realizable, the corresponding framed 4-graph can be just drawn on the plane (with virtual crossings) and the algorithm becomes a "real redrawing algorithm", or we can just apply the constructed equivalence between the set of homotopy classes and the set of graph-knots, see further. The redrawing algorithm shows that the statement is true in the realizable case. In the general case, the statement follows immediately from the theorems in [127, 128]. These theorems allow us to switch from Gauss diagrams to rotating diagrams and vice versa, whenever we are proving some non-realizability theorems. We shall show that there are graphs which are not equivalent to looped interlacement graphs by Reidemeister moves and it follows from the equivalence that the situation is the same for graph-links. The examples of "non-realizable" looped interlacement graphs first appeared in [233, 234]. So, the theory of graph-links is interesting for various reasons:

(1) In some cases, it exhibits purely combinatorial ways of extracting invariants for knots.
(2) In some cases, it produces heuristic approaches to new "knot theories".
(3) It highlights some "graphical" effects which are hardly visible in usual or virtual knot theory.

Note that the theory of Khovanov homology is constructed for graph-knots and, therefore, for homotopy classes of looped interlacement graphs [36, 256, 257].

We call a graph-link (respectively, a homotopy class of looped interlacement graphs) *non-realizable* if it has no realizable representative. We conclude the introduction part by a couple of examples of non-realizable *free graph-links* and *free homotopy classes of looped graphs*. Here we do not indicate any crossing decoration (but, of course, the structure of opposite edges) because any graph-link (any homotopy class of looped graphs) with this underlying graph is non-realizable. The homotopy class generated by the graph depicted in Fig. 9.5 (left) gives us a non-realizable *free homotopy class of looped graphs*. Having a virtual link, we may forget about over/under information and cyclic order of half-edges at each vertex, and take care only about the underlying framed 4-graph with the structure of opposite edges. This leads us to the notion of a *free knot* and *free link*, see Chap. 8. It turns out that some information about virtual knots can be caught just from the underlying framed 4-graph. This information is enough to prove the non-triviality of many free knots and free links. The same trick works for graph-links and homotopy classes of looped graphs: However, here instead of the underlying framed 4-graph we consider an abstract graph which plays the role of the intersection graph of the *non-existing chord diagram*.

The graph shown in Fig. 9.8 (left) is itself non-realizable (in the sense of looped interlacement graphs and Gauss diagrams), but what if we decorate its crossings in some way and then try to apply Reidemeister moves hoping to make it realizable. For some graph-links it is possible, see, e.g. Fig. 9.8. The homotopy class generated by the looped graph shown in Fig. 9.8 (left) is realizable. Indeed, the second Reidemeister move translated into the language of Gauss diagrams is an addition/removal of two "parallel" chords. In the language of intersection graphs, chords correspond to vertices, and "parallel" chords correspond to vertices having the same set of adjacent vertices. So, the vertices A and A' in Fig. 9.8 are adjacent, and the removal of these two vertices makes our graph realizable.

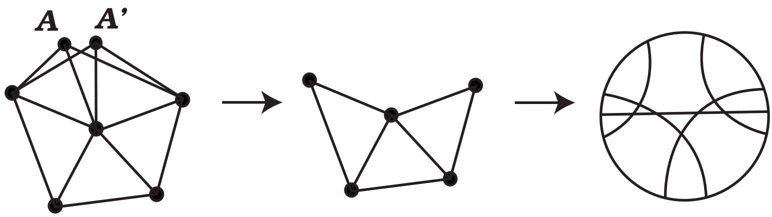


Fig. 9.8 A non-realizable graph representing a trivial graph-link.

The problem of finding free graph-links and free homotopy classes of looped graphs having no representative realizable by a chord diagram is a problem similar to the problem of constructing virtual knots not equivalent to classical knots. Surprisingly, the solution in the case of free links can be achieved by using *parity considerations* (Theorem 9.9): All the vertices shown in Fig. 9.5 (left) are *odd* (each of them is adjacent to an odd number of other vertices) and there is no immediate way to contract any two of them by using a second Reidemeister moves. This is indeed sufficient for a homotopy class to be non-realizable in a very strong sense: Any representative of this class has a subgraph isomorphic to the initial graph (we disregard the writhe number information), which is non-realizable, and, the homotopy class is, in turn, itself non-realizable. Also we have an example of non-realizable graph with all vertices being even, see Fig. 9.9.

This chapter is organized as follows. We first give definitions of graph-links and homotopy classes of looped graphs. Further, we introduce parity and prove non-triviality results. In particular, the parity arguments allow one to construct graph-valued invariants of graph-links.

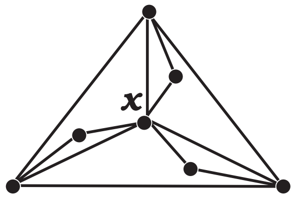


Fig. 9.9 An even non-realizable graph representing a non-realizable homotopy class.

In Sec. 9.4, we briefly describe the way of extending the Kauffman bracket and some other invariants. We also formulate some minimality results for graph-links.

9.2 Graph-links and looped graphs

9.2.1 *Chord diagrams*

Since any two equivalent (in the class of all virtual diagrams) connected (see Definition 4.13) virtual diagrams are equivalent in the class of connected virtual diagrams [129], without loss of generality, all virtual diagrams are assumed to be connected and contain at least one classical crossing [129, 130].

Definition 9.1. A chord diagram is *labeled* if every chord is endowed with a label (a, α), where $a \in \{0, 1\}$ is the *framing* of the chord, and $\alpha \in \{\pm\}$ is the *sign* of the chord. If no labels are indicated, we assume the chord diagram has all chords with label $(0, +)$.

Remark 9.1. Thus, we have two types of chord diagrams: framed (see Definition 4.3) and labeled.

Let D be a labeled chord diagram. One can construct a virtual link diagram $K(D)$ (up to virtualization) in such a way that the chord diagram D coincides with the chord diagram of a rotating circuit on $K(D)$. Let us immerse this diagram D in $\mathbb{R}^2$ by taking an embedding of the core circle and placing some chords inside the circle and the others outside the circle. After that we remove neighborhoods of each of the chord ends and replace them by a pair of lines connecting four points on the circle which are obtained after removing neighborhoods. The new chords form only a classical crossing (with each other) if the chord is framed by 0, and form classical and virtual crossings if the chord is framed by 1, see Fig. 9.10

(intersections of chords from different pair form virtual crossings). We also require that the initial piece of the circle corresponds to the A-smoothing if the chord is positive and to the B-smoothing if it is negative.

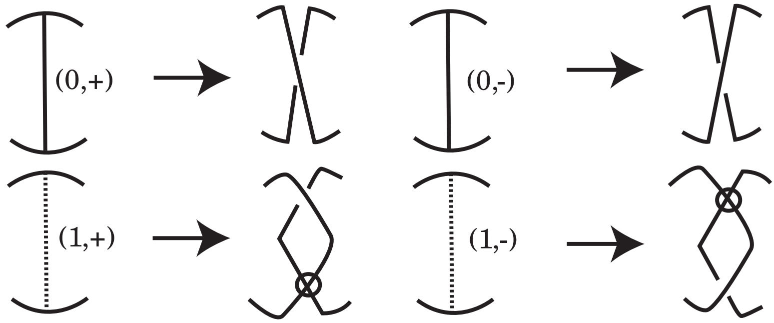


Fig. 9.10 Replacing a chord with a pair of lines.

Conversely, having a connected virtual diagram K, one can get a labeled chord diagram $D_C(K)$. Indeed, one takes a rotating circuit C of K (we pass transversally at each virtual crossing) and construct the labeled chord diagram as in Sec. 7.8.2. The sign of the chord is + (respectively, $-$) if the circuit locally agrees with the A-smoothing (respectively, the B-smoothing), and the framing of a chord is 0 (respectively, 1) if two opposite half-edges have the opposite (respectively, the same) orientation, see Fig. 9.11. It can be easily checked that this operation is indeed inverse to the operation of constructing a virtual link diagram out of a chord diagram: If we take a chord diagram D, and construct a virtual diagram $K(D)$ out of it, then for some circuit C the chord diagram $D_C(K(D))$ will coincide with D. This proves the following.

Theorem 9.1 ([221]). *For any connected virtual diagram K' there is a certain labeled chord diagram D such that $K' = K(D)$.*

9.2.2 *Reidemeister moves for looped interlacement graphs and graph-links*

Now we are describing moves on graphs obtained from the Reidemeister moves on virtual diagrams by using a rotating circuit [129, 130] and the Gauss circuit [293]. These moves in both cases will correspond to "real" Reidemeister moves when applied to realizable diagrams. Then we shall

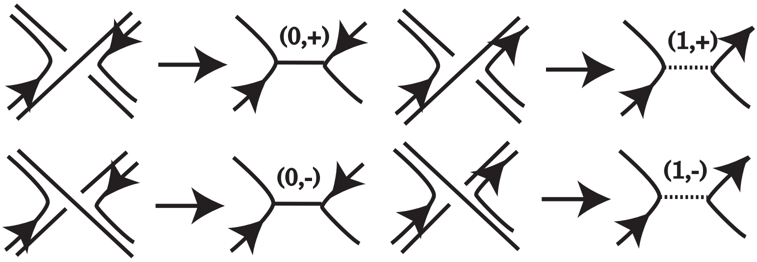


Fig. 9.11 Replacing a classical crossing with the labeled chord.

extend these moves to all graphs (not only to realizable ones). As a result, we get two new objects: a *graph-link* and a *homotopy class of looped interlacement graphs*. Thus, virtual diagrams are represented (with loss of some information) by graphs. We get two new objects in a way similar to the generalization of classical knots to virtual knots. The passage from realizable Gauss diagrams (classical knots) to arbitrary chord diagrams leads to the concept of a virtual knot, and the passage from realizable (by means of chord diagrams) graphs to arbitrary graphs leads to the concept of two new objects, a *graph-link* and a *homotopy class of looped interlacement graphs* (here "looped" corresponds to the writhe number, if the writhe number is -1, then the corresponding vertex has a loop). To construct the first object we shall use simple labeled graphs, and for the second one we shall use (unlabeled) graphs without multiple edges, but loops are allowed.

Definition 9.2. A graph is *labeled* if every vertex v of it is endowed with a pair (a, α), where $a \in \{0, 1\}$ is the *framing* of v, and $\alpha \in \{\pm\}$ is the *sign* of v.

Let D be a labeled chord diagram. The *labeled intersection graph* (cf. [56, 270]) $G(D)$ of D is the intersection graph whose vertices are endowed with the corresponding labels. A simple graph H is called *realizable* if there is a chord diagram D such that $H = G(D)$. Otherwise, a graph is called *non-realizable*.

Remark 9.2. We shall also consider simple graphs whose vertices have only one label, 0 or 1. We call these graphs *framed*. Thus we have two types of framed graphs: 4-graphs and simple graphs. From the context it will always be clear what framed graph is under consideration.

In the realizable case, framed graphs are intersection graphs of framed chord diagrams.

The following lemma is evident.

Lemma 9.1. *A simple graph is realizable if and only if each of its connected components is realizable.*

Definition 9.3. Let G be a graph and let $v \in V(G)$. The set of all vertices adjacent to v is called the *neighborhood* of v and denoted by $N(v)$ or $N_G(v)$.

Let us define two operations on simple unlabeled graphs whose definitions in the matrix language coincide with Definitions 7.21 and 7.22 up to labels.

Definition 9.4 (Local Complementation). Let G be a graph. The *local complementation* of G at $v \in V(G)$ is the operation which toggles adjacencies between $a, b \in N(v)$, $a \neq b$, and does not change the rest of G. Denote the graph obtained from G by the local complementation at a vertex v by $\mathrm{LC}(G; v)$.

Definition 9.5 (Pivot). Let G be a graph with distinct vertices u and v. The *pivoting operation* of a graph G at u and v is the operation which toggles adjacencies between x, y such that $x, y \notin \{u, v\}$, $x \in N(u)$, $y \in N(v)$ and either $x \notin N(v)$ or $y \notin N(u)$, and does not change the rest of G. Denote the graph obtained from G by the pivoting operation at vertices u and v by $\mathrm{piv}(G; u, v)$.

Example 9.1. In Fig. 9.12 the graphs G, $\mathrm{LC}(G; u)$ and $\mathrm{piv}(G; u, v)$ are depicted.

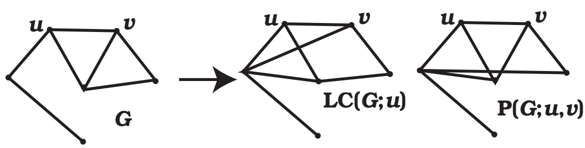


Fig. 9.12 Local complementation and pivot.

From Definition 7.22, we have the following lemma.

Lemma 9.2. *If u and v are adjacent, then there is an isomorphism*

$$\mathrm{piv}(G; u, v) \cong \mathrm{LC}(\mathrm{LC}(\mathrm{LC}(G; u); v); u).$$

Let us define *graph-moves*, i.e. moves on labeled graphs. We consider labeled chord diagrams constructed by using rotating circuits and moves on them which originate from "real" Reidemeister moves on virtual diagrams. Then we extend these moves to arbitrary labeled graphs by using intersection graphs of chord diagrams. As a result, we obtain a new object: an equivalence class of labeled graphs under formal moves. These moves were defined in [129, 130].

Definition 9.6. $\Omega_g 1$. *The first Reidemeister graph-move* is an addition/removal of an isolated vertex labeled $(0,\alpha)$, $\alpha \in \{\pm\}$.

$\Omega_g 2$. *The second Reidemeister graph-move* is an addition/removal of two non-adjacent (respectively, adjacent) vertices labeled $(0,\pm\alpha)$ (respectively, $(1,\pm\alpha)$) and the same adjacencies with other vertices.

$\Omega_g 3$. *The third Reidemeister graph-move* is defined as follows. Let u, v, w be three vertices of G all having label $(0,-)$ so that u is adjacent only to v and w in G, and v and w are not adjacent to each other. Then we only change the adjacency of u with the vertices v, w and $t \in \big(N(v) \setminus N(w)\big) \cup \big(N(w) \setminus N(v)\big)$ (for other pairs of vertices we do not change their adjacency). In addition, we switch the signs of v and w to $+$. The inverse operation is also called the third Reidemeister graph-move.

$\Omega_g 4$. *The fourth graph-move* for G is defined as follows. We take two adjacent vertices u and v labeled $(0,\alpha)$ and $(0,\beta)$, respectively. Replace G with $\mathrm{piv}(G;u,v)$ and change the signs of u and v so that the sign of u becomes $-\beta$ and the sign of v becomes $-\alpha$.

$\Omega_g 4'$. In this fourth graph-move we take a vertex v with the label $(1,\alpha)$. Replace G with $\mathrm{LC}(G;v)$ and change the sign of v and the framing for each $u \in N(v)$.

Remark 9.3. The third Reidemeister graph-move does not exhaust all the possibilities for representing the third Reidemeister move on chord diagrams constructed by rotating circuits. The other versions of the third Reidemeister move are combinations of the second, third and fourth Reidemeister graph-moves.

Remark 9.4. The fourth graph-moves $\Omega_g 4$ and $\Omega_g 4'$ in the realizable case correspond to a rotating circuit change on a virtual diagram. Sometimes, applying these graph-moves we shall just say that we change the circuit.

Remark 9.5. We have defined the graph-moves for labeled graphs. If we consider framed graphs, then graph-moves for them are obtained from the graph-moves $\Omega_g 1 - \Omega_g 4'$ by forgetting the sign, i.e. the second component of the label. In this case we use the same notation.

The comparison of the graph-moves with the Reidemeister moves yields the following theorem.

Theorem 9.2. *Let G_1 and G_2 be two labeled intersection graphs corresponding to virtual diagrams K_1 and K_2, respectively. If K_1 and K_2 are equivalent in the class of connected diagrams, then G_1 and G_2 are obtained from one another by a sequence of $\Omega_g 1 - \Omega_g 4'$.*

Definition 9.7. A *graph-link* is an equivalence class of simple labeled graphs modulo $\Omega_g 1 - \Omega_g 4'$ graph-moves.

Remark 9.6. Having a chord diagram, we can construct an *atom* (see Chap. 4) corresponding to the chord diagram. In fact, chord diagrams in the sense of rotating circuits with all chords having framing 0 encode all orientable atoms. Chord diagrams with all positive chords encode all atoms with one white cell: This white cell corresponds to the A-state of the virtual diagram.

For graph-links having representatives with orientable atoms, there are two formally different equivalence relations. The first relation is described in [129] (which includes only diagrams with orientable atoms) and the last one defines graph-links. We know that these equivalence relations coincide [129], therefore, we use the same term "graph-link" for the object introduced here.

The next definition is similar to the definition of free knots. We consider only one label-framing on a graph, which in the realizable case is responsible for the structure of opposite edges.

Definition 9.8. A *free framed graph* is an equivalence classes of simple labeled graphs with labels having only framings, modulo $\Omega_g 4$ and $\Omega_g 4'$ graph-moves up to signs of labels (we disregard the sign of each vertex). A *free graph-link* is an equivalence class of free framed graphs modulo $\Omega_g 1 - \Omega_g 3$ considered up to signs of labels.

Definition 9.9. We call a free framed graph *realizable* if each representative of it is realized by a chord diagram.

Remark 9.7. It is not difficult to show that a free framed graph is realizable if and only if it has a realizable representative (we just redraw the picture).

Let us consider another approach based on Gauss circuits. Let $D_G(K)$ be the Gauss diagram of a virtual diagram K. Let us construct the graph obtained from the intersection graph of $D_G(K)$ by adding loops to vertices corresponding to chords with the negative writhe number, i.e. crossings with negative writhe number [293]. We refer to this graph as a *looped interlacement graph* or *looped graph*. Let us describe the set of moves on looped graphs. These moves are similar to the moves for graph-links and also correspond to "real" Reidemeister moves on virtual diagrams.

Definition 9.10. Ω1. *The first Reidemeister move* for looped interlacement graphs is an addition/removal of an isolated looped or unlooped vertex.

Ω2. *The second Reidemeister move* for looped interlacement graphs is an addition/removal of two vertices having the same adjacencies with other vertices and, moreover, one of which is looped and the other one is unlooped.

Ω3. *The third Reidemeister move* for looped interlacement graphs is defined as follows. Let u, v, w be three vertices such that v is looped, w is unlooped, v and w are adjacent, u is adjacent to neither v nor w, and every vertex $x \notin \{u, v, w\}$ is adjacent to either 0 or precisely two of u, v, w. Then we only remove all three edges uv, uw and vw, see Fig. 9.13. The inverse operation is also called the third Reidemeister move.

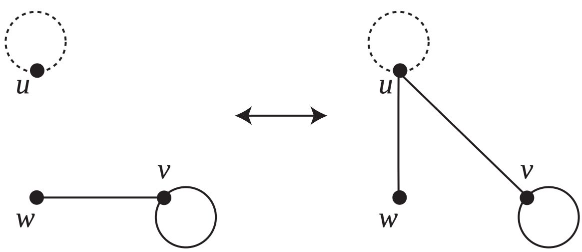


Fig. 9.13 The third Reidemeister move.

Remark 9.8. The two third Reidemeister moves (the first one is the move with the third vertex being looped, and the second one is the move with the third vertex being unlooped) do not exhaust all the possibilities for representing the third Reidemeister move on Gauss diagrams [293]. It can be shown that all the other versions of the third Reidemeister move, depicted in Fig. 9.14 (we toggle adjacencies between only the three vertices), are combinations of the second and third Reidemeister moves described in Definition 9.10, see [260] for details.

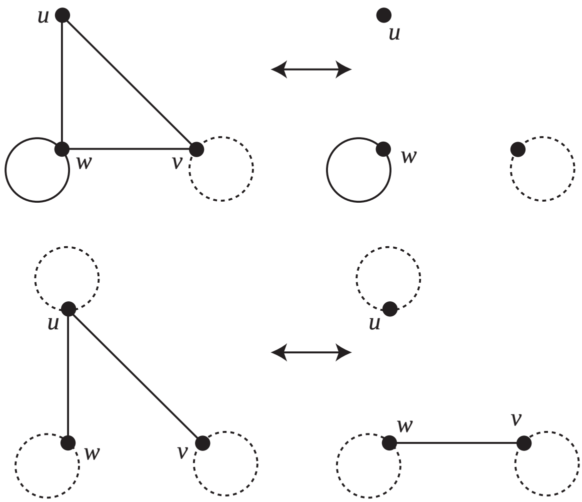


Fig. 9.14 The possible configurations of the third Reidemeister move.

Definition 9.11. We call an equivalence class of graphs (without multiple edges, but loops are allowed) modulo the three moves listed in Definition 9.10 a *homotopy class* of looped interlacement graphs. A *free homotopy class* is an equivalence class of simple graphs modulo the Reidemeister moves for looped interlacement graphs up to loops, i.e. we forget about loops.

Remark 9.9. The equivalence relation from Definition 9.11 is called the *Reidemeister equivalence* in [293], and it differs from the classical homotopy of links.

Looped interlacement graphs encode only knot diagrams but graph-links can encode virtual diagrams with any number of components. The approach using a rotating circuit has an advantage in this sense. One may generalize these two approaches and consider any Euler tour of a virtual diagram. This was initiated by Traldi [291], where he introduced the notion of a marked graph.

Definition 9.12. We call a (free) graph-link (respectively, (free) homotopy class of looped graphs) *realizable* if there exists its representative which can be realized by a chord diagram.

9.2.3 *Looped graphs and graph-links*

Let G be a labeled graph on $V(G) = \{v_1, \ldots, v_n\}$.

Definition 9.13. The *adjacency matrix* $A(G)$ of a labeled graph G is the matrix over $\mathbb{Z}_2$ defined as follows: a_{ii} is equal to the framing of v_i, $a_{ij} = 1$, $i \neq j$, if and only if v_i is adjacent to v_j and $a_{ij} = 0$ otherwise.

Statement 9.1 ([129, 130]). *If G and G' represent the same graph-link, then* $\mathrm{corank}_{\mathbb{Z}_2}(A(G) + E) = \mathrm{corank}_{\mathbb{Z}_2}(A(G') + E)$, *where E is the identity matrix. Thus, the number* $\mathrm{corank}_{\mathbb{Z}_2}(A(G)+E)$, *where G is a representative of a graph-link $\mathfrak{F}$, is an invariant of the graph-link $\mathfrak{F}$.*

Definition 9.14. The *number of components* in a graph-link $\mathfrak{F}$ is $\mathrm{corank}_{\mathbb{Z}_2}(A(G) + E) + 1$, where G is a representative of $\mathfrak{F}$. A graph-link $\mathfrak{F}$ with $\mathrm{corank}_{\mathbb{Z}_2}(A(G) + E) = 0$ for any representative G of $\mathfrak{F}$ is called a *graph-knot*.

Assume $\mathrm{corank}_{\mathbb{Z}_2}(A(G) + E) = 0$, and set $B_i(G) = A(G) + E + E_{ii}$ for each vertex $v_i \in V(G)$; here E_{ii} is the matrix with the only one non-zero element equal to 1 in the ith column and ith row. The *writhe number* w_i of G (with $\mathrm{corank}_{\mathbb{Z}_2}(A(G) + E) = 0$) at v_i is $w_i = (-1)^{\mathrm{corank}_{\mathbb{Z}_2} B_i(G)} \mathrm{sign}\, v_i$, and the *writhe number* of G is

$$w(G) = \sum_{i=1}^{n} w_i.$$

Remark 9.10. If G is a realizable graph by a chord diagram and, therefore, by a virtual diagram, then we can find the number of components of the link by using the formula from Theorem 7.8. The obtained formula may be used for the definition of the number of components of a non-realizable graph. As a result, we get Definition 9.14.

Note if G is a realizable graph, then w_i is the "real" writhe number of the crossing corresponding to v_i.

Note that in this section we use the following *main principle. Assume there is an equality concerning numbers of circles in some states of chord diagrams, which can be formulated in terms of the intersection graph. Then the corresponding equality usually holds even for non-realizable graphs.*

The reason for this principle to hold is the following. Every time we have a picture where some two numbers of circles are equal to each other (or differ by a constant), this can be expressed in terms of the corresponding adjacency matrices, and the proof does not generally depend on the behavior of the matrix outside of the crossings in question. This means that the equality holds true for generic matrices, thus, it works for general intersection graphs. This principle has lots of consequences. We shall demonstrate it for two examples.

The first example comes from Definition 9.14. It shows that the number of components defined for non-realizable graphs by the same formula as for realizable ones does not change under the Reidemeister moves and the writhe number of a graph does not change under the second and third Reidemeister graph-moves and changes by ± 1 under the first Reidemeister graph-move.

The second example is as follows. Assume we have a framed 4-graph K with a vertex v and we would like to know whether this vertex belongs to one component (in the sense of links) or it belongs to different components of the corresponding graph-link. Then we may take the two *smoothings* K_a, K_b of K at v and see how many components we get. If v belongs to two branches of the same component of K, then the number of components of one of K_a, K_b is equal to that of K, and the number of components of the other one is equal to that of $K+1$. If v belongs to two different components of K, then the number of components of each of K_a, K_b is that of $K-1$. Now, turning to graph-links, by taking appropriate matrix ranks, we may see whether each vertex belongs to one component or to two different components of the graph-link. This method is used for proving the invariance of the Kauffman bracket polynomial for graph-links, see further.

Let us show that there is an equivalence between the set of homotopy classes of looped graphs and the set of graph-links, i.e. we prove the following theorem.

Theorem 9.3. *There is a one-to-one correspondence between the set of all looped graphs and the set of all equivalence classes of labeled graphs G with* $\operatorname{corank}_{\mathbb{Z}_2}(A(G)+E)=0$ *under two fourth graph-moves. This correspondence generates an equivalence between the set of all homotopy classes of looped graphs and the set of all graph-knots. Moreover, if K is a virtual knot diagram, and $\mathfrak{F}$ is the graph-knot constructed from K and $\mathfrak{L}$ is the homotopy class of looped graphs constructed from K, then $\mathfrak{F}$ and $\mathfrak{L}$ are related by this equivalence.*

First, we shall prove some lemmas and give necessary definitions.

Remark 9.11. Indeed, we have constructed this map, see Theorem 7.11. We just have to show that this map is well defined for graph-links and homotopy classes.

By $\widehat{C}_{i,j,\ldots,k}$ denote the matrix obtained from a matrix C by deleting the $i,j,\ldots,k$th rows and $i,j,\ldots,k$th columns. We shall write $\widehat{B}_{i,j,\ldots,k}(G)$ instead of $\widehat{B(G)}_{i,j,\ldots,k}$.

Lemma 9.3. *The following holds*:

$$w_i(G) = (-1)^{\operatorname{corank}_{\mathbb{Z}_2}\widehat{B}_i(G)+1}\operatorname{sign} v_i.$$

Proof. Without loss of generality we prove the claim of the lemma for w_1. Let

$$A(G)+E = \begin{pmatrix} a & \mathbf{b}^\top \\ \mathbf{b} & C \end{pmatrix}$$

and

$$\operatorname{corank}_{\mathbb{Z}_2}(A(G)+E) = 0 \Longleftrightarrow \det(A(G)+E) = 1,$$

where bold characters indicate column vectors. We have

$$\det B_1(G) = \det\begin{pmatrix} a+1 & \mathbf{b}^\top \\ \mathbf{b} & C \end{pmatrix} = \det\begin{pmatrix} a & \mathbf{b}^\top \\ \mathbf{b} & C \end{pmatrix} + \det\begin{pmatrix} 1 & \mathbf{0}^\top \\ \mathbf{b} & C \end{pmatrix}$$

and

$$\widehat{B}_1(G) = C, \qquad \det(B_1(G)) = \det(A(G)+E) + \det C = 1 + \det\widehat{B}_1(G).$$

The last equality gives us the claim of the lemma. □

Lemma 9.4. *Let G be a labeled graph with $\det B(G) = 1$ and $B(G)^{-1} = (b^{ij})$. Then*

$$b^{ii} = \frac{1 - w_i(G)\operatorname{sign} v_i}{2}.$$

Proof. We have

$$\begin{aligned} w_i(G) &= (-1)^{\operatorname{corank}_{\mathbb{Z}_2}\widehat{B}_i(G)+1}\operatorname{sign} v_i \\ &\Longleftrightarrow w_i(G)\operatorname{sign} v_i = (-1)^{\operatorname{corank}_{\mathbb{Z}_2}\widehat{B}_i(G)+1} \\ &\Longleftrightarrow \operatorname{corank}_{\mathbb{Z}_2}\widehat{B}_i(G) = \frac{w_i(G)\operatorname{sign} v_i + 1}{2} \\ &\Longleftrightarrow b^{ii} = \det\widehat{B}_i(G) = 1 - \operatorname{corank}_{\mathbb{Z}_2}\widehat{B}_i(G) = \frac{1 - w_i(G)\operatorname{sign} v_i}{2}. \end{aligned}$$

□

Definition 9.15. Define the *adjacency matrix* $A(L) = (a_{ij})$ over $\mathbb{Z}_2$ for a looped graph L with enumerated set of vertices as: $a_{ii} = 1$ if and only if the vertex numbered i is looped, and $a_{ii} = 0$ otherwise; $a_{ij} = 1$, $i \neq j$, if and only if the vertex with the number i is adjacent to the vertex with the number j, and $a_{ij} = 0$ otherwise.

Let us construct a map χ from the set of all graph-knots to the set of all homotopy classes of looped graphs. We shall show that this map has an inverse map.

Let G be a representative of a graph-knot $\mathfrak{F}$. Let us consider the simple graph H having the adjacency matrix coinciding with $(A(G)+E)^{-1}$ up to diagonal elements, see Definition 7.20. Let us construct the graph $L(G)$ from H by just adding loops to any vertex of H corresponding to a vertex of G with the negative writhe number (see Definition 9.14). By definition, put $\chi(\mathfrak{F}) = \mathfrak{L}$, where $\mathfrak{L}$ is the homotopy class of $L(G)$.

Theorem 9.4. *The map χ is well defined.*

Proof. Let G_1, G_2 be two representatives of $\mathfrak{F}$, and let $B(G_i) = A(G_i) + E$, $B(G_i)^{-1} = (b_i^{kl})$. We have to show that the homotopy classes of $L(G_1)$ and $L(G_2)$ are the same, i.e. $L_1 = L(G_1)$ and $L_2 = L(G_2)$ are related to each other by the Reidemeister moves on looped graphs.

Let us consider four cases.

(1) We know that if G_1 and G_2 are obtained from each other by $\Omega_g 4$ and/or $\Omega_g 4'$ (these moves correspond to changing of a rotating circuit in the case of realizable graphs), then the writhe numbers of corresponding vertices of G_1 and G_2 are the same (see [129, 130]), and the graphs L_1 and L_2 are isomorphic up to loops, i.e. their adjacency matrices coincide with each other up to diagonal elements, see Theorem 7.11. From these two statements we get that L_1 and L_2 are isomorphic.

(2) Let G_1 and G_2 be obtained from each other by $\Omega_g 1$ (we remove the vertex with the number 1). We have

$$B(G_1) = \begin{pmatrix} 1 & \mathbf{0}^\top \\ \mathbf{0} & A(G_2)+E \end{pmatrix}, \qquad B(G_2) = A(G_2) + E,$$

$$B(G_1)^{-1} = \begin{pmatrix} 1 & \mathbf{0}^\top \\ \mathbf{0} & (A(G_2)+E)^{-1} \end{pmatrix}, \qquad B(G_2)^{-1} = (A(G_2)+E)^{-1},$$

where $\mathbf{0}$ indicates the column vector with all entries 0. Therefore, L_2 is obtained from L_1 by $\Omega 1$.

(3) Let G_1 and G_2 be obtained from each other by $\Omega_g 2$ (we remove the vertices with the numbers 1 and 2). We have two cases.

The first case

$$B(G_1) = \begin{pmatrix} 1 & 0 & \mathbf{a}^\top \\ 0 & 1 & \mathbf{a}^\top \\ \mathbf{a} & \mathbf{a} & A(G_2)+E \end{pmatrix}, \qquad B(G_2) = A(G_2)+E,$$

and the second case

$$B(G_1) = \begin{pmatrix} 0 & 1 & \mathbf{a}^\top \\ 1 & 0 & \mathbf{a}^\top \\ \mathbf{a} & \mathbf{a} & A(G_2)+E \end{pmatrix}, \qquad B(G_2) = A(G_2)+E.$$

Let us consider only the first case. We know that $w_1(G_1) = -w_2(G_1)$ in G_1 [129, 130]. Moreover,

$$\det \begin{pmatrix} 1 & 0 & \widetilde{\mathbf{a}}^\top \\ 0 & 1 & \widetilde{\mathbf{a}}^\top \\ \widetilde{\mathbf{a}} & \widetilde{\mathbf{a}} & C \end{pmatrix} = \det \begin{pmatrix} 1 & 0 & \widetilde{\mathbf{a}}^\top \\ 1 & 1 & \mathbf{0}^\top \\ \widetilde{\mathbf{a}} & \widetilde{\mathbf{a}} & C \end{pmatrix} = \det \begin{pmatrix} 1 & 0 & \widetilde{\mathbf{a}}^\top \\ 1 & 1 & \mathbf{0}^\top \\ \mathbf{0} & \mathbf{0} & C \end{pmatrix} = \det C,$$

so

$$B(G_1)^{-1} = \begin{pmatrix} b & c & \mathbf{d}^\top \\ c & b & \mathbf{d}^\top \\ \mathbf{d} & \mathbf{d} & (A(G_2)+E)^{-1} \end{pmatrix}, \qquad B(G_2)^{-1} = (A(G_2)+E)^{-1}.$$

This means that L_1 and L_2 are obtained from each other by $\Omega 2$.

(4) Now assume that G_1 and G_2 are obtained from each other by $\Omega_g 3$. The corresponding vertices of G_1 and G_2 under $\Omega_g 3$ have the same numbers (in [129] we have another enumeration). We shall prove that L_1 and L_2 are obtained from each other by a sequence of $\Omega 2$, $\Omega 3$ moves.

We have

$$B(G_1) = \begin{pmatrix} 1 & 1 & 1 & \mathbf{0}^\top \\ 1 & 1 & 0 & \mathbf{a}^\top \\ 1 & 0 & 1 & \mathbf{b}^\top \\ \mathbf{0} & \mathbf{a} & \mathbf{b} & C \end{pmatrix},$$

$$1 = \det B(G_1) = \det \begin{pmatrix} 1 & 1 & 1 & \mathbf{0}^\top \\ 1 & 1 & 0 & \mathbf{a}^\top \\ 1 & 0 & 1 & \mathbf{b}^\top \\ \mathbf{0} & \mathbf{a} & \mathbf{b} & C \end{pmatrix} = \det \begin{pmatrix} 0 & 1 & \mathbf{a}^\top \\ 1 & 0 & \mathbf{b}^\top \\ \mathbf{a} & \mathbf{b} & C \end{pmatrix},$$

$$B(G_2) = \begin{pmatrix} 1 & 0 & 0 & (\mathbf{a}+\mathbf{b})^\top \\ 0 & 1 & 0 & \mathbf{b}^\top \\ 0 & 0 & 1 & \mathbf{a}^\top \\ \mathbf{a}+\mathbf{b} & \mathbf{b} & \mathbf{a} & C \end{pmatrix}, \qquad \det B(G_2) = 1.$$

Let us show that we have a structure either for $v_1, v_2, v_3 \in V(L_1)$ or $\widetilde{v}_1, \widetilde{v}_2, \widetilde{v}_3 \in V(L_2)$ as in Fig. 9.14.

We have ([129])

$$w_i(G_1) = w_i(G_2), \qquad i = 1, 2, 3,$$

$$\det \widehat{B}_1(G_1) = \det \begin{pmatrix} 1 & 0 & \mathbf{a}^\top \\ 0 & 1 & \mathbf{b}^\top \\ \mathbf{a} & \mathbf{b} & C \end{pmatrix} = \det \begin{pmatrix} 1 & 0 & \mathbf{a}^\top \\ 1 & 1 & \mathbf{b}^\top \\ \mathbf{0} & \mathbf{b} & C \end{pmatrix} + \det \begin{pmatrix} 0 & 0 & \mathbf{a}^\top \\ 1 & 1 & \mathbf{b}^\top \\ \mathbf{a} & \mathbf{b} & C \end{pmatrix}$$

$$= \det \begin{pmatrix} 1 & 0 & \mathbf{a}^\top \\ 1 & 1 & \mathbf{b}^\top \\ \mathbf{0} & \mathbf{b} & C \end{pmatrix} + \det \begin{pmatrix} 0 & 1 & \mathbf{a}^\top \\ 1 & 1 & \mathbf{b}^\top \\ \mathbf{a} & \mathbf{0} & C \end{pmatrix} + \det \begin{pmatrix} 0 & 1 & \mathbf{a}^\top \\ 1 & 0 & \mathbf{b}^\top \\ \mathbf{a} & \mathbf{b} & C \end{pmatrix},$$

$$\det \widehat{B}_2(G_1) = \det \begin{pmatrix} 1 & 1 & \mathbf{0}^\top \\ 1 & 1 & \mathbf{b}^\top \\ \mathbf{0} & \mathbf{b} & C \end{pmatrix} = \det \begin{pmatrix} 0 & \mathbf{b}^\top \\ \mathbf{b} & C \end{pmatrix},$$

$$\det \widehat{B}_3(G_1) = \det \begin{pmatrix} 1 & 1 & \mathbf{0}^\top \\ 1 & 1 & \mathbf{a}^\top \\ \mathbf{0} & \mathbf{a} & C \end{pmatrix} = \det \begin{pmatrix} 0 & \mathbf{a}^\top \\ \mathbf{a} & C \end{pmatrix},$$

$$b_1^{12} = \det \begin{pmatrix} 1 & 0 & \mathbf{a}^\top \\ 1 & 1 & \mathbf{b}^\top \\ \mathbf{0} & \mathbf{b} & C \end{pmatrix} = \det \begin{pmatrix} 1 & \mathbf{a}^\top + \mathbf{b}^\top \\ \mathbf{b} & C \end{pmatrix}$$

$$= \det \begin{pmatrix} 1 & \mathbf{a}^\top \\ \mathbf{b} & C \end{pmatrix} + \det \begin{pmatrix} 0 & \mathbf{b}^\top \\ \mathbf{b} & C \end{pmatrix},$$

$$b_1^{13} = \det \begin{pmatrix} 1 & 1 & \mathbf{a}^\top \\ 1 & 0 & \mathbf{b}^\top \\ \mathbf{0} & \mathbf{a} & C \end{pmatrix} = \det \begin{pmatrix} 1 & \mathbf{a}^\top + \mathbf{b}^\top \\ \mathbf{a} & C \end{pmatrix}$$

$$= \det \begin{pmatrix} 1 & \mathbf{b}^\top \\ \mathbf{a} & C \end{pmatrix} + \det \begin{pmatrix} 0 & \mathbf{a}^\top \\ \mathbf{a} & C \end{pmatrix},$$

$$b_1^{23} = \det \begin{pmatrix} 1 & 1 & \mathbf{0}^\top \\ 1 & 0 & \mathbf{b}^\top \\ \mathbf{0} & \mathbf{a} & C \end{pmatrix} = \det \begin{pmatrix} 1 & \mathbf{b}^\top \\ \mathbf{a} & C \end{pmatrix},$$

$$b_2^{12} = \det \begin{pmatrix} 0 & 0 & \mathbf{b}^\top \\ 0 & 1 & \mathbf{a}^\top \\ \mathbf{a}+\mathbf{b} & \mathbf{a} & C \end{pmatrix} = \det \begin{pmatrix} 0 & 0 & \mathbf{b}^\top \\ 1 & 1 & \mathbf{a}^\top \\ \mathbf{b} & \mathbf{a} & C \end{pmatrix}$$

$$= \det \begin{pmatrix} 0 & 1 & \mathbf{b}^\top \\ 1 & 0 & \mathbf{a}^\top \\ \mathbf{b} & \mathbf{a} & C \end{pmatrix} + \det \begin{pmatrix} 0 & 1 & \mathbf{b}^\top \\ 1 & 1 & \mathbf{a}^\top \\ \mathbf{b} & \mathbf{0} & C \end{pmatrix} = 1 + b_1^{12},$$

$$b_2^{13} = \det \begin{pmatrix} 0 & 1 & \mathbf{b}^\top \\ 0 & 0 & \mathbf{a}^\top \\ \mathbf{a}+\mathbf{b} & \mathbf{b} & C \end{pmatrix} = \det \begin{pmatrix} 1 & 1 & \mathbf{b}^\top \\ 0 & 0 & \mathbf{a}^\top \\ \mathbf{a} & \mathbf{b} & C \end{pmatrix}$$

$$= \det \begin{pmatrix} 1 & 0 & \mathbf{b}^\top \\ 0 & 1 & \mathbf{a}^\top \\ \mathbf{a} & \mathbf{b} & C \end{pmatrix} + \det \begin{pmatrix} 1 & 1 & \mathbf{b}^\top \\ 0 & 1 & \mathbf{a}^\top \\ \mathbf{a} & \mathbf{0} & C \end{pmatrix} = 1 + b_1^{13},$$

$$b_2^{23} = \det \begin{pmatrix} 1 & 0 & (\mathbf{a}+\mathbf{b})^\top \\ 0 & 0 & \mathbf{a}^\top \\ \mathbf{a}+\mathbf{b} & \mathbf{b} & C \end{pmatrix} = \det \begin{pmatrix} 1 & 0 & \mathbf{b}^\top \\ 0 & 0 & \mathbf{a}^\top \\ \mathbf{a} & \mathbf{b} & C \end{pmatrix}$$

$$= \det \begin{pmatrix} 1 & 0 & \mathbf{b}^\top \\ 0 & 1 & \mathbf{a}^\top \\ \mathbf{a} & \mathbf{b} & C \end{pmatrix} + \det \begin{pmatrix} 1 & 0 & \mathbf{b}^\top \\ 0 & 1 & \mathbf{a}^\top \\ \mathbf{a} & \mathbf{0} & C \end{pmatrix} = 1 + b_1^{23},$$

$$b_1^{12} = b_1^{23} + \det \widehat{B}_2(G_1), \qquad b_1^{13} = b_1^{23} + \det \widehat{B}_3(G_1),$$
$$b_1^{12} + b_1^{13} = \det \widehat{B}_1(G_1) + 1,$$
$$b_2^{12} = b_1^{12} + 1, \qquad b_2^{13} = b_1^{13} + 1, \qquad b_2^{23} = b_1^{23} + 1.$$

It is not difficult to show that the last equalities guarantee us that either $v_1, v_2, v_3 \in V(L_1)$ or $\widetilde{v}_1, \widetilde{v}_2, \widetilde{v}_3 \in V(L_2)$ have a structure as in Fig. 9.14. The structure of the other triple is obtained from the first triple by toggling the non-loop edges.

Denote by $\overline{\mathbf{f}}^i$ the column vector obtained from $\mathbf{f}$ by deleting the ith element and denote by $\overline{C}_j$ (respectively, $\overline{C}_j^i$) the matrix obtained from C by deleting the jth column (respectively, the ith row and the jth column).

Let us show that two vertices in L_1 which are different from $v_1, v_2, v_3 \in V(L_1)$ are adjacent to each other if and only if the corresponding vertices in L_2 are adjacent, i.e. the corresponding elements of the matrices $B(G_1)^{-1}$ and $B(G_2)^{-1}$ coincide. For $i, j > 3$ we have

$$b_1^{ij} = \det \begin{pmatrix} 1 & 1 & 1 & \mathbf{0}^\top \\ 1 & 1 & 0 & (\overline{\mathbf{a}}^{j-3})^\top \\ 1 & 0 & 1 & (\overline{\mathbf{b}}^{j-3})^\top \\ \mathbf{0} & \overline{\mathbf{a}}^{i-3} & \overline{\mathbf{b}}^{i-3} & \overline{C}_{j-3}^{i-3} \end{pmatrix}$$

$$= \det \begin{pmatrix} 1 & 0 & 0 & \left(\overline{(\mathbf{a}+\mathbf{b})}^{j-3}\right)^\top \\ 0 & 1 & 0 & (\overline{\mathbf{b}}^{j-3})^\top \\ 0 & 0 & 1 & (\overline{\mathbf{a}}^{j-3})^\top \\ \overline{(\mathbf{a}+\mathbf{b})}^{i-3} & \overline{\mathbf{b}}^{i-3} & \overline{\mathbf{a}}^{i-3} & \overline{C}_{j-3}^{i-3} \end{pmatrix} = b_2^{ij},$$

$$b_2^{1j} = \det\begin{pmatrix} 0 & 1\ 0\ (\overline{\mathbf{b}}^{j-3})^\top \\ 0 & 0\ 1\ (\overline{\mathbf{a}}^{j-3})^\top \\ (\mathbf{a}+\mathbf{b}) & \mathbf{b}\ \mathbf{a}\ \ \overline{C}_{j-3} \end{pmatrix} = \det\begin{pmatrix} 1\ 1\ 0\ (\overline{\mathbf{b}}^{j-3})^\top \\ 1\ 0\ 1\ (\overline{\mathbf{a}}^{j-3})^\top \\ \mathbf{0}\ \mathbf{b}\ \mathbf{a}\ \ \overline{C}_{j-3} \end{pmatrix} = b_1^{1j},$$

$$b_2^{2j} = \det\begin{pmatrix} 1 & 0\ 0 & \left(\overline{(\mathbf{a}+\mathbf{b})}^{j-3}\right)^\top \\ 0 & 0\ 1 & (\overline{\mathbf{a}}^{j-3})^\top \\ \mathbf{a}+\mathbf{b} & \mathbf{b}\ \mathbf{a} & \overline{C}_{j-3} \end{pmatrix} = \det\begin{pmatrix} 1\ 0\ 0 & \left(\overline{(\mathbf{a}+\mathbf{b})}^{j-3}\right)^\top \\ 1\ 0\ 1 & (\overline{\mathbf{a}}^{j-3})^\top \\ \mathbf{0}\ \mathbf{b}\ \mathbf{a} & \overline{C}_{j-3} \end{pmatrix}$$

$$= \det\begin{pmatrix} 1\ 0\ 0 & (\overline{\mathbf{b}}^{j-3})^\top \\ 1\ 0\ 1 & \mathbf{0}^\top \\ \mathbf{0}\ \mathbf{b}\ \mathbf{a} & \overline{C}_{j-3} \end{pmatrix} = \det\begin{pmatrix} 1\ 1\ 0 & (\overline{\mathbf{b}}^{j-3})^\top \\ 1\ 1\ 1 & \mathbf{0}^\top \\ \mathbf{0}\ \mathbf{b}\ \mathbf{a} & \overline{C}_{j-3} \end{pmatrix} = b_1^{2j},$$

analogously $b_1^{3j} = b_2^{3j}$.

We have to verify that every vertex $x \notin \{v_1, v_2, v_3\}$ is adjacent to either none of v_1, v_2, v_3 or precisely two of them in L_1 and analogously for L_2. This statement is equivalent to both equalities

$$b_1^{1j} + b_1^{2j} + b_1^{3j} = 0, \qquad b_2^{1j} + b_2^{2j} + b_2^{3j} = 0.$$

Using the above equalities, it is enough to verify only the first equality.

We have

$$b_1^{2j} = \det\begin{pmatrix} 1\ 1\ 1 & \mathbf{0}^\top \\ 1\ 0\ 1 & (\overline{\mathbf{b}}^{j-3})^\top \\ \mathbf{0}\ \mathbf{a}\ \mathbf{b} & \overline{C}_{j-3} \end{pmatrix} = \det\begin{pmatrix} 1\ 0 & (\overline{\mathbf{b}}^{j-3})^\top \\ \mathbf{a}\ \mathbf{b} & \overline{C}_{j-3} \end{pmatrix},$$

$$b_1^{3j} = \det\begin{pmatrix} 1\ 1\ 1 & \mathbf{0}^\top \\ 1\ 1\ 0 & (\overline{\mathbf{a}}^{j-3})^\top \\ \mathbf{0}\ \mathbf{a}\ \mathbf{b} & \overline{C}_{j-3} \end{pmatrix} = \det\begin{pmatrix} 0\ 1 & (\overline{\mathbf{a}}^{j-3})^\top \\ \mathbf{a}\ \mathbf{b} & \overline{C}_{j-3} \end{pmatrix},$$

$$b_1^{1j} = \det\begin{pmatrix} 1\ 1\ 0 & (\overline{\mathbf{a}}^{j-3})^\top \\ 1\ 0\ 1 & (\overline{\mathbf{b}}^{j-3})^\top \\ \mathbf{0}\ \mathbf{a}\ \mathbf{b} & \overline{C}_{j-3} \end{pmatrix} = \det\begin{pmatrix} 1\ 1 & \left(\overline{(\mathbf{a}+\mathbf{b})}^{j-3}\right)^\top \\ \mathbf{a}\ \mathbf{b} & \overline{C}_{j-3} \end{pmatrix} = b_1^{2j} + b_1^{3j}.$$

We have proven that our triples of vertices are related to each other as in Remark 9.8, therefore, L_1 and L_2 are obtained from each other by a sequence of $\Omega 2$ and $\Omega 3$ moves. □

Let us define the map ψ from the set of all homotopy classes of looped graphs to the set of all graph-knots. Let $\mathfrak{L}$ be the homotopy class of L. Using Lemma 7.13, we can construct a symmetric matrix $A = (a_{ij})$ over $\mathbb{Z}_2$ coinciding with the adjacency matrix of L up to diagonal elements and having $\det A = 1$. Let $G(L)$ be the labeled simple graph having the matrix $(A^{-1} + E)$ as its adjacency matrix. Therefore, the first component of the label of a vertex is equal to the corresponding diagonal element of $(A^{-1} + E)$, the second component of the label of the vertex with the number i is $w_i(1 - 2a_{ii})$, where $w_i = 1$ if the vertex of L with the number i does not have a loop, and $w_i = -1$ otherwise. Set $\psi(\mathfrak{L}) = \mathfrak{F}$, where $G(L)$ is a representative of $\mathfrak{F}$.

Lemma 9.5. *The graph-knot $\mathfrak{F}$ does not depend on the choice of diagonal elements under the construction of A.*

Proof. The independence of $\mathfrak{F}$ (up to the sign, i.e. the second component of the label of a vertex) on the choice of diagonal elements follows from Theorem 7.11.

Namely, let A_1 and A_2 be two symmetric matrices over $\mathbb{Z}_2$ coinciding with the adjacency matrix of L up to diagonal elements and having $\det A_1 = \det A_2 = 1$. Then the matrices $(A_1^{-1} + E)$ and $(A_2^{-1} + E)$ are obtained from each other by $\Omega_g 4$ and/or $\Omega_g 4'$ graph-moves (up to the sign of a vertex). It remains to note that we have defined the sign of a vertex in such a way that looped vertices correspond to vertices with the negative writhe number and unlooped vertices correspond to vertices with the positive writhe number (Lemma 9.4). Moreover, the writhe number does not change under $\Omega_g 4$, $\Omega_g 4'$ graph-moves, and both the writhe number and the framing allow one to determine the sign of a vertex. □

Theorem 9.5. *The map ψ is well defined.*

Proof. Let L_1, L_2 be two representatives of $\mathfrak{L}$. We have to show that the graph-knots having representatives $G_1 = G(L_1)$ and $G_2 = G(L_2)$, respectively, are the same. Using Lemma 9.5 it suffices to show that G_1 and G_2 are related to each other by a sequence of Reidemeister graph-moves.

Let us consider three cases.

(1) Let L_1 and L_2 be obtained from each other by $\Omega 1$ (we remove the vertex with the number 1). We have

$$A(L_1) = \begin{pmatrix} a & \mathbf{0}^\top \\ \mathbf{0} & A(L_2) \end{pmatrix}, \qquad a \in \{0, 1\}.$$

Assume $\det \widetilde{A}(L_2) = 1$, where $\widetilde{A}(L_2)$ coincides with $A(L_2)$ up to diagonal elements, and

$$\widetilde{A}(L_1) = \begin{pmatrix} 1 & \mathbf{0}^\top \\ \mathbf{0} & \widetilde{A}(L_2) \end{pmatrix}.$$

Then

$$\widetilde{A}(L_1)^{-1} = \begin{pmatrix} 1 & \mathbf{0}^\top \\ \mathbf{0} & \widetilde{A}(L_2)^{-1} \end{pmatrix}.$$

Therefore, G_1 and G_2 are related by a sequence of $\Omega_g 1$, $\Omega_g 4$, $\Omega_g 4'$ graph-moves.

(2) Let L_1 and L_2 be obtained from each other by $\Omega 2$ (we remove the vertices with the numbers 1 and 2). We have

$$A(L_1) = \begin{pmatrix} 0 & b & \mathbf{a}^\top \\ b & 1 & \mathbf{a}^\top \\ \mathbf{a} & \mathbf{a} & A(L_2) \end{pmatrix}, \qquad b \in \{0, 1\}.$$

Assume $\det \widetilde{A}(L_2) = 1$, where $\widetilde{A}(L_2)$ coincides with $A(L_2)$ up to diagonal elements, and

$$\widetilde{A}(L_1) = \begin{pmatrix} 1+b & b & \mathbf{a}^\top \\ b & 1+b & \mathbf{a}^\top \\ \mathbf{a} & \mathbf{a} & \widetilde{A}(L_2) \end{pmatrix}, \qquad \det \widetilde{A}(L_1) = 1.$$

As

$$\det \begin{pmatrix} 1+b & b & \widetilde{\mathbf{a}}^\top \\ b & 1+b & \widetilde{\mathbf{a}}^\top \\ \widetilde{\mathbf{a}} & \widetilde{\mathbf{a}} & C \end{pmatrix} = \det \begin{pmatrix} 1+b & b & \widetilde{\mathbf{a}}^\top \\ 1 & 1 & \mathbf{0}^\top \\ \widetilde{\mathbf{a}} & \widetilde{\mathbf{a}} & C \end{pmatrix}$$

$$= \det \begin{pmatrix} 1+b & b & \widetilde{\mathbf{a}}^\top \\ 1 & 1 & \mathbf{0}^\top \\ \mathbf{0} & \mathbf{0} & C \end{pmatrix} = \det C,$$

$$\det \begin{pmatrix} 1+b & \mathbf{a}^\top \\ \mathbf{a} & \widetilde{A}(L_2) \end{pmatrix} = \det \begin{pmatrix} b & \mathbf{a}^\top \\ \mathbf{a} & \widetilde{A}(L_2) \end{pmatrix} + \det \widetilde{A}(L_2),$$

then

$$\widetilde{A}(L_1)^{-1} = \begin{pmatrix} f & f+1 & \mathbf{d}^\top \\ f+1 & f & \mathbf{d}^\top \\ \mathbf{d} & \mathbf{d} & \widetilde{A}(L_2)^{-1} \end{pmatrix}.$$

From the structure of the matrix $\widetilde{A}(L_1)^{-1}$ it follows that the two vertices (which do not belong to G_2) have the same framing and the necessary adjacencies, and from the structure of the matrix $A(L_1)$ and the coincidence of the vertices' framings it follows that the two vertices have different signs. This means that G_1 and G_2 are obtained from each other by a sequence of $\Omega_g 2$, $\Omega_g 4$, $\Omega_g 4'$ graph-moves.

(3) Now assume that L_1 and L_2 are obtained from each other by $\Omega 3$. Let us enumerate all the vertices of $V(L_i) = \{v_1^i, \ldots, v_n^i\}$ in such a way that corresponding vertices of L_1 and L_2 under $\Omega 3$ move have the same number, and without loss of generality we assume that v_1^1 and v_3^1 are looped, v_2^1 is unlooped, and v_1^1 is adjacent to v_2^1; v_3^1 is adjacent to neither v_1^1 nor v_2^1 in L_1. The case when v_3^1 is unlooped is obtained from the first case by applying second and third Reidemeister moves.

We have

$$A(L_1) = \begin{pmatrix} 1 & 1 & 0 & \mathbf{a}^\top \\ 1 & 0 & 0 & \mathbf{b}^\top \\ 0 & 0 & 1 & \mathbf{c}^\top \\ \mathbf{a} & \mathbf{b} & \mathbf{c} & D \end{pmatrix}, \qquad A(L_2) = \begin{pmatrix} 1 & 0 & 1 & \mathbf{a}^\top \\ 0 & 0 & 1 & \mathbf{b}^\top \\ 1 & 1 & 1 & \mathbf{c}^\top \\ \mathbf{a} & \mathbf{b} & \mathbf{c} & D \end{pmatrix},$$

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}.$$

Without loss of generality (if necessary we apply second Reidemeister moves), we may assume that $\mathbf{c} \neq \mathbf{0}$. Using the proof of Lemma 7.13, we construct the matrix

$$\widetilde{A}(L_1) = \begin{pmatrix} 0 & 1 & 0 & \mathbf{a}^\top \\ 1 & 1 & 0 & \mathbf{b}^\top \\ 0 & 0 & 0 & \mathbf{c}^\top \\ \mathbf{a} & \mathbf{b} & \mathbf{c} & \widetilde{D} \end{pmatrix}$$

with $\det \widetilde{A}(L_1) = 1$.

As

$$\det \begin{pmatrix} 0 & 0 & 1 & \mathbf{a}^\top \\ 0 & 0 & 1 & \mathbf{b}^\top \\ 1 & 1 & 1 & \mathbf{c}^\top \\ \mathbf{a} & \mathbf{b} & \mathbf{c} & \widetilde{D} \end{pmatrix} = \det \begin{pmatrix} 0 & 0 & 1 & \mathbf{a}^\top \\ 0 & 0 & 0 & \mathbf{c}^\top \\ 1 & 0 & 1 & \mathbf{c}^\top \\ \mathbf{a} & \mathbf{c} & \mathbf{c} & \widetilde{D} \end{pmatrix}$$

$$= \det \begin{pmatrix} 0 & 0 & 1 & \mathbf{a}^\top \\ 0 & 0 & 0 & \mathbf{c}^\top \\ 1 & 0 & 1 & \mathbf{b}^\top \\ \mathbf{a} & \mathbf{c} & \mathbf{b} & \widetilde{D} \end{pmatrix} = \det \widetilde{A}(L_1) = 1, \qquad (9.1)$$

we have

$$\widetilde{A}(L_2) = \begin{pmatrix} 0 & 0 & 1 & \mathbf{a}^\top \\ 0 & 0 & 1 & \mathbf{b}^\top \\ 1 & 1 & 1 & \mathbf{c}^\top \\ \mathbf{a} & \mathbf{b} & \mathbf{c} & \widetilde{D} \end{pmatrix}.$$

Let $\widetilde{A}(L_1)^{-1} = (\widetilde{a}_1^{ij})$, $\widetilde{A}(L_2)^{-1} = (\widetilde{a}_2^{ij})$, and let $G_i = G(L_i)$, $i = 1, 2$, be the two labeled graphs having the adjacency matrices $\widetilde{A}(L_i)^{-1} + E$. Let us show that G_1 and G_2 are obtained from each other by a sequence of $\Omega_g 2$, $\Omega_g 3$, $\Omega_g 4$, $\Omega_g 4'$ graph-moves.

Performing the same elementary manipulations as in (9.1), we have $\widetilde{a}_1^{ij} = \widetilde{a}_2^{ij}$ for $i, j \geqslant 3$. Further, we get

$$1 = \det \widetilde{A}(L_1) = \det \begin{pmatrix} 0 & 1 & 0 & \mathbf{a}^\top \\ 1 & 1 & 0 & \mathbf{b}^\top \\ 0 & 0 & 0 & \mathbf{c}^\top \\ \mathbf{a} & \mathbf{b} & \mathbf{c} & \widetilde{D} \end{pmatrix} = \det \begin{pmatrix} 0 & 1 & 0 & \mathbf{a}^\top \\ 1 & 1 & 0 & \mathbf{b}^\top \\ 1 & 0 & 0 & \mathbf{0}^\top \\ \mathbf{a} & \mathbf{b} & \mathbf{c} & \widetilde{D} \end{pmatrix}$$

$$= \det \begin{pmatrix} 1 & 0 & \mathbf{a}^\top \\ 1 & 0 & \mathbf{b}^\top \\ \mathbf{b} & \mathbf{c} & \widetilde{D} \end{pmatrix} = \det \begin{pmatrix} 1 & 0 & \mathbf{b}^\top \\ 0 & 0 & \mathbf{c}^\top \\ \mathbf{b} & \mathbf{c} & \widetilde{D} \end{pmatrix} = \widetilde{a}_1^{11},$$

$$\widetilde{a}_1^{12} = \det \begin{pmatrix} 1 & 0 & \mathbf{b}^\top \\ 0 & 0 & \mathbf{c}^\top \\ \mathbf{a} & \mathbf{c} & \widetilde{D} \end{pmatrix} = \det \begin{pmatrix} 1 & 0 & \mathbf{b}^\top \\ 0 & 0 & \mathbf{c}^\top \\ \mathbf{b} & \mathbf{c} & \widetilde{D} \end{pmatrix} = 1,$$

$$\widetilde{a}_1^{13} = \det \begin{pmatrix} 1 & 1 & \mathbf{b}^\top \\ 0 & 0 & \mathbf{c}^\top \\ \mathbf{a} & \mathbf{b} & \widetilde{D} \end{pmatrix} = \det \begin{pmatrix} 0 & 1 & \mathbf{b}^\top \\ 0 & 0 & \mathbf{c}^\top \\ \mathbf{c} & \mathbf{b} & \widetilde{D} \end{pmatrix} = 1,$$

$$\widetilde{a}_1^{1j} = \det \begin{pmatrix} 1 & 1 & 0 & (\overline{\mathbf{b}}^{j-3})^\top \\ 0 & 0 & 0 & (\overline{\mathbf{c}}^{j-3})^\top \\ \mathbf{a} & \mathbf{b} & \mathbf{c} & \overline{\widetilde{D}}_{j-3} \end{pmatrix} = 0, \qquad j \geqslant 3 \qquad (\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}),$$

$$\widetilde{a}_1^{2j} = \det \begin{pmatrix} 0 & 1 & 0 & (\overline{\mathbf{a}}^{j-3})^\top \\ 0 & 0 & 0 & (\overline{\mathbf{c}}^{j-3})^\top \\ \mathbf{a} & \mathbf{b} & \mathbf{c} & \overline{\widetilde{D}}_{j-3} \end{pmatrix} = \det \begin{pmatrix} 0 & 1 & 0 & (\overline{\mathbf{a}}^{j-3})^\top \\ 0 & 0 & 0 & (\overline{\mathbf{c}}^{j-3})^\top \\ \mathbf{a} & \mathbf{0} & \mathbf{c} & \overline{\widetilde{D}}_{j-3} \end{pmatrix},$$

$$\widetilde{a}_1^{3j} = \det \begin{pmatrix} 0 & 1 & 0 & (\overline{\mathbf{a}}^{j-3})^\top \\ 1 & 1 & 0 & (\overline{\mathbf{b}}^{j-3})^\top \\ \mathbf{a} & \mathbf{b} & \mathbf{c} & \overline{\widetilde{D}}_{j-3} \end{pmatrix} = \det \begin{pmatrix} 0 & 1 & 0 & (\overline{\mathbf{a}}^{j-3})^\top \\ 1 & 0 & 0 & (\overline{\mathbf{b}}^{j-3})^\top \\ \mathbf{a} & \mathbf{0} & \mathbf{c} & \overline{\widetilde{D}}_{j-3} \end{pmatrix}.$$

If either $\widetilde{a}_1^{22} = 0$ or $\widetilde{a}_1^{33} = 0$, we can apply the same second Reidemeister graph-moves to G_1 and G_2, and then after applying the $\Omega_g' 4$ graph-move, we get that the corresponding vertices have framing 0. Analogously, if $\widetilde{a}_1^{32} = 1$, we can apply the same second Reidemeister graph-moves to G_1 and G_2, and then after applying the $\Omega_g 4$ graph-move, we get that v_2^1 and v_3^1 are not adjacent to each other. Therefore, without loss of generality, we may assume that $\widetilde{a}_1^{22} = \widetilde{a}_1^{33} = 1$, $\widetilde{a}_1^{32} = 0$. Using the last equalities, we get

$$\widetilde{a}_1^{22} = \det\begin{pmatrix} 0 & 0 & \mathbf{a}^\top \\ 0 & 0 & \mathbf{c}^\top \\ \mathbf{a} & \mathbf{c} & \widetilde{D} \end{pmatrix} = \det\begin{pmatrix} 0 & 0 & \mathbf{b}^\top \\ 0 & 0 & \mathbf{c}^\top \\ \mathbf{b} & \mathbf{c} & \widetilde{D} \end{pmatrix}$$

$$= \det\begin{pmatrix} 1 & 0 & \mathbf{b}^\top \\ 0 & 0 & \mathbf{c}^\top \\ \mathbf{b} & \mathbf{c} & \widetilde{D} \end{pmatrix} + \det\begin{pmatrix} 1 & 0 & \mathbf{0}^\top \\ 0 & 0 & \mathbf{c}^\top \\ \mathbf{b} & \mathbf{c} & \widetilde{D} \end{pmatrix} = 1 + \det\begin{pmatrix} 0 & \mathbf{c}^\top \\ \mathbf{c} & \widetilde{D} \end{pmatrix} = 1,$$

$$\widetilde{a}_1^{33} = \det\begin{pmatrix} 0 & 1 & \mathbf{a}^\top \\ 1 & 1 & \mathbf{b}^\top \\ \mathbf{a} & \mathbf{b} & \widetilde{D} \end{pmatrix} = \det\begin{pmatrix} 1 & 0 & \mathbf{c}^\top \\ 0 & 1 & \mathbf{b}^\top \\ \mathbf{c} & \mathbf{b} & \widetilde{D} \end{pmatrix} = 1 + \det\begin{pmatrix} 1 & \mathbf{b}^\top \\ \mathbf{b} & \widetilde{D} \end{pmatrix} = 1,$$

$$\widetilde{a}_1^{23} = \det\begin{pmatrix} 0 & 1 & \mathbf{a}^\top \\ 0 & 0 & \mathbf{c}^\top \\ \mathbf{a} & \mathbf{b} & \widetilde{D} \end{pmatrix} = \det\begin{pmatrix} 1 & 1 & \mathbf{b}^\top \\ 0 & 0 & \mathbf{c}^\top \\ \mathbf{c} & \mathbf{b} & \widetilde{D} \end{pmatrix} = 1 + \det\begin{pmatrix} 0 & \mathbf{c}^\top \\ \mathbf{b} & \widetilde{D} \end{pmatrix} = 0.$$

Let us find the remaining elements of $\widetilde{A}(L_2)^{-1}$. We have

$$\widetilde{a}_2^{11} = \det\begin{pmatrix} 0 & 1 & \mathbf{b}^\top \\ 1 & 1 & \mathbf{c}^\top \\ \mathbf{b} & \mathbf{c} & \widetilde{D} \end{pmatrix} = \det\begin{pmatrix} 0 & 1 & \mathbf{b}^\top \\ 1 & 0 & \mathbf{a}^\top \\ \mathbf{b} & \mathbf{c} & \widetilde{D} \end{pmatrix} = \det\begin{pmatrix} 1 & 1 & \mathbf{b}^\top \\ 1 & 0 & \mathbf{a}^\top \\ \mathbf{b} & \mathbf{c} & \widetilde{D} \end{pmatrix} + \det\begin{pmatrix} 0 & \mathbf{a}^\top \\ \mathbf{c} & \widetilde{D} \end{pmatrix}$$

$$= \det\begin{pmatrix} 1 & 1 & \mathbf{b}^\top \\ 0 & 1 & \mathbf{c}^\top \\ \mathbf{b} & \mathbf{c} & \widetilde{D} \end{pmatrix} + \det\begin{pmatrix} 0 & \mathbf{b}^\top \\ \mathbf{c} & \widetilde{D} \end{pmatrix} + \det\begin{pmatrix} 0 & \mathbf{c}^\top \\ \mathbf{c} & \widetilde{D} \end{pmatrix}$$

$$= \det\begin{pmatrix} 1 & 1 & \mathbf{b}^\top \\ 0 & 0 & \mathbf{c}^\top \\ \mathbf{b} & \mathbf{c} & \widetilde{D} \end{pmatrix} + \det\begin{pmatrix} 1 & \mathbf{b}^\top \\ \mathbf{b} & \widetilde{D} \end{pmatrix} + 1 = 1,$$

$$\widetilde{a}_2^{22} = \det\begin{pmatrix} 0 & 1 & \mathbf{a}^\top \\ 1 & 1 & \mathbf{c}^\top \\ \mathbf{a} & \mathbf{c} & \widetilde{D} \end{pmatrix} = \det\begin{pmatrix} 1 & 1 & \mathbf{b}^\top \\ 1 & 1 & \mathbf{c}^\top \\ \mathbf{b} & \mathbf{c} & \widetilde{D} \end{pmatrix} = 1,$$

$$\widetilde{a}_2^{33} = \det\begin{pmatrix} 0 & 0 & \mathbf{a}^\top \\ 0 & 0 & \mathbf{b}^\top \\ \mathbf{a} & \mathbf{b} & \widetilde{D} \end{pmatrix} = \det\begin{pmatrix} 0 & 0 & \mathbf{b}^\top \\ 0 & 0 & \mathbf{c}^\top \\ \mathbf{b} & \mathbf{c} & \widetilde{D} \end{pmatrix} = 1,$$

$$\widetilde{a}_2^{12} = \det\begin{pmatrix} 0 & 1 & \mathbf{b}^\top \\ 1 & 1 & \mathbf{c}^\top \\ \mathbf{a} & \mathbf{c} & \widetilde{D} \end{pmatrix} = \det\begin{pmatrix} 1 & 1 & \mathbf{b}^\top \\ 0 & 1 & \mathbf{c}^\top \\ \mathbf{b} & \mathbf{c} & \widetilde{D} \end{pmatrix}$$

$$= \det\begin{pmatrix} 1 & 1 & \mathbf{b}^\top \\ 0 & 0 & \mathbf{c}^\top \\ \mathbf{b} & \mathbf{c} & \widetilde{D} \end{pmatrix} + \det\begin{pmatrix} 1 & \mathbf{b}^\top \\ \mathbf{b} & \widetilde{D} \end{pmatrix} = 0,$$

$$\widetilde{a}_2^{13} = \det\begin{pmatrix} 0 & 0 & \mathbf{b}^\top \\ 1 & 1 & \mathbf{c}^\top \\ \mathbf{a} & \mathbf{b} & \widetilde{D} \end{pmatrix} = \det\begin{pmatrix} 0 & 0 & \mathbf{b}^\top \\ 0 & 1 & \mathbf{c}^\top \\ \mathbf{c} & \mathbf{b} & \widetilde{D} \end{pmatrix}$$

$$= \det\begin{pmatrix} 0 & 0 & \mathbf{b}^\top \\ 0 & 0 & \mathbf{c}^\top \\ \mathbf{c} & \mathbf{b} & \widetilde{D} \end{pmatrix} + \det\begin{pmatrix} 0 & \mathbf{b}^\top \\ \mathbf{c} & \widetilde{D} \end{pmatrix} = 0,$$

$$\widetilde{a}_2^{23} = \det\begin{pmatrix} 0 & 0 & \mathbf{a}^\top \\ 1 & 1 & \mathbf{c}^\top \\ \mathbf{a} & \mathbf{b} & \widetilde{D} \end{pmatrix} = \det\begin{pmatrix} 0 & 1 & \mathbf{b}^\top \\ 0 & 1 & \mathbf{c}^\top \\ \mathbf{c} & \mathbf{b} & \widetilde{D} \end{pmatrix} = 0,$$

$$\widetilde{a}_2^{1j} = \det\begin{pmatrix} 0 & 0 & 1 & (\overline{\mathbf{b}}^{j-3})^\top \\ 1 & 1 & 1 & (\overline{\mathbf{c}}^{j-3})^\top \\ \mathbf{a} & \mathbf{b} & \mathbf{c} & \overline{\widetilde{D}}_{j-3} \end{pmatrix} = \det\begin{pmatrix} 0 & 0 & 1 & (\overline{\mathbf{b}}^{j-3})^\top \\ 1 & 1 & 0 & (\overline{\mathbf{a}}^{j-3})^\top \\ \mathbf{a} & \mathbf{b} & \mathbf{c} & \overline{\widetilde{D}}_{j-3} \end{pmatrix}$$

$$= \det\begin{pmatrix} 1 & 0 & 1 & (\overline{\mathbf{b}}^{j-3})^\top \\ 0 & 1 & 0 & (\overline{\mathbf{a}}^{j-3})^\top \\ \mathbf{0} & \mathbf{b} & \mathbf{c} & \overline{\widetilde{D}}_{j-3} \end{pmatrix} = \det\begin{pmatrix} 1 & 0 & (\overline{\mathbf{a}}^{j-3})^\top \\ \mathbf{b} & \mathbf{c} & \overline{\widetilde{D}}_{j-3} \end{pmatrix} = \widetilde{a}_1^{2j} + \widetilde{a}_1^{3j},$$

$$\widetilde{a}_2^{2j} = \det\begin{pmatrix} 0 & 0 & 1 & (\overline{\mathbf{a}}^{j-3})^\top \\ 1 & 1 & 1 & (\overline{\mathbf{c}}^{j-3})^\top \\ \mathbf{a} & \mathbf{b} & \mathbf{c} & \overline{\widetilde{D}}_{j-3} \end{pmatrix} = \det\begin{pmatrix} 0 & 1 & 1 & (\overline{\mathbf{a}}^{j-3})^\top \\ 1 & 0 & 0 & (\overline{\mathbf{b}}^{j-3})^\top \\ \mathbf{a} & \mathbf{0} & \mathbf{c} & \overline{\widetilde{D}}_{j-3} \end{pmatrix} = \widetilde{a}_1^{3j},$$

$$\widetilde{a}_2^{3j} = \det\begin{pmatrix} 0 & 0 & 1 & (\overline{\mathbf{a}}^{j-3})^\top \\ 0 & 0 & 1 & (\overline{\mathbf{b}}^{j-3})^\top \\ \mathbf{a} & \mathbf{b} & \mathbf{c} & \overline{\widetilde{D}}_{j-3} \end{pmatrix} = \det\begin{pmatrix} 0 & 0 & 1 & (\overline{\mathbf{a}}^{j-3})^\top \\ 0 & 0 & 0 & (\overline{\mathbf{c}}^{j-3})^\top \\ \mathbf{a} & \mathbf{b} & \mathbf{0} & \overline{\widetilde{D}}_{j-3} \end{pmatrix} = \widetilde{a}_1^{2j}.$$

We see that

$$\widetilde{A}(L_1)^{-1} + E = \begin{pmatrix} 0 & 1 & 1 & \mathbf{0}^\top \\ 1 & 0 & 0 & \mathbf{f}^\top \\ 1 & 0 & 0 & \mathbf{g}^\top \\ \mathbf{0} & \mathbf{f} & \mathbf{g} & H \end{pmatrix},$$

$$\widetilde{A}(L_2)^{-1} + E = \begin{pmatrix} 0 & 0 & 0 & \mathbf{f}^\top + \mathbf{g}^\top \\ 0 & 0 & 0 & \mathbf{g}^\top \\ 0 & 0 & 0 & \mathbf{f}^\top \\ \mathbf{f} + \mathbf{g} & \mathbf{g} & \mathbf{f} & H \end{pmatrix}.$$

It is easy to see that the corresponding vertices have the structure as in Definition 9.6. Therefore, G_2 is obtained from G_1 by a sequence of $\Omega_g 2$, $\Omega_g 3$, $\Omega_g 4$, $\Omega_g 4'$ graph-moves. □

From Theorems 9.4, 9.5 and from the definitions of the maps χ and ψ it follows that these maps are inverse. Therefore, we have proved Theorem 9.3.

We conclude this section with an example of a non-realizable graph-knot.

Definition 9.16. A graph-link (a homotopy class of looped graphs) is called *non-realizable* if it has no realizable representative.

Corollary 9.1. *A graph-link $\mathfrak{F}$ is non-realizable if and only if $\chi(\mathfrak{F})$ is non-realizable.*

Let G be the labeled graph depicted in Fig. 9.15, whose vertices have framing 1 and arbitrary signs.

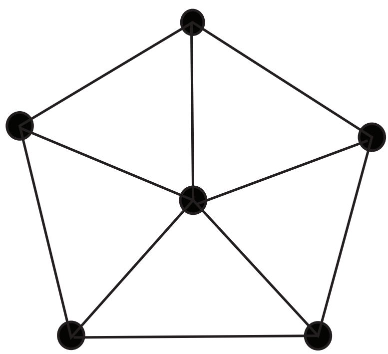


Fig. 9.15 The first Bouchet graph gives a non-realizable graph-knot.

Corollary 9.2. *The graph-knot $\mathfrak{F}$ generated by G is non-realizable.*

Proof. Let $\mathfrak{L} = \chi(\mathfrak{F})$. It is not difficult to see that the looped graph isomorphic to G (loops are available) is a representative of $\mathfrak{L}$. Therefore, the looped graph $\mathfrak{L}$ (see Theorem 9.9 and Corollary 9.4) is non-realizable, and, in turn, $\mathfrak{F}$ is a non-realizable graph-knot. □

9.2.4 *Smoothing operations and Turaev's* $\boldsymbol{\Delta}$

Let us define a smoothing operation for free graph-links. As usual we first mimic the definition of a smoothing for framed 4-graphs for the case of realizable free graph-links, see Sec. 8.4.1, and then use the same definition for all graphs.

Let $\mathcal{G}$ be a free framed (simple) graph, i.e. an equivalence class of labeled simple graphs under the fourth graph-moves. All representatives of a free framed graph have the same number of vertices. Therefore, the notion of a vertex for a free framed graph is well defined.

Let v be a vertex of $\mathcal{G}$. Let us consider two cases. In the first case, there exists a representative H of $\mathcal{G}$ for which v has either framing 1 or the degree greater than 0. It is not difficult to see that v has the same property for each representative of $\mathcal{G}$, and there are two representatives H_1 and H_2 of $\mathcal{G}$ which differ from each other by $\Omega_g 4$ or $\Omega_g 4'$ at v. By a *smoothing* of the free framed graph $\mathcal{G}$ at the vertex v we mean any of the two free framed graphs having the representatives $H_1 \setminus \{v\}$ and $H_2 \setminus \{v\}$, respectively. In the realizable case this means that the framed 4-graph has a rotating circuit having any of the two possible connection types at the vertex. Then the smoothing at the vertex corresponds to the removal of the chord of the framed chord diagram (the vertex of the intersection graph) corresponding to the vertex. If a smoothing of a framed 4-graph leads to a disconnected graph, this may be repaired by taking another representative of the same graph-link. We get the second case: the vertex v has framing 0 and is isolated for a representative, and, therefore, for any representative, of $\mathcal{G}$. Let H be a representative of $\mathcal{G}$. Let us construct the new graph H' obtained from H by adding a new vertex u with framing 0 to H which is adjacent only to v, see Fig. 9.16 for the case of realizable graphs (the dashed line is a rotating circuit). By a *smoothing* of the free framed graph $\mathcal{G}$ at the vertex v we mean any of the two free framed graphs having the representatives $H \setminus \{v\}$ and H', respectively.

Generally, a *smoothing* of a free framed graph in a collection of vertices is a free framed graph obtained by a sequence of smoothings.

Remark 9.12. Sometimes after applying a smoothing, additional vertices can appear, which were absent in the original graph. We do not apply any smoothing to these new vertices.

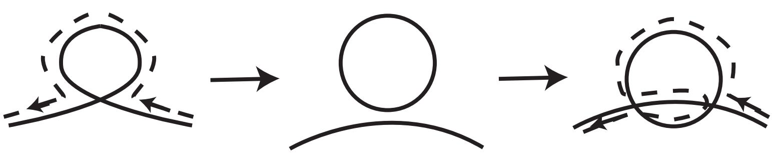


Fig. 9.16 One of the two smoothings at an isolated vertex.

Proposition 9.1. *Assume $\mathcal{G}$, $\mathcal{G}'$ are free framed graphs, and $\mathcal{G}$ can be obtained from $\mathcal{G}'$ as a result of smoothings at some vertices. If $\mathcal{G}'$ is realizable by a chord diagram, then so is $\mathcal{G}$.*

Proof. If $\mathcal{G}'$ is realizable and $\mathcal{G}$ is obtained from $\mathcal{G}'$ by applying a fourth graph-move and/or deleting a vertex, then one can draw the corresponding framed 4-graph, and take the corresponding resmoothing which will lead us to the chord diagram for $\mathcal{G}$. In the other case, if $\mathcal{G}'$ is realizable, then the realizability of $\mathcal{G}$ follows from Lemma 9.1. □

Let us construct the operation Δ. Let $i > 1$ be a natural number. Define the set $\mathbb{Z}_2\mathcal{G}_i$ to be the $\mathbb{Z}_2$-linear space generated by the set $\mathcal{G}_i$ of free framed graphs having i components modulo the following relations:

(1) the second Reidemeister graph-move;
(2) $\mathcal{G} = 0$, if $\mathcal{G}$ has two vertices with framing 0 which are adjacent only to each other.

The meaning of (2) is that a free framed graph equals zero if it has a "component not linked with others" (see Definition 9.14). To have two vertices with framing 0 and being adjacent only to each other means the corresponding free framed graph has a unicursal component not linked with others.

For $i = 1$, we define $\mathbb{Z}_2\mathcal{G}_1$ analogously with respect to equivalence (1) and not (2).

Let us define the map $\Delta\colon \mathbb{Z}_2\mathcal{G}_1 \to \mathbb{Z}_2\mathcal{G}_2$. We take a free framed graph $\mathcal{G}$ with $\operatorname{corank}_{\mathbb{Z}_2}(A(\mathcal{G}) + E) = 0$ and construct an element $\Delta(\mathcal{G})$ from $\mathbb{Z}_2\mathcal{G}_2$ as follows. For each vertex v of $\mathcal{G}$, there are two ways of smoothing it. One way gives us a graph from $\mathbb{Z}_2\mathcal{G}_1$, and the other smoothing gives us a free framed graph $\mathcal{G}_v$ from $\mathbb{Z}_2\mathcal{G}_2$. We take $\mathcal{G}_v$ and set

$$\Delta(\mathcal{G}) = \sum_v \mathcal{G}_v \in \mathbb{Z}_2\mathcal{G}_2.$$

Theorem 9.6. *Δ is a well-defined mapping from $\mathbb{Z}_2\mathcal{G}_1$ to $\mathbb{Z}_2\mathcal{G}_2$.*

Using the main principle from Sec. 9.2.3, we can define whether a vertex belongs to "one component" or "different components" of a free framed graph.

Definition 9.17. We call a vertex v_i of a free framed graph $\mathcal{G}$ *oriented* if

$$\operatorname{corank}_{\mathbb{Z}_2}(A(\mathcal{G}) + E) \leqslant \operatorname{corank}_{\mathbb{Z}_2}(B_i(\mathcal{G})).$$

It is not difficult to show that

$$\operatorname{corank}_{\mathbb{Z}_2} B_i(\mathcal{G}) \neq \operatorname{corank}_{\mathbb{Z}_2} B(\mathcal{G} \setminus \{v_i\})$$

if and only if v_i is oriented.

Using the notion of an oriented vertex we can define the map Δ^i (iteration of the map Δ) by considering smoothings at oriented vertices and taking that smoothing which has more components than other in each step.

Corollary 9.3. *Δ^i is a well-defined mapping from $\mathbb{Z}_2\mathcal{G}_1$ to $\mathbb{Z}_2\mathcal{G}_{i+1}$.*

9.3 Parity, minimality and non-trivial examples

In this section we consider the parity for free graph-knots and free graph-links in the spirit of Chap. 8. Since we have constructed the equivalence between the set of graph-knots and the set of homotopy classes of looped interlacement graphs, it is sufficient to construct a parity for free homotopy classes of looped interlacement graphs and for graph-links with more than one component.

9.3.1 *Definition of parity*

Definition 8.4 can be straightforwardly generalized for the case of looped graphs and for the case of graph-links.

Let $\mathfrak{L}$ (respectively, $\mathfrak{F}$) be a free homotopy class of graphs or homotopy class of looped interlacement graphs (respectively, a free graph-link of graph-link).

Let us define the category $\mathfrak{C}(\mathfrak{L})$ (respectively, $\mathfrak{C}(\mathfrak{F})$) of graphs of the (free) homotopy class $\mathfrak{L}$ (respectively, the (free) graph-link $\mathfrak{F}$). The objects of $\mathfrak{C}(\mathfrak{L})$ (respectively, $\mathfrak{C}(\mathfrak{F})$) are graphs of $\mathfrak{L}$ (respectively, labeled graphs of $\mathfrak{F}$) and morphisms of the category are (formal) compositions of *elementary morphisms*. By an elementary morphism we mean

- an isomorphism of graphs;
- a Reidemeister move.

Let us denote by $\mathcal{V}$ the vertex functor on $\mathfrak{C}(\mathfrak{L})$ (respectively, $\mathfrak{C}(\mathfrak{F})$), i.e. the functor from $\mathfrak{C}(\mathfrak{L})$ (respectively, $\mathfrak{C}(\mathfrak{F})$) to the category; objects of which are finite sets and morphisms are partial bijections. For each graph G we define $\mathcal{V}(G)$ to be the set of the vertices of G. Any elementary morphism $f\colon G \to G'$ naturally induces a partial bijection $f_*\colon \mathcal{V}(G) \to \mathcal{V}(G')$.

Let A be an abelian group.

Definition 9.18. A *parity* p on graphs of a (free) homotopy class $\mathfrak{L}$ (respectively, a (free) graph-link $\mathfrak{F}$) with coefficients in A is a family of maps $p_G\colon \mathcal{V}(G) \to A$, where $G \in \mathrm{ob}(\mathfrak{C}(\mathfrak{L}))$ (respectively, $G \in \mathrm{ob}(\mathfrak{C}(\mathfrak{F}))$) is an object of the category, such that for any elementary morphism $f\colon G \to G'$ the following conditions hold:

(1) $p_{G'}(f_*(v)) = p_G(v)$ provided that $v \in \mathcal{V}(G)$ and there exists $f_*(v) \in \mathcal{V}(G')$, i.e. the parity of the corresponding vertices is the same;
(2) $p_G(v_1) + p_G(v_2) = 0$ if f is a decreasing second Reidemeister move and v_1, v_2 are the disappearing vertices;
(3) $p_G(v_1) + p_G(v_2) + p_G(v_3) = 0$ if f is a third Reidemeister move and v_1, v_2, v_3 are the vertices participating in this move.

Analogously to Lemma 8.2, one can prove the following lemma.

Lemma 9.6. *Let p be any parity and G be a (labeled) graph. Then $p_G(v) = 0$ if f is the decreasing first Reidemeister move applied to G and v is the disappearing vertex of G.*

Let us consider two examples of parity with coefficients from $\mathbb{Z}_2$.

Example 9.2. Let $\mathfrak{L}$ be a (free) homotopy class of looped graphs, and L be its representative.

Definition 9.19. Define the map $\mathrm{gp}_L\colon \mathcal{V}(L) \to \mathbb{Z}_2$ by putting $\mathrm{gp}_L(v) = 0$ if the degree of v is even (*an even vertex*), and $\mathrm{gp}_G(v) = 1$ otherwise (*an odd vertex*).

Lemma 9.7 ([132]). *The map* gp *is a parity for $\mathfrak{L}$. This parity is called the* Gaussian parity.

Example 9.3. Let $\mathfrak{F}$ be a (free) two-component graph-link, and G be its representative. Using Definition 9.17, we can define the notion of oriented vertex for G.

Definition 9.20. Define the map $p_G\colon \mathcal{V}(G) \to \mathbb{Z}_2$ by putting $p_G(v) = 0$ if v is an oriented vertex (*an even vertex*), and $p_G(v) = 1$ otherwise (*an odd vertex*).

Lemma 9.8 ([132]). *The map p is a parity for $\mathfrak{F}$.*

9.3.2 *The universal parity*

In this subsection we want to classify parity for free homotopy classes of looped graphs, i.e. we want to prove a theorem analogous to Theorem 8.3. Using Definition 8.8, we can define the universal parity for homotopy classes of looped graphs and graph-links.

Theorem 9.7. *Let a free homotopy class be given. Then the Gaussian parity on its graphs is the universal parity.*

This theorem will follow from Lemmas 9.9–9.11 which are analogous to Lemmas 8.6–8.8.

Definition 9.21. Let G be a (simple) graph with the set of vertices $V(G)$ and the set of edges $E(G)$, and let $H \subseteq G$ be an induced subgraph of the graph G, where $V(H)$ is the set of vertices of H and $E(H)$ is the set of edges of H, and "induced" means that $\{u, v\} \in E(H)$, where u, $v \in V(H)$, if and only if $\{u, v\} \in E(G)$. We shall say that H *is even with respect to* H if each vertex from $V(G) \setminus V(H)$ is adjacent to even number of vertices from $V(H)$.

Lemma 9.9. *Let H be an even subgraph with respect to a graph G. Then for any parity on graphs of the free homotopy class generated by G the sum of the parities of all vertices from $V(H)$ is equal to* 0.

Proof. Let p be an arbitrary parity on graphs of the free homotopy class $\mathfrak{L}$ generated by G. We proceed by induction on the number of vertices of H. For any finite set M we denote by $\#M$ the number of elements of M.

The induction base. The claim holds for $V(H)$ with $\#V(H) = 1$, 2 or 3 according to Lemma 9.6 and Definition 9.18, respectively.

The induction step. Assume that the claim is true for any subgraph H such that H is even with respect to G and $\#V(H) = k - 1$.

Let us consider any subgraph H being even with respect to G and $\#V(H) = k$.

Let $V(G) = \{v_1, \ldots, v_n\}$, and let v_1 and v_2 be two vertices of H. We have $N_G(v_i) = N_H(v_i) \sqcup A_i$, $i = 1, 2$, where A_i is the set of vertices from $V(G) \setminus V(H)$ adjacent to v_i. Denote by $B = N_G(v_1) \cap N_G(v_2)$.

Let us apply the second Reidemeister move $f\colon G \to G'$ to G by adding two vertices u_1 and u_2 as follows. The vertices u_1 and u_2 are adjacent to $N_G(v_i) \setminus B$, $i = 1, 2$, the vertices u_1 and u_2 are not adjacent to a vertex v_i if $\#N_H(v_i) \equiv 0 \pmod 2$ and adjacent otherwise, and u_1 and u_2 are not adjacent if $\#\big((N_H(v_1) \cup N_H(v_2)) \setminus B\big) \equiv 0 \pmod 2$ and adjacent otherwise.

Let $H' = H \setminus \{v_1, v_2\} \cup \{u_1\}$ be an induced subgraph of G'. We claim that H' is even with respect to G'. Let us consider three cases.

(1) Let $v_s \in V(G)\setminus V(H)$. If $N_G(v_s) \cap \{v_1, v_2\} = \emptyset$, then $N_{G'}(v_s) \cap V(H') = N_G(v_s) \cap V(H)$. If $N_G(v_s) \cap \{v_1, v_2\} = \{v_i\}$ for some $i = 1, 2$, then

$$N_{G'}(v_s) \cap V(H') = \big(N_G(v_s) \cap V(H) \cup \{u_1\}\big) \setminus \{v_i\}$$

and

$$\#\big(N_{G'}(v_s) \cap V(H')\big) = \#\big(N_G(v_s) \cap V(H)\big).$$

If $N_G(v_s) \cap \{v_1, v_2\} = \{v_1, v_2\}$, then

$$N_{G'}(v_s) \cap V(H') = \big(N_G(v_s) \cap V(H)\big) \setminus \{v_1, v_2\}$$

and

$$\#\big(N_{G'}(v_s) \cap V(H')\big) = \#\big(N_G(v_s) \cap V(H)\big) - 2.$$

(2) Consider the vertex v_i. We have $N_{G'}(v_i) \cap V(H') = N_H(v_i)$ if $\#N_H(v_i) \equiv 0 \pmod 2$, or $N_{G'}(v_i) \cap V(H') = N_H(v_i) \cup \{u_1\}$ otherwise. As a result, we get $\#\big(N_{G'}(v_i) \cap V(H')\big) \equiv 0 \pmod 2$.

(3) Consider the vertex u_2. We have $N_{G'}(u_2) \cap V(H') = (N_H(v_1) \cup N_H(v_2)) \setminus B$ if $\#\big((N_H(v_1) \cup N_H(v_2)) \setminus B\big) \equiv 0 \pmod 2$ or $N_{G'}(u_2) \cap V(H') = (N_H(v_1) \cup N_H(v_2) \cup \{u_1\}) \setminus B$ otherwise. As a result we get $\#\big(N_{G'}(u_2) \cap V(H')\big) \equiv 0 \pmod 2$.

By the induction hypothesis, we have

$$p_{G'}(u_1) + \sum_{v_{i_j} \in V(H)\setminus\{v_1, v_2\}} p_{G'}(f_*(v_{i_j})) = 0.$$

Since the three equalities $\#N_H(v_1) \equiv 1 \pmod 2$, $\#N_H(v_2) \equiv 1 \pmod 2$ and $\#\big((N_H(v_1) \cup N_H(v_2)) \setminus B\big) \equiv 1 \pmod 2$ cannot hold simultaneously, we get that the three vertices u_2, v_1, v_2 form the third Reidemeister move. We have

$$p_{G'}(u_2) + p_{G'}(f_*(v_1)) + p(f_*(v_2)) = 0$$

and

$$p_{G'}(u_1) + p_{G'}(u_2) = 0.$$

Therefore,

$$\sum_{v_{i_j} \in V(H)} p_{G'}(f_*(v_{i_j})) = \sum_{v_{i_j} \in V(H)} p_G(v_{i_j}) = 0.$$

□

Lemma 9.10. *Let G be a graph. For any parity p on graphs of the free homotopy class generated by G and any vertex $v \in V(G)$ we have $p_G(v) = 0$ if $\mathrm{gp}_G(v) = 0$.*

Proof. We prove the lemma by induction on the degree of the vertex v.

The induction base: If $\deg(v) = 0$, then v forms the first Reidemeister move and $p_G(v) = 0$ in accordance with Lemma 9.6.

The induction step: Assume that for any vertex $u \in V(G)$ of G with $\deg(u) < 2k$ and $\mathrm{gp}_G(u) = 0$, we have $p_G(u) = 0$. Let v be a vertex of degree $2k$ and $N_G(v) = \{v_1, \ldots, v_{2k}\}$. Apply two consecutive decreasing second Reidemeister moves $f\colon G \to G_1$ and $g\colon G_1 \to G_2$. We add two vertices u_1 and u_2 which are not adjacent and adjacent only to vertices $v_3, \ldots, v_{2k}$, and we add two vertices w_1 and w_2 which are not adjacent and adjacent only to vertices from $\big(N_{G_1}(v_1) \cup N_{G_1}(v_2)\big) \setminus \big(N_{G_1}(v_1) \cap N_{G_1}(v_2)\big)$.

We have

$$p_{G_1}(u_1) + p_{G_1}(u_2) = 0 \Longleftrightarrow p_{G_2}(g_*(u_1)) + p_{G_2}(g_*(u_2)) = 0$$

and

$$p_{G_2}(w_1) + p_{G_2}(w_2) = 0.$$

By construction, the three vertices w_1, v_1, v_2 form the third Reidemeister move, therefore,

$$p_{G_2}(w_1) + p_{G_2}((gf)_*(v_1)) + p_{G_2}((gf)_*(v_2)) = 0.$$

Since any vertex distinct from v_1 and v_2 are adjacent either to both vertices v and u_1 or none of v, u_1, the induced subgraph $H \subseteq G_2$ with $V(H) = \{v, u_1, w_1, v_1, v_2\}$ is even with respect to G. Therefore, by Lemma 9.9,

$$p_{G_2}((gf)_*(v)) + p_{G_2}(g_*(u_1)) + p_{G_2}(w_1) + p_{G_2}((gf)_*(v_1)) + p_{G_2}((gf)_*(v_2)) = 0.$$

By the induction hypothesis, we get

$$p_G(v) = p_{G_2}((gf)_*(v)) = p_{G_2}(g_*(u_2)) = 0.$$

□

Lemma 9.11. *Let p be an arbitrary parity (with coefficients from a group A) on graphs of the free homotopy class represented by a graph G. Then for any two vertices v_1, v_2 such that $\mathrm{gp}_G(v_1) = \mathrm{gp}_G(v_2) = 1$ we have $p_G(v_1) = p_G(v_2) = x \in A$ and $2x = 0$.*

Proof. Let us show that $p_G(v_1) = -p_G(v_2)$.

Let us apply the second Reidemeister move $f\colon G \to G'$ by adding two vertices u_1, u_2 such that they are not adjacent and adjacent only to vertices from $\big(N_G(v_1) \cup N_G(v_2)\big) \setminus \big(N_G(v_1) \cap N_G(v_2)\big)$.

Since the vertices $\mathrm{gp}_G(v_1) = \mathrm{gp}_G(v_2) = 1$, we have, Lemma 9.10, $\mathrm{gp}_{G'}(u_1) = \mathrm{gp}_{G'}(u_2) = 0$ and the vertices u_1, v_1, v_2 form the third Reidemeister move. Therefore,

$$p_{G'}(u_1) + p_{G'}(f_*(v_1)) + p_{G'}(f_*(v_2)) = 0.$$

Finally, we get

$$p_G(v_1) = p_{G'}(f_*(v_1)) = -p_{G'}(f_*(v_2)) = -p_G(v_2).$$

Applying second Reidemeister moves, we can assume that there are more than two odd vertices. Considering three odd vertices v_1, v_2, v_3, we get $p_G(v_1) = p_G(v_2) = p_G(v_3)$. This completes the proof. □

Using Lemmas 9.10 and 9.11 for any parity p (with coefficients from a group A) on graphs of the free homotopy class having a graph G, we can construct the homomorphism $\rho\colon A \to \mathbb{Z}_2$ by taking $\rho(x) = 1$, where $p_G(v) = x$ and $\mathrm{gp}_G(v) = 1$. This concludes the proof of Theorem 9.7.

Remark 9.13. Let p be a parity on a free homotopy class $\mathfrak{L}$. It is not possible that there exist two graphs G_1 and G_2 of $\mathfrak{L}$, both having vertices being odd in the Gaussian parity such that p is trivial on G_1, and p is the Gaussian parity on G_2. It follows from the fact that there is a sequence of Reidemeister moves transforming G_1 to G_2 such that any graph in this sequence has vertices being odd in the Gaussian parity.

9.3.3 *Minimality*

Using the Gaussian parity, we can define the map Δ^i_{odd}, where the sum is taken over all odd oriented vertices, or Δ^i_{even}, where the sum is taken over all even oriented vertices. We have to define the notion of *even* and *odd* vertex for free framed graphs with many components.

Definition 9.22. We call a vertex v of a free framed graph $\mathcal{G}$ with one component *even* (respectively, *odd*) if the vertex corresponding to v of the looped interlacement graph $\chi(\mathcal{G})$ is even (respectively, odd).

Let us consider the free framed graph $\mathcal{G}_{v_1,\ldots,v_{k-1}}$ with k components which is obtained from $\mathcal{G}$ by smoothing $\mathcal{G}$ consequently at $v_1, \ldots, v_{k-1}$, where v_1 is oriented in $\mathcal{G}$ and v_i, $i = 2, \ldots, k-1$, is an oriented vertex in $\mathcal{G}_{v_1,\ldots,v_{i-1}}$. An oriented vertex u of $\mathcal{G}_{v_1,\ldots,v_{k-1}}$ is *even with respect to the smoothing at* $v_1, \ldots, v_{k-1}$ (respectively, *odd with respect to the smoothing at* $v_1, \ldots, v_{k-1}$) if the number of oriented vertices in $\mathcal{G}_{v_1,\ldots,v_{k-1}}$ which are incident to u in $\chi(\mathcal{G})$ is even (respectively, odd).

Remark 9.14. We have defined even vertices only for those free framed graphs with many components which originate from given free framed graphs with one component. First, it is sufficient to define the iteration $\Delta^i_{\text{even(odd)}}$. Second, we have done it so as not to complicate the construction, but it is possible to do in the general case.

Proposition 9.2. *Δ^i_{odd} is a well-defined mapping from $\mathbb{Z}_2\mathcal{G}_1$ to $\mathbb{Z}_2\mathcal{G}_{i+1}$.*

Now, we can define the brackets $[\cdot]$ for graph-knots and $\{\cdot\}$ for graph-links analogously to Sec. 8.4.2. For a graph G representing a free graph-knot and for a graph H representing a free two-component graph-link consider the following sums:

$$[G] = \sum_{s \text{ even,1 comp.}} G_s \in \mathbb{Z}_2\mathcal{G}_1 \quad \text{and} \quad \{H\} = \sum_{s \text{ even, non-trivial}} H_s \in \mathbb{Z}_2\mathcal{G}_2.$$

In the first formula, the sums are taken over all smoothings at all even vertices, and only those summands are taken into account where $\operatorname{corank}_{\mathbb{Z}_2}(A(G_s) + E) = 0$. In the second formula, the sums are taken over all smoothings at all even vertices, and H_s is not equivalent to any simple graph having two vertices with framing 0 which are adjacent only to each other. Thus, if G has k even vertices, then $[G]$ will contain at most 2^k summands, and if all vertices of G are odd, then we shall have exactly one summand, the graph G itself. The same is true for H and $\{H\}$.

Theorem 9.8. *If G and G' represent the same free graph-knot, then the following equality holds: $[G] = [G']$. If H and H' represent the same free graph-link with two components, then $\{H\} = \{H'\}$.*

The proof of Theorem 9.8 verbally reproduces the proof of Theorem 8.9 according to the main principle or, maybe, a slight modification of it.

Definition 9.23. We call a labeled graph G (respectively, a looped graph L) *minimal* if there is no representative of the graph-link corresponding to G (respectively, the homotopy class of L) having strictly smaller number of vertices than G (respectively, L) has.

Theorem 9.9 (cf. Theorem 8.1). *Let G (respectively, H) be a simple labeled graph representing a free graph-knot (a two-component graph-link) with all odd vertices in the sense of Definition* 9.19 (*respectively, Definition* 9.20), *such that no decreasing second Reidemeister move is applicable to G (respectively, H). Then G (respectively, H) is minimal.*

As a consequence of this theorem we may deduce the following

Corollary 9.4. *The graph G shown in Fig.* 9.5 (*left*), *representing a free graph-knot, is minimal. In particular, the corresponding free graph-knot is non-trivial and has no realizable representatives.*

The first claim of the corollary is trivial: we just check the conditions of the theorem and see that G is minimal. To see that the second claim indeed holds, we shall need to go through the proof of Theorem 9.8, where we see that any representative G' of the graph-knot $\mathfrak{F}$ has G as a *smoothing*, that is, G *lies inside* each representative G' of the same graph-link, and if G is not realizable, then so is G' according to Proposition 9.1. Analogously, one sees that the free two-component graph-link $\mathfrak{H}$ with the representative H shown in Fig. 9.17 has no realizable representative because $\{H\} = H$. Note that H in Fig. 9.17 is equivalent to the Bouchet graph shown in the right part of the picture by Ω_4-graph move; so they represent the same two-component free graph-link.

Let us consider one more example.

Proposition 9.3. *The looped graph K shown in Fig.* 9.9 *is minimal and non-realizable.*

The proof consists of the following steps.

First, note that $\Delta(K)$ consists of seven summands $L + \sum_i L_i$, where only one summand (corresponding to the vertex x) is a two-component free graph-link with all *odd* vertices; for each of the remaining summands L_i, there is at least one even vertex.

Now, the two-component free graph-link generated by L has the representative shown in Fig. 9.17; all framings of the vertices are 0. To see it, one should consecutively perform the following operations for K. First, we "smooth" it at the crossing x (it can be done only at the expense of changing our circuit to the rotating one at some vertex different from x as a Gauss circuit cannot represent link). After that, we have to make the circuit rotating at all other vertices. All these steps are shown in Fig. 9.18 (of course one can apply the map from Theorem 9.3 and immediately get

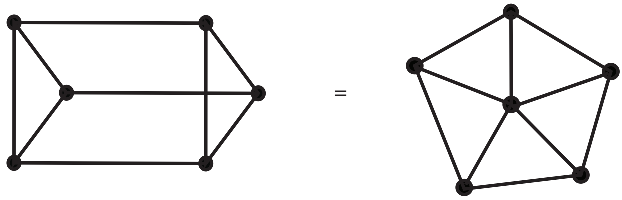


Fig. 9.17 The two-component free link.

the final graph, but we have decided to perform all transformations for clearness).

Now, consider the bracket $\{\Delta(K)\} = L + \sum_i \{L_i\}$. Note that all the graph-links generated by summands $\{L_i\}$ have representatives with strictly less than six vertices since each of L_i has at least one even vertex; on the other hand, the graph-link generated by L has no representative with less than six crossings; so, this element L is not canceled in the sum. Since it is not realizable, the free framed knot K is not realizable either.

9.4 A generalization of Kauffman's bracket and other invariants. Minimality theorems

We have already considered the minimality theorem for graph-links (see Theorem 9.9) and the minimality theorems for links (see Secs. 4.5 and 5.10). In this section we present minimality theorems which use Kauffman's bracket generalization for graph-links. Let us generalize the Kauffman bracket polynomial and some notions for the case of graph links.

Let $s \subset V(G)$ be a subset of the set $V(G)$ of vertices of G. Set $G(s)$ to be the induced subgraph of the graph G with the set of vertices $V(G(s)) = s$ and the set of edges $E(G(s))$ such that $\{u, v\} \in E(G(s))$, where u, $v \in s$, if and only if $\{u, v\} \in E(G)$.

Definition 9.24. We call a subset of $V(G)$ a *state* of G. The *A-state* is the state consisting of all the vertices of G labeled $(a, -)$, $a \in \{0, 1\}$, and no vertex labeled $(b, +)$, $b \in \{0, 1\}$. Analogously, the *B-state* is the state consisting of all vertices of G labeled $(b, +)$ and no vertex labeled $(a, -)$.

The *opposite* state to a state s is the set of vertices complementary to s (the opposite state to the A-state is the B-state). Two states are called *neighboring* if they differ only in one vertex, which belongs to one state

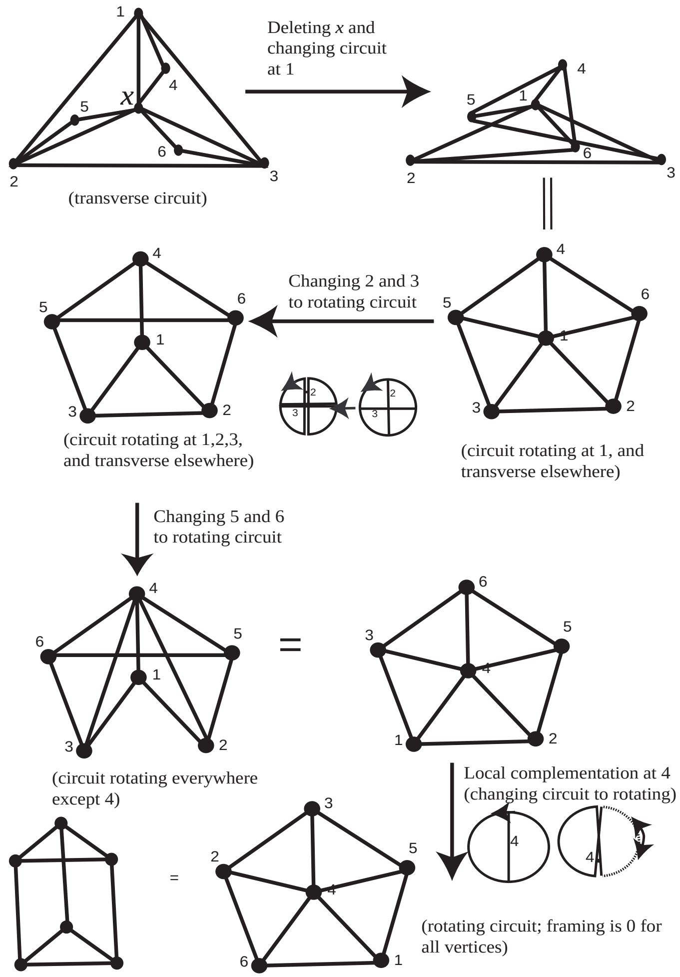


Fig. 9.18 The two-component free link.

and not to the other state. The *distance* between two states is equal to the number of the vertices in which the two states differ. We define the *number of circles* in a state s as $\operatorname{corank}_{\mathbb{Z}_2} A(G(s)) + 1$.

The *Kauffman bracket polynomial* of a labeled graph G is

$$\langle G\rangle(a) = \sum_s a^{\alpha(s)-\beta(s)}(-a^2 - a^{-2})^{\operatorname{corank}_{\mathbb{Z}_2} A(G(s))},$$

where the sum is taken over all states s of G, $\alpha(s)$ is equal to the sum of the vertices labeled $(a,-)$ from s and the vertices labeled $(b,+)$ from $V(G)\setminus s$, $\beta(s)=|V(G)|-\alpha(s)$.

Theorem 9.10. *The Kauffman bracket polynomial of a labeled graph is invariant under* $\Omega_g 2-\Omega_g 4'$ *and gets multiplied by* $(-a^{\pm 3})$ *under* $\Omega_g 1$.

Definition 9.25. Define the *Jones polynomial* for a labeled graph G with $\mathrm{corank}_{\mathbb{Z}_2}(A(G)+E)=0$ as $X(G)=(-a)^{-3w(G)}\langle G\rangle(a)$.

Remark 9.15. The Jones polynomial can be defined for any graph-link, but first we have to define the notion of an "oriented" graph-link. For simplicity we are not going to do it in this book.

Theorem 9.11. *The Jones polynomial is an invariant of graph-knots.*

The main results concerning minimality theorems in the classical case come from the well-known Kauffman–Murasugi–Thistlethwaite theorem, Theorem 4.6.

Recall that a *splitting point* of a connected virtual diagram is a classical crossing v such that after deleting the corresponding vertex v' from the frame of the corresponding atom, the frame breaks up into more connected components than one.

In [211, 221], Theorem 4.6 is generalized for virtual diagrams. The estimate $\mathrm{span}\langle K\rangle\leqslant 4n$ can be sharpened to $\mathrm{span}\langle K\rangle\leqslant 4n-4g$, where g is the genus of the atom.

From the "atomic" point of view, alternating link diagrams are those having atom genus zero (more precisely, diagram has genus zero if it is a connected sum of several alternating diagrams).

For virtual links, we have a notion of a *quasialternating* diagram [221]. This is a virtual diagram obtained from classical alternating diagrams by detours and virtualizations. Note the works [317, 318] where the authors consider virtual alternating diagrams.

Definition 9.26. *Non-split* diagram is a connected virtual diagram without splitting points.

Let us generalize the notions introduced above for graph-links.

Definition 9.27. A labeled graph G on n vertices is *alternating* if $k+l=n+2$, where k is the number of circles in the A-state s_1, i.e. $k=\mathrm{corank}_{\mathbb{Z}_2}A(G(s_1))+1$, and l is the number of circles in the B-state s_2 of G, i.e. $l=\mathrm{corank}_{\mathbb{Z}_2}A(G(s_2))+1$.

A labeled graph G is *adequate* if the number of circles k in the A-state is locally minimal, that is, there is no neighboring state for the A-state with $k+1$ circles, and the same is true for the number of circles in the B-state.

Remark 9.16. This definition of an adequate graph generalizes (see, e.g. [286]) the classical definition of an adequate diagram: Neither circle of the A-state nor the B-state splits into a pair of circles after one resmoothing.

From the definition of an alternating graph we have the following proposition.

Proposition 9.4. *The framing of each vertex of an alternating graph is zero.*

Definition 9.28. A labeled graph G is *non-split* if it has no isolated vertices.

Definition 9.29. For a labeled graph G the *atom genus* is $1-(k+l-n)/2$, where k and l are the numbers of circles in the A-state s_1 and the B-state s_2 of G, respectively, i.e. $k = \operatorname{corank}_{\mathbb{Z}_2} A(G(s_1)) + 1$ and $l = \operatorname{corank}_{\mathbb{Z}_2} A(G(s_2)) + 1$.

Note that this number agrees with the atom genus in the general case. We just use $\chi = 2 - 2g$, where χ is the Euler characteristic, and count χ by using the number of crossings n, number of edges $2n$ and the number of 2-cells (A-state circles and B-state circles).

Proposition 9.5. *A labeled graph G is alternating if and only if its atom genus is equal to* 0.

Lemma 9.12. *For any labeled graph G on n vertices we have*

$$\operatorname{span}\langle G\rangle \leqslant 4n - 4g(G),$$

where $g(G)$ is the genus of the corresponding atom.

Proof. Indeed, the assertion of this lemma comes from the definition of the Kauffman bracket and the atom genus. Denote the number of circles in the A-state of G by k, and denote the number of circles in the B-state of G by l.

It is easy to see that there is no state giving a term with a degree more than the degree of the leading term of the Kauffman bracket coming from the A-state, and there is no state giving a term with a degree less than the degree of the leading term of the Kauffman bracket coming from the B-state.

The leading term of the Kauffman bracket coming from the A-state has degree $n+2(k-1)$, and the lowest term coming from the B-state has degree $-n-2(l-1)$.

It is obvious that the span of the Kauffman bracket $\langle G\rangle$ is less than or equal to $2n+2(k+l-2)$.

Since $g(G) = 1-(k+l-n)/2$ we get the claim of the lemma. □

Lemma 9.13. *For an adequate labeled graph G on n vertices we have*

$$\mathrm{span}\langle G\rangle = 4n - 4g(G),$$

where $g(G)$ is the genus of the corresponding atom.

Proof. Indeed, it is sufficient to check that the leading term coming from the A-state of G is not canceled by any other term coming from another state (the argument for the lowest term coming from the B-state is the same).

To do that, let us consider the term $a^{\alpha(s)-\beta(s)}(-a^2)^{\mathrm{corank}_{\mathbb{Z}_2} A(G(s))}$ for a state s. For the A-state, we have $\alpha = n$, $\beta = 0$, $\gamma = k$. If we switch one crossing to the B-smoothing, then α is decreased by 1, β is increased by 1 and hence the degree of $a^{\alpha-\beta}$ is decreased by 2. We may compensate this only if $\mathrm{corank}_{\mathbb{Z}_2} A(G(s))$ is increased by 1. This may happen only if there is a state $\widetilde{s}$ neighboring to the A-state, with $\mathrm{corank}_{\mathbb{Z}_2} A(G(\widetilde{s})) = k$. Thus, the diagram is inadequate.

Terms corresponding to other states have the degree equal to or less than the degree of the term corresponding to the state s. This completes the proof. □

Lemma 9.14. *An alternating labeled graph G is adequate if and only if it is non-split.*

Proof. The direction ($\Longrightarrow$) is obvious.

Now, assume that the diagram G is inadequate, alternating and has no isolated vertices. Denote the number of circles of the A-state by k, and that of the B-state by l. Without loss of generality, assume that there is a state s with $\alpha(s) = n-1$, $\beta(s) = 1$ and $\mathrm{corank}_{\mathbb{Z}_2} A(G(s)) = k$. Consider the state $\widetilde{s}$ opposite to the state s. Obviously, the number of circles in this state $\widetilde{s}$ is $l-1$ (the total number cannot exceed $k+l$). Denote the vertex of G where the A-state differs from s by v. Thus, the labeled graph G' obtained by changing the label of the vertex v has genus 0, too.

Since G is alternating, all single-circle states are at the same distance from the A-state. On the other hand, all single-circle states are at the same distance from s. This means that these single-circle states (as subsets of $\{1,\ldots,n\}$) either all contain v or all do not contain v.

Assume they all contain v. We argue that v is an isolated vertex. Indeed, if there were a vertex w adjacent to v, then, starting from a single-circle state containing v and changing it at v and w, we would get another single-circle state not containing v. This completes the proof. □

Lemmas 9.12–9.14 together yield the following theorem.

Theorem 9.12. *An alternating non-split labeled graph is minimal.*

Proof. Let G be an alternating non-split labeled graph with n vertices. Assume the contrary. Then we have

$$4n = \text{span}\langle G\rangle = \text{span}\langle G'\rangle \leqslant 4n' - 4g(G'),$$

where n' is the number of crossings of G', and $g(G')$ is the atom genus of G'. The inequality $n' < n$ leads to a contradiction, which completes the proof. □

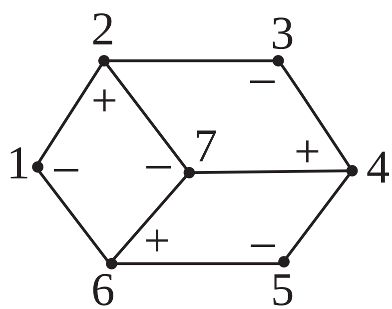


Fig. 9.19 The Bouchet graph BW_3.

Example 9.4. Consider the graph BW_3, depicted in Fig. 9.19, consisting of seven vertices with the following incidences: i is connected to j if and only if $i - j \equiv \pm 1 \pmod 6$, $i, j = 1,\ldots,6$, and 7 is connected to 2, 4, 6. Label all even vertices by $(0,+)$, and label all odd vertices by $(0,-)$. BW_3 is alternating. By Theorem 9.12, BW_3 is minimal.

Theorem 9.13. *The graph-link generated by the labeled graph BW_3 is non-realizable.*

Here we give a sketch of the proof of this fact.

Before proving this result, we give some definitions and formulate some assertions.

Let $\mathfrak{F}$ be a graph-link with more than one component. Let G be any representative of $\mathfrak{F}$ with $V(G) = \{v_1, \ldots, v_n\}$.

Definition 9.30. We say that a vertex $v_i \in V(G)$ *lies on one component* of G if $\operatorname{corank}_{\mathbb{Z}_2} B_i(G) \neq \operatorname{corank}_{\mathbb{Z}_2} \widehat{B}_i(G)$. Otherwise, we say that v_i *belongs to two different components.*

Remark 9.17. Let $G' = G \backslash \{v_i\}$, i.e. G' is obtained from G by deleting the vertex v_i and all the edges incident to this vertex. In the realizable case, this operation corresponds to a smoothing of the crossing corresponding to the vertex v_i. Since $\widehat{B}_i(G) = B(G')$, we get that $\operatorname{corank}_{\mathbb{Z}_2} \widehat{B}_i(G)$ is equal to the number of components of the link after this smoothing. It is not difficult to see that $\operatorname{corank}_{\mathbb{Z}_2} B_i(G)$ equals the number of components of the link obtained by smoothing the crossing corresponding to v_i in the other way.

Therefore, in the realizable case, Definition 9.30 means that v_i lies on one component if after two smoothings we have links with different number of components. This definition coincides with the "real" definition of a vertex when it lies on one component.

Remark 9.18. Note that in Definition 9.17 we called a vertex lying on one component, oriented.

Let us define the following relation on the set of all vertices belonging to two different components.

Definition 9.31. Let v_i and v_j be two vertices from $V(G)$ belonging to two different components. We say that at these vertices *two components meet* if either $v_i = v_j$, or v_i lies on one component of the labeled graph $G \setminus \{v_j\}$.

Remark 9.19. In the realizable case, Definition 9.31 means that after a smoothing of one vertex, the other one lies on the component obtained by "joining" two components sharing the first vertex.

It is not difficult to prove the following lemma.

Lemma 9.15. *The relation from Definition* 9.31 *is an equivalence relation on the set of vertices belonging to two different components.*

Let us consider equivalence classes and define the number $\vartheta(G)$ to be the number of equivalence classes having an odd number of vertices.

Theorem 9.14. *The number $\vartheta(G)$ is invariant under Reidemeister graph-moves.*

This theorem can be proved by following the main principle. In the realizable case, the claim of the theorem follows straightforwardly from the definitions, and in the general case, we have the following definitions reformulated in the language of adjacent matrices.

It is easy to show that the graph-link generated by BW_3 has four components and $\vartheta(BW_3) = 7$, and for any realizable graph-link with four components we have ϑ to be strictly less than 7.

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