2021-10-27 Kähler geometry

Recall we defined "stable orbits" in a vector space V with a group action of $G \subset SL(V)$. ①

Blow-up at the origin $0 \in V$. Then we get

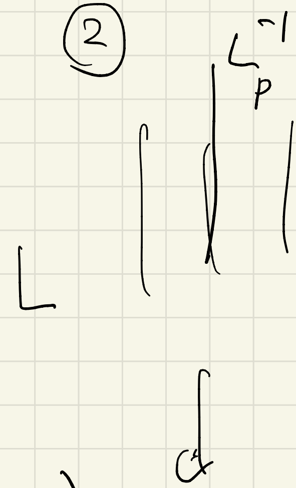
$$\mathcal{O}_{\mathbb{P}^N}(-1) \longrightarrow \mathbb{P}^N$$

Consider a submanifold $Z \subset \mathbb{P}^N$ and

$$\begin{array}{ccc} \mathcal{O}_{\mathbb{P}^N}(-1) & \longrightarrow & \mathbb{P}^N \\ & & \cup \\ \mathcal{O}_Z(-1) & \longrightarrow & Z \\ \| & & \\ \mathcal{O}_{\mathbb{P}^N}(-1)\Big|_Z . & & \end{array}$$

Think of Z as a parameter space of $\{$varieties$\}$ or $\{$vector bdles$\}$ or $\{$Kähler manifolds$\}$ you are interested in.

Z can be infinite dimensional, but for the moment we assume $\dim Z < \infty$.



<u>Def</u> $p \in Z$ (semi) stable

$\underset{\text{def}}{\iff}$ $\forall x \neq 0$ in L_p^{-1}

$G \cdot x$ closed

and finite stabilizer

(resp. $\overline{Gx} \cap$ zero section $= \phi$)

$\mathbb{P}(V) \supset Z \ni p$

We next describe Kempf-Ness picture.
We replace Z by a symplectic manifold N.
($Z = N$ but Z algebraic variety, N symplectic mfd)

Let (N, Ω) be a symplectic manifold N,
i.e. Ω satisfies the following

- Ω closed 2-form.
- $\Omega : TN \times TN \longrightarrow \mathbb{R}$
non-degenerate
i.e. if $\Omega(X, Y) = 0$ for $\forall Y$ then $X = 0$.

Ω is called a symplectic form.

<u>Example</u> If (M,ω) is Kähler, ω Kähler form. $(M,\omega) = (N,\Omega)$

Let K be a Lie group acting (N,Ω) as symplectomorphisms

(symplectic diffeomorphism)

i.e. $\forall g \in K$, $g^*\Omega = \Omega$.

Let k be the Lie algebra of K. Then $\forall X \in k$ defines a vector field on N which we denote by the same letter X and

$$L_X \Omega = 0. \qquad L_X \text{ Lie derivative}$$

(∵)

$$\left.\frac{d}{dt}\right|_{t=0} \operatorname{Exp}(tX)^* \Omega = L_X\Omega$$

$$= \left.\frac{d}{dt}\right|_{t=0} \Omega = 0$$

(∴)

Since $L_X = d \circ i(X) + i(X)\circ d$, $d\Omega = 0$.

we have $0 = L_X\Omega = d(i(X)\Omega)$.

So $i(X)\Omega$ is closed.

<u>Def</u> $X \in \mathfrak{k}$ is a Hamiltonian vector field if $i(X)\Omega$ is <u>exact</u>, i.e.

$\exists\, u_X \in C^\infty(N)$ s.t. $i(X)\Omega = -du_X$.

u_X is called the <u>Hamiltonian function</u> of X.

$\left(\begin{array}{l} \text{Instead you can start with } u \in C^\infty(N) \\ \text{then, since } \Omega \text{ is non-degenerate,} \\ \quad \exists X \text{ s.t. } i(X)\Omega = -du \\ \quad X =: X_u \text{ is called the \underline{Hamiltonian vector} } \\ \quad \text{\underline{field of } } u. \end{array} \right)$

<u>Def</u> Let $\mathfrak{k}^*$ be the dual of $\mathfrak{k}$.

$\mu : N \longrightarrow \mathfrak{k}^*$ is a <u>moment map</u> for the action of K

$\underset{\text{def}}{\Longleftrightarrow}$ (1) $\langle d\mu, X \rangle = -i(X)\Omega$

(2) $g^*\mu = \mu \circ Ad(g)$: K-equivariant.

$\mu_{g(p)}(X) = \mu_p(Ad(g)X)$

Set $u_X(p) = \langle \mu(p), X \rangle$. Then

$$du_X = \langle d\mu, X \rangle = -i(X)\Omega$$

$\therefore$ u_X is a Hamiltonian function.

Suppose $[\Omega] \in H^2_{DR}(N)$ is an integral class.

$$\left(\begin{array}{l} 0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0 \\ H^2(N;\mathbb{Z}) \to H^2(N;\mathbb{R}) \cong H^2_{DR}(N) \\ [\Omega] \in \text{Image of this map} \end{array} \right)$$

$\exists$ $\Lambda \to N$ complex line bundle with

$$c_1(\Lambda) = [\Omega].$$

Let ∇ be a connection of Λ.

e local frame.

$\nabla e = e \cdot \theta$ $\qquad$ θ local 1-form.

(connection form)

Suppose $\frac{1}{2\pi} d\theta = \Omega$.

(General fact: If $[\Omega]$ is an integral class then such L and ∇ always exist.)

Let $\Lambda^* = \Lambda$ - zero section.

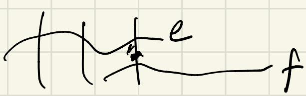
locally on U $\quad \Lambda^*|_U \cong U \times \mathbb{C}^*$

$$z e(p) \longleftrightarrow (p, z), \quad z \neq 0$$

$$\tilde{\theta} := \frac{dz}{z} + \theta$$

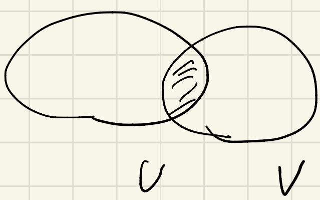
<u>Proposition</u>: $\tilde{\theta}$ defines a global 1-form on Λ^*.

(This is the connection form for the principal $\mathbb{C}^*$-bundle Λ^*. $\frac{dz}{z}$ is the Maurer-Cartan form.)



frame fields.

$f = e\sigma \qquad \sigma \in C^\infty(U \cap V, \mathbb{C}^*)$



$= ez \qquad = fw.$

$p = z e = wf = e\sigma w$

$\therefore z = \sigma w$

$\nabla e = e\theta$ on U, $\quad \nabla f = f\theta'$ on V.

$$\nabla f = \nabla e \cdot \sigma + e \cdot d\sigma = e\theta\sigma + e\,d\sigma$$

$$\nabla f = e\sigma \cdot \theta'$$

$$\theta\sigma + d\sigma = \sigma\theta'$$

$$\theta' = \theta + \sigma^{-1} d\sigma.$$