The formal moment map geometry of the space of symplectic connection

Day 3:

(M,ω) symplectic mfd

$\boxed{\nabla \in \mathcal{E}(M,\omega) \text{ ie } \begin{array}{l} \nabla\omega = 0 \\ \nabla \text{ torsion free} \end{array}}$

$\downarrow$

∂ Covariant derivative on ΓW

elements in ΓW (locally): $a(x,y,\nu) = \sum_{\substack{2k+r\geq 0 \\ k\geq 0\ r\geq 0}} \nu^k a_{k,i_1\dots i_r}(x)\, y^{i_1}\dots y^{i_r}$.

$\circ$- product: $a\circ b = \left(\exp\left(\frac{\nu}{2}\Lambda^{ij}\partial_{y^i}\partial_{z^j}\right) a(x,y,\nu)\, b(x,z,\nu)\right)_{z=y}$

$\downarrow$ Choose $\underline{\Omega} \in \Gamma\Lambda^2 M[[\nu]]$ closed

$D = \partial - \delta + \frac{1}{\nu}[r,\cdot]_\circ$ flat connect°

$(\Gamma W_D = \{f \in \Gamma W \mid Df = 0\}, \circ)$

D) Fedosov star products.

<u>Theorem 2:</u> (Fedosov)

$\forall F \in C^\infty(M)[[\nu]] : \exists ! f \in \Gamma W_D$ with $\underset{f|_{y=0}}{\sigma(f)} = F$.

Write $Q : C^\infty(M)[[\nu]] \longrightarrow \Gamma W_D$ for $(\sigma|_{\Gamma W_D})^{-1}$

<u>Def 10:</u> The Fedosov star product $*$ determined by $\nabla \in \mathcal{E}(M,\omega)$ and $\Omega \in \nu \Gamma \Lambda^2 M[[\nu]]$ closed formal 2-form is

$$F * G := \sigma(Q(F) \circ Q(G)), \text{ for } F, G \in C^\infty(M)[[\nu]].$$

Proof of Th 2:

If $Df=0$ with $\sigma(f)=F$

$\Rightarrow \delta f = \partial f + \frac{1}{\hbar}[r,f]_\circ$, $\delta^{-1} f = 0$ because f is a 0-form

Hodge decomposition: $f = f_{00} + \delta^{-1}\delta f$

$$f = F + \delta^{-1}\left(\partial f + \tfrac{1}{\hbar}[r,f]_\circ\right) \qquad (a)$$

Because δ^{-1} increase the W-degree, r is of W-degree at least 3, Equation (a) has a unique solution constructed recursively.

Conversely, if f is a solution of (a), we check $f \in \Gamma W_D$.

$A := Df$

Then: • $DA = D^2 f = 0$

$$\bullet\ \delta^{-1} A = \underbrace{\delta^{-1}\left(\partial f + \tfrac{1}{\hbar}[r,f]_\circ\right)}_{\substack{\shortparallel \\ f - f_{00} = f - F}} - \underbrace{\delta^{-1}\delta f}_{f-F} = 0$$

Hodge decomposition: $A = \cancel{A_{00}} + \delta^{-1}\delta A = \delta^{-1}\left(\partial A + \tfrac{1}{\hbar}[r,A]\right)$.

is a 1-form.

Hence $A=0$, with a similar argument as what we did in Th 1.

$\Rightarrow Df = 0$. $\square$

Exercise 4: Define $\widetilde{Q} : \Gamma W \longrightarrow \Gamma W$

$$f \longmapsto \widetilde{Q}(f) := \sum_{r\geq 0} \left[\delta^{-1}\left(\partial + \frac{1}{\nu}[r,\cdot]_\circ\right)\right]^r (f)$$

ω - $\widetilde{Q}|_{C^\infty(M)[[\nu]]} - Q$

Fedosov star product:

$$F * G = \sigma(Q(F) \circ Q(G)) = \sum_{r\geq 0} \nu^r C_r(F,G)$$

- associativity of $*$ comes from associativity of $\circ$.
- Compute $Q(F)$: For simplicity, take $F \in C^\infty(M)$.

$$\text{Eq }(\alpha) \Rightarrow Q(F) = F + \delta^{-1}\Big(\partial Q(F) + \underbrace{\frac{1}{\nu}[r, Q(F)]_\circ}\Big)$$

because r has at least deg 3 it will not contribute before $Q(F)^{(3)}$.

$$Q(F)^{(0)} = F.$$

$$Q(F)^{(1)} = \delta^{-1}(\partial F) = \delta^{-1}(dF) = \partial_i F y^i.$$

$$Q(F)^{(2)} = \delta^{-1}(\partial(Q(F)^{(1)})) = \frac{1}{2} D^2_{ek} F y^k y^e.$$

$Q(F)^{(3)}$ more complicate

$$F * G = F.G + \frac{\nu}{2} \underbrace{\partial_i F \Lambda^{ij} \partial_j G}_{\{F,G\}.} + O(\nu^2)$$

$C_0(F,G) = F.G$ ✓

$C_1(F,G) - C_1(G,F) = \{F,G\}$ ✓

- $Q(1) = 1 \Rightarrow F * 1 = F$

E) Equivalence of star product:

Def 11: Two star product $*$ and $*'$ on (M,ω) are equivalent if there exists $E = \mathrm{Id} + \sum_{r\geq 1} \nu^r E_r$ with E_r being differential operators st

$$E(F * G) = E(F) *' E(G) \quad \forall F, G \in C^\infty(M)[[\nu]].$$

We write $* \sim *'$.

Theorem 3: (Gutt, Nest-Tsygan)

On (M,ω) symplectic mfd, equivalence classes of $*$-products are parametrised by $\nu H^2_{dR}(M)[[\nu]]$.

Theorem 4: (Fedosov)

$\nabla, \nabla' \in \mathcal{E}(M,\omega)$ and $\Omega, \Omega' \in \nu\Gamma\Lambda^2 M[[\nu]]$ closed

Then $*_{\nabla,\Omega} \sim *_{\nabla',\Omega'}$ iff $[\Omega]_{dR} = [\Omega']_{dR}$

Fedosov star prod attached to ∇ and Ω

IV) Formal connections:

A) Definitions and 1^{st} properties.

Let (M,ω) be a closed symplectic mfd and $\mathcal{T}$ a mfd

$\mathcal{T}$ is possibly of infinite dimension.

Assume that there is on (M,ω) a family of $*$-products $\{*_\sigma\}_{\sigma\in\mathcal{T}}$ smoothly parametrised by $\mathcal{T}$.

We associate to it the product fibration:

$$\begin{array}{c} \mathcal{T}\times C^\infty(M)[[\nu]] \\ \downarrow p \text{ projection on } 1^{st} \text{ factor} \\ \mathcal{T} \end{array}$$

We endow the fiber above $\sigma\in\mathcal{T}$: $p^{-1}(\sigma)$ with the star product $*_\sigma$.

(Andersen–Masulli–Schätz)

Def 12: A formal connection for $\{*_\sigma\}_{\sigma\in\mathcal{T}}$ is a connection D in the bundle $\begin{array}{c}\mathcal{T}\times C^\infty(M)[[\nu]] \\ \downarrow \\ \mathcal{T}\end{array}$ of the form:

in the $\mathcal{T}$-direction

$$D_X F = dF(X) + \beta(X)\,F$$

$F\in\Gamma(\mathcal{T}\times C^\infty(M)[[\nu]])$

$X\in T\mathcal{T}$

with $\beta(X)=\sum_{k=1}^{\infty}\nu^k\beta_k(X)$ for β_k being 1-forms on $\mathcal{T}$ with values in differential operators on M.

Moreover, $D(F*_\sigma G)=(DF)*_\sigma G+F*_\sigma DG$.

$F,G\in\Gamma(\mathcal{T}\times C^\infty(M)[[\nu]])$

Rk about smoothness:

- $F \in \Gamma(T \times C^\infty(M)[[\nu]])$ is a map $F : T \longrightarrow C^\infty(M)[[\nu]]$

$\sigma \mapsto F_\sigma := \sum_{r \geq 0} \nu^r F_{r,\sigma}$ smooth wrt σ

- Same idea $f : T \longrightarrow \underline{\Gamma W}$

Infinite-dimensional mfd: Lecture notes by Karl-Hermann Neeb.

Denote by $\Gamma_{sympl} TM[[\nu]]$ the space of formal power series of symplectic v.f on (M, ω) $X = \sum_{r \geq 0} \nu^r \underbrace{X_r}_{\text{sympl. v.f.}}$

Theorem 5: (Andersen-Masulli-Schätz)

- $\exists$ a formal connection for $\{\star_\sigma\}_{\sigma \in T}$ iff $\forall \sigma, \sigma' \in T : \star_\sigma \sim \star_{\sigma'}$.
- The space of formal connections for $\{\star_\sigma\}_{\sigma \in T}$ is the affine space:

$$D_0 + \Omega^1(T, \Gamma_{sympl} TM[[\nu]])$$

for D_0 a given formal connection.

B) A formal connection for $\{*_\nabla\}_{\nabla\in\mathcal{E}(M,\omega)}$

$*_\nabla$ = Fedosov star product build with $\nabla\in\mathcal{E}(M,\omega)$ $\boxed{\Omega=0}$

Def 13: The tautological star product bundle over $\mathcal{E}(M,\omega)$ is

$$\begin{array}{c}\mathcal{V}:=\mathcal{E}(M,\omega)\times C^\infty(M)[[\nu]]\\ \big\downarrow p\\ \mathcal{E}(M,\omega)\end{array}$$

with the fiber over ∇ endowed with $*_\nabla$.

In Fedosov's construction: $*_\nabla$ can be compared with $*_{\nabla'}$.

Lemma 1: For $b\in\Gamma\mathcal{W}\otimes\Lambda^1M$ with $Db=0$

$\Rightarrow$ Equation $Da=b$ admits a unique solution $a\in\Gamma\mathcal{W}$ with $a_{|y=0}=0$

It is given by

$$a=D^{-1}b:=-\widetilde{Q}(\delta^{-1}b) \quad (\widetilde{Q} \text{ coming from Ex 4})$$

$$=-\sum_{r\geq 0}\left[\delta^{-1}\left(\partial+\tfrac{1}{\nu}[r,\cdot]\right)\right]^r(\delta^{-1}b).$$

For $\nabla\in\mathcal{E}(M,\omega)$, I emphasize the dependence in ∇ of $*_\nabla$, Γ^∇, ∂^∇, D^∇, r^∇

$\Omega=0$.

Theorem 5: (Fedosov)

Consider a smooth path $t \in [0,1] \longmapsto D^t \in \mathcal{E}(M,\omega)$

$\Rightarrow \exists$ isomorphisms $B_t : \Gamma W_{D^{0}} \longrightarrow \Gamma W_{D^{t}}$.

Moreover, B_t can be chosen in the canonical form:

$$B_t a = v_t \circ a \circ v_t^{-1}$$

for $v_t \in \Gamma W^{+}$ is the unique solution to:

$$\begin{cases} \frac{d}{dt} v_t = \frac{1}{\nu} h_t \circ v_t \\ v_0 = 1 \end{cases} \qquad (\text{Eq } 1)$$

v_t is some kind of exponential for $\circ$-product.

$$\text{with } \boxed{h_t} = -(D^{D^t})^{-1}\left(\frac{d}{dt}\Gamma^{D^t} + \frac{d}{dt} r^{D^t}\right). \qquad (\text{Eq } 2)$$

check one can use Lemma 1.

We want to build a formal connection $\mathcal{D}$ st

$$\mathcal{D}_{\frac{d}{dt}\nabla^t}\left(\left(\underset{\sim}{v_t \circ Q^{\nabla^0}(F) \circ v_t^{-1}} \right)\Big|_{y=0} \right) = 0$$

v_t coming from Theorem

<u>Def 16</u>: Define for $A \in T_\nabla \mathcal{E}(M,\omega)$:

① The connection 1-form $\alpha \in \Omega^1(\mathcal{E}(M,\omega), \Gamma^\infty W^3)$ by

$$\alpha_\nabla(A) = (D^\nabla)^{-1}\left(\overline{A} + \frac{d}{dt}\Big|_0 r^{\nabla+tA}\right)$$

for $\overline{A} = \frac{1}{2}\omega_{\ell k} A^k_{ij}\, y^\ell y^j dx^i = \frac{d}{dt}\Big|_0 \overline{\Gamma}^{\nabla+tA}$.

② The 1-form β with values in diff operators

$$\beta_\nabla(A)(F) := \left(\frac{1}{\nu}\left[\alpha_\nabla(A), Q^\nabla(F)\right]_\circ\right)\Big|_{y=0}, \text{ for } F \in C^\infty(M)[[\nu]]$$

③ The formal connection $\mathcal{D} = d + \beta$ on $\mathcal{V}$.

<u>Theorem 6</u>: (Andersen-Masulli-Schätz)

$\mathcal{D}$ is indeed a formal connection for the family $\{\star_\nabla\}_{\nabla\in\mathcal{E}(M,\omega)}$

IV) Curvature and a formal symplectic form on $\mathcal{E}(M,\omega)$.

A) The curvature of $\mathcal{D}$

For A, B v.f. over $\mathcal{E}(M,\omega)$

$F \in \Gamma \mathcal{W}$

The curvature of $\mathcal{D}$ is defined as

$$\mathcal{R}(A,B)F = \mathcal{D}_A(\mathcal{D}_B F) - \mathcal{D}_B(\mathcal{D}_A F) - \mathcal{D}_{[A,B]}F$$

Theorem 7:

$$\mathcal{R}(A,B)F = \left(\frac{1}{\nu}\left[\underline{\tilde{R}(A,B)}, Q(F)\right]_\circ\right)\Big|_{y=0}$$

with $\tilde{R}$ being a 2-form on $\mathcal{E}(M,\omega)$ with values in $\Gamma \mathcal{W}^3$

given by

$$\tilde{R}(A,B) := (d\alpha)(A,B) + \frac{1}{\nu}\left[\alpha(A), \alpha(B)\right]_\circ.$$

Moreover, at $\nabla \in \mathcal{E}(M,\omega)$:

- $\underline{\tilde{R}_\nabla(A,B)}$ is a D^∇-flat section.
- $\tilde{R}_\nabla(A,B)\big|_{y=0} = \frac{\nu^2}{24}\boxed{\Lambda^{i_1 j_1}\Lambda^{i_2 j_2}\Lambda^{i_3 j_3}\underline{A}_{i_1 i_2 i_3}\underline{B}_{j_1 j_2 j_3}} + O(\nu^3).$

$$\underline{A}_{ijk} = \underline{\omega_{il}} A^l_{jk}$$

Computation of $\tilde{R}_\nu(A,B)|_{y=0}$.

Take $A, B \in \Gamma S^3 T^*M$ seen as cst v.f.

$$d\alpha_\nu(A,B)|_{y=0} = A(\alpha(B)) - B(\alpha(A))|_{y=0} = 0$$

but $\alpha_\nu(A) = (D^\nu)^{-1}(\bar{A} + \frac{d}{dt} e^{\nu + tA})$ always contains y

Now $[\alpha(A), \alpha(B)]_\nu$.

$$\alpha_\nu(A) = D^{\nu^{-1}}(\bar{A} + \frac{d}{dt} e^{\nu + tA}) = S^{-1}(\bar{A}) + (\text{terms in } \Gamma W^4)$$

$$= \frac{1}{6} \underbrace{\omega_{\ell k} A^k_{ij}}_{= \underline{A}_{\ell ij}} y^\ell y^j y^i + (\text{terms in } \Gamma W^4)$$

$$\Rightarrow \frac{1}{\nu}[\alpha(A), \alpha(B)]_\nu|_{y=0} = \underline{\frac{\nu^2}{24} \Lambda^{i_1 j_1} \Lambda^{i_2 j_2} \Lambda^{i_3 j_3} \underline{A}_{i_1 i_2 i_3} \underline{B}_{j_1 j_2 j_3}} + O(\nu^3)$$

$\square$

B) A formal symplectic form on $\mathcal{E}(M,\omega)$

Defn 7: A formal sympl form on a mfld N is

$$\sigma := \sigma_0 + \nu\sigma_1 + \nu^2\sigma_2 + \dots \in \Omega^2(N)[[\nu]]$$

which is closed and starts with σ_0 a sympl form.

is infi. dimensional

not doable.

R curvature of D

$$\begin{tikzcd} (V, D) \arrow[d] \arrow[r, squiggly] & \cancel{(\Lambda^{top} V, D^{\Lambda^{top}})} \arrow[d] \\ \mathcal{E}(M,\omega) & \mathcal{E}(M,\omega) \end{tikzcd}$$

$R^{\Lambda^{top}} = \text{Trace of } R$

There is a not° of trace for $\star$-prod

Prop 3: Let (M,ω) be a closed mfld and $\star$ a star product

$\exists!$ map $Tr^\star$ called the trace st

$$Tr^\star : C^\infty(M)[[\nu]] \longrightarrow \nu^{-n}\mathbb{R}[[\nu]]$$

st:
- $Tr^\star([H,K]_\star) = 0$
- $Tr^\star$ satisfies a normalisation condition.

We consider $Tr^\star(\underbrace{\tilde{R}}_{}(A,B)|_{\nu=0})$

Theorem 8:

The formal 2-form

$$\widetilde{\Omega}^{\mathcal{E}}_{\nabla}(A,B) = 2\pi^{n}\nu^{n-2}\, 24\, \mathrm{Tr}^{*_\nabla}\left(\widetilde{R}_{\nabla}(A,B)|_{y=0}\right),$$

is a formal symplectic form on $\mathcal{E}(M,\omega)$.

Moreover, the action of $\mathrm{Ham}(M,\omega)$ preserves $\widetilde{\Omega}^{\mathcal{E}}$

Rk. The leading term of $\mathrm{Tr}^{*_\nabla}(\underset{C^\infty(M)[[\nu]]}{F}) = \frac{1}{(2\pi\nu)^n}\int F \frac{\omega^n}{n!} + O(\nu^{-n+1})$

combined with the leading term of $\widetilde{R}_\nabla$

$$\Rightarrow \widetilde{\Omega}^{\mathcal{E}}_{\nabla}(A,B) = \underline{\Omega^{\mathcal{E}}(A,B)} + O(\nu).$$

Theorem 9:

The trace Tr^{*_∇} gives a formal moment map for the action of $\mathrm{Ham}(M,\omega)$ on $(\mathcal{E}(M,\omega), \widetilde{\Omega}^{\mathcal{E}})$:

For $H \in C^\infty_0(M) \sim \mathrm{Lie}(\mathrm{Ham}(M,\omega))$

- $\frac{d}{dt}\Big|_0 \mathrm{Tr}^{*_{\nabla+tA}}(H) = \widetilde{\Omega}^{\mathcal{E}}(\underbrace{\mathcal{L}_{X_H}\nabla}_{\text{infinitesimal act° on } \mathcal{E}}, A) \qquad \underline{A \in T_\nabla \mathcal{E}}$

- $\mathrm{Tr}^{*_{\varphi \cdot \nabla}}(H) = \mathrm{Tr}^{*_\nabla}(\varphi^* H) \qquad \varphi \in \mathrm{Ham}(M,\omega)$