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A note on Tukey's polyefficiency

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Summary

The efficiency of an estimate S_n is usually computed at some underlying distribution. Because it is rare that a real situation can be represented by a single assumed model, Tukey proposed to compute instead the polyefficiency of S_n, that is the infimum of the efficiencies of S_n at a reasonable collection of distributions called corners. It is shown that high polyefficiency over a finite number of corners, seen as possible distributions of the n dimensional sample, implies at least as high an efficiency at any convex combination of these corners.

Some key words: Efficiency; Robustness; Tukey's corners.

The efficiency of an estimate S_n of θ, within a class C of estimates, is usually defined as $\inf\{R(\theta, \delta_n); \delta_n \in C\}/R(\theta, S_n)$, where $R(\theta, \delta_n)$ is the risk of δ_n with respect to a specific underlying distribution, and n is the sample size. For example, when θ is a location parameter, if S_n is unbiased, one is usually comparing its variance with the Cramér–Rao bound. Unfortunately estimates that have high efficiency for the assumed underlying distribution do not necessarily have high efficiency for the true distribution. This fact motivated several notions of robustness most relying on the asymptotic properties of the estimators (Huber, 1964; Hampel, 1971).

A different notion of robustness, based on the finite sample behaviour of estimates, was introduced by Tukey (1987) and Mosteller & Tukey (1977, § 10B, p. 206). We are interested in having high efficiency in a variety of situations rather than in one situation. Thus a finite number of alternative distributions is considered representing relatively diverse circumstances. These distributions constitute the so-called corners. The polyefficiency of an estimate S_n is defined as the infimum of the efficiencies at the corners. Interest is restricted to estimates with high polyefficiency. An estimate that achieves the supremum of the polyefficiencies is called efficient-robust or polyefficient optimal with respect to the corners. Tukey usually considers three corners: the normal $N(0, 1)$ distribution; the slash distribution obtained by dividing a standard normal random variable with an independent uniform $U(0, 1)$ random variable; the one-wild distribution obtained sampling 95% $N(0, 1)$ random variables and 5% $N(0, 100)$ random variables. Estimates with high polyefficiency have been constructed, under different circumstances, in unpublished Princeton theses by R. Guarino and G. S. Easton among others.

Major criticisms of polyefficiency are the arbitrariness in the choice of the corners, and that nothing is known about interpolation among these corners (Tukey, 1987, p. 5). The idea behind this notion of robustness is that if an estimate performs well at each of the corners chosen to represent the extreme kinds of data it should also perform well at distributions that lie in between them, in a sense not yet specified.

The corners used to compute the polyefficiency of an estimate could also be seen as possible distributions of the n dimensional sample. Adopting this point of view it is shown that if an estimate S_n has high polyefficiency it will also have at least equally high performance over all mixtures of the corners, thus relaxing the second criticism. Moreover, by enlarging the set of possible corners to be used in a given situation the first criticism is also relaxed.

The following elementary lemma is used; it can be proved by induction.

LEMMA. *Let* a_i, x_i, y_i $(i=1,\ldots,k)$ *be all positive. Then*

$$\frac{a_1x_1+\ldots+a_kx_k}{a_1y_1+\ldots+a_ky_k} \geqslant \min_{1\leqslant i\leqslant k} (x_i/y_i).$$

Assume now that the n dimensional random vector X is observed with density

$$f_{\theta,a}(x) = a_1 g_{\theta,1}(x)+\ldots+a_k g_{\theta,k}(x), \quad a=(a_1,\ldots,a_k), \quad a_i>0,$$

$g_{\theta,i}(x)$ is a density, for $i=1,\ldots,k$. Let $S=S(X)$ be an estimate of θ and $R_a(\theta, S)$ be the risk function of S with respect to $f_{\theta,a}$. If in the Lemma we identify x_i with the risk $R_{e_i}(\theta, T)$ and y_i with the risk $R_{e_i}(\theta, S)$, where $(e_i)_j=1$ if $j=i$ and 0 otherwise, we immediately have the following.

PROPOSITION. *For each* a,

$$\inf_{T\in C} \{R_a(\theta, T)\}/R_a(\theta, S) \geqslant \min_{1\leqslant i\leqslant k} [\inf_{T\in C} \{R_{e_i}(\theta, T)\}/R_{e_i}(\theta, S)].$$

The proposition can be rephrased: high polyefficiency implies at least as high an efficiency for any element of the smallest convex set containing the corners. An example can be found in my unpublished Montreal technical report, showing that for a family of distributions with an infinite number of extreme points, one cannot necessarily find a lower bound on the efficiency of an estimate based on its polyefficiency at a finite number of corners.

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