Lecture 8. Quadrisecants of knots

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We shall consider knots as embeddings of a circle in three-space.

Definition 1.1

We define an *n-secant* of the knot K as an oriented line intersecting K at exactly n points. By an *n-secant* we mean an ordered n-tuple of points on the knot K (no two of them belong to a straight segment of the knot K), which lie on an n-secant line in the given order.

We shall call 2-secant simply *secants*, we shall call 3-secants *trisecants*, and we shall call 4-secants *quadrisecants*.
The set of n-secants is $S_n = K^n \setminus \tilde{\Delta}$, where $\tilde{\Delta}$ denotes the set of n-tuples where some two of n points form a segment belonging completely to K.
Note that the set of secants $S = S_2$ is homeomorphic to the ring $S^1 \times (0,1)$.

The types of trisecants

Denote the set of trisecants $\mathcal{T} \subset K^3 \setminus \tilde{\Delta}$. Each trisecant *abc* may have two cyclic orders depending on the orientation of the knot.We denote them by the smallest element in the lexicographic order: *abc* or *acb* and call them the *direct* and the *reversed* order respectively. The two types of quadrisecants are given in Fig. 1. Changing the orientation of the knot to the opposite one changes the trisecant type.

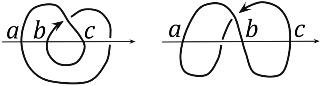

Figure 1: The *reversed* (left) and the *direct* (right) trisecants

Denote the set of direct trisecants by $\mathcal{T}^d$ and denote the set of reversed trisecants by $\mathcal{T}^r$. It is clear that $\mathcal{T}^d \cap \mathcal{T}^r = \emptyset$. Changing the orientation of the knot or of the secant line permutes the sets $\mathcal{T}^d$ and $\mathcal{T}^r$.

Types of quadrisecants

Points of the quadrisecant *abcd* appear on the knot K in some cyclic order. There are 3 different orders of quadrisecant points on the knot, if we don't pay attention to the orientation of the knot K. Each order is given by its lexicographically minima representative (type) *abcd*, *abdc*, or *acbd*.

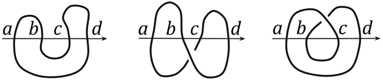

Figure 2: Simple, reversed, and alternating quadrisecants

Definition 1.2

Quadrisecants of types *acbd*, *abcd*, and *abdc* are called *alternating*, *simple*, and *reversed quadrisecants*, respectively.

When considering quadrisecants *abcd* we shall, as usual, orient the knot K in such a way that $b \in \gamma_{ad}$. Then the cyclic order of points on K will be *abcd*, *abcd*, or *acbd* depending on the type.

Essential secants

Definition 1.3

Let α, β, and γ be three simple curves with ends *a* and *b* forming a linked Θ-graph, see Fig. 3. Let$X = \mathbb{R}^3 \setminus (\alpha \cup \gamma)$, and δ be a curve parallel to $\alpha \cup \beta$ in *X*. (By *parallel* we mean that $\alpha \cup \beta$ and δ cobound an embedded annulus in *X*.) We assume that δ is homologically trivial in *X*, i.e., the linking index between δ and $\alpha \cup \gamma$ equals zero. Let $h = h(\alpha, \beta, \gamma) \in \pi_1(X)$ be the (free) homotopy class of the curve δ. We call the triple (α, β, γ) *inessential* if the class *h* is trivial. Otherwise we call the triple (α, β, γ) *essential*.

In other words, the oriented triple (α, β, γ) is *inessential*, if there exists a disc *D* with boundary $\alpha \cup \beta$ without interior intersections with the knot $\alpha \cup \gamma$ (self-intersections of the disc *D* and its interior intersections with β are allowed).

Essential secants

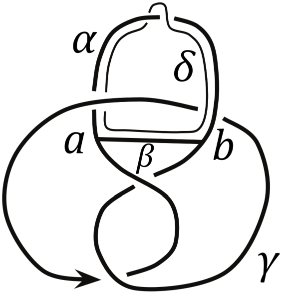

Figure 3: In the knotted Θ-graph $\alpha \cup \beta \cup \gamma$ the triple (α, β, γ) is essential. To see this, let us consider the curve δ parallel to $\alpha \cup \beta$ and having zero intersection index with $\alpha \cup \gamma$ and note that it is homotopically non-trivial in the complement $\mathbb{R}^3 \setminus (\alpha \cup \gamma)$. In the figure β is the segment $\overline{ab}$; hence we can say that the arc $\alpha = \gamma_{ab}$ of the knot $\alpha \cup \gamma$ is essential.

Essential secants

Let $a, b \in K$. Denote the knot arc from the point a to the point b (according to the knot orientation) by γ_{ab}, and denote its length by ℓ_{ab}. The secant from a to b will be denoted by $\overline{ab}$.

Definition 1.4

1. Assume the knot K is non-trivial, $a, b \in K \quad \ell = \overline{ab}$. We say that the arc γ_{ab} is *essential*, if for each $\epsilon > 0$ there exists an ϵ- perturbation ℓ' of the segment ℓ (with the same ends), such that $K \cup \ell'$ form a Θ-graph, for which the triple $(\gamma_{ab}, \ell', \gamma_{ba})$ is essential.
2. The secant ab of the knot K is *essential* if both arcs γ_{ab} and γ_{ba} are essential. Otherwise the arc is *inessential*. Denote the *set of essential secants* by $ES \subset S$.
3. We say that the n-secant $a_1 a_2 \dots a_n$ is *essential*, if the secant $a_i a_{i+1}$ is essential for each i such that on one of the arcs $\gamma_{a_i a_{i+1}}$ or $\gamma_{a_{i+1} a_i}$ there are no other points a_j.

Existence of quadrisecants

Theorem (E. Pannwitz, 1933)

In each non-trivial generic polygonal knot there exists at least $2u^2$ quadrisecants where u is the unknotting number.

Theorem (G. Kuperberg, 1994)

Each non-trivial smooth knot has an essential quadrisecant.

Theorem (E. Denne, 2004)

Each non-trivial smooth knot has at least one alternating quadrisecant.

Theorem (A. Cruz-Cota and T. Ramirez-Rosas, 2015)

Let K be a generic polygonal knot with n edges. Then K has no more than $\frac{n}{12}(n-3)(n-4)(n-5)$ quadrisecants.

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Existence of quadrisecants

Theorem 2.1 (E. Denne, 2004)

Each nontrivial knot has at least one alternating quadrisecant.

Remark 2.2

Note that the quadrisecant appears when one has some trisecants with common points. The quadrisecant *abcd* contains the three trisecants: (1)*abc*, (2)*abd*, (3)*acd*, (4)*bcd*. Pannwitz [Pann] proved the existence of a quadrisecant by considering pairs of trisecants (1)*abc* and (3)*acd*. Kuperberg [Kup] proved that quadrisecants exists by using pairs of trisecants (2)*abd* and (3)*acd*. Schmitz [Schm] was proving the existence of alternating quadrisecants by considering families (1)*abc* and (2)*abd*, however in his proof some quadrisecants can degenerate into a trisecant. Later we shall use the approach of Schmitz.

Existence of trisecants

Lemma 2.3 ([Pann])

Every point of a non-trivial knot K is an initial point of some trisecant.

Proof.

Assume $a \in K$ does not serve as the first point of any trisecant. Then the union of segments $\overline{ab}$, $b \in K$, is a disc with boundary K. If two chords ab and ac intersect at a point distinct from a, then one of them is a subset of the other. Then they form a trisecant (abc or acb), contradiction. Thus, the disc is embedded, hence, the knot K is trivial. □

Existence of quadrisecants

Let us consider the projection $\pi_{12} : K^3 \to K^2$, $\pi_{12}(xyz) = xy$, and let $T = \pi_{12}(\mathcal{T}) \subset S$ be the image of the set of trisecants $\mathcal{T}$. Let us introduce the notation $T^d := \pi_{12}(\mathcal{T}^d)$ and $T^r := \pi_{12}(\mathcal{T}^r)$.

Lemma 2.4

Let $ab \in T^d \cap T^r$ in S. This meanst that there exist points c, d such that $abc \in \mathcal{T}^r$ $abd \in \mathcal{T}^d$. Then either $abcd$ or $abdc$ is an alternating quadrisecant.

Thus, in order to prove the existence of an alternating quadrisecant, it suffices to check that $T^d \cap T^r \neq \emptyset$ in S. To this end we shall first consider generic polygonal knots. We shall prove the existence of alternating quadrisecant by passing to the limit, and then extend the result to smooth knots.

A generic polygonal knot

Definition 2.5

A polygonal knot K in $\mathbb{R}^3$ is *generic*, if the following conditions hold:

- No four vertices of the knot K are coplanar, no three vertices are collinear.
- For any three pairwise skew edges of the knot K there exist no other edge K lying in the quadric generated by these edges.
- There knot has no n-secants for $n \geq 5$.

Proposition 2.6

The set of generic n-vertex polygonal knots is open and dense in $\mathbb{R}^{3n}$.

Trisecants for a polygonal knot: adjacent edges

Let e_i and e_{i+1} be adjacenet edges of a knot. If some edge e_j intersects some part of the plane generated by e_i and e_{i+1}, then we get a one-parameter family of quadrisecants. This family is homeomorphic to $[0,1]$ or to $[0,1)$ depending on the domain intersected by e_j (see Fig. 6). A quadrisecant appears when the fourth edge intersects one of the trisecants. By genericiry, two non-adjacent edges can not be coplanar, hence, in this case there is no more than one quadrisecant.

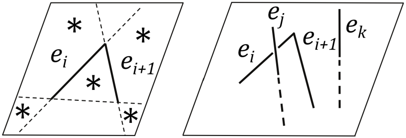

Figure 4: The plane is generated by edges e_i, e_{i+1}. A trisecant intersecting e_i, e_{i+1} exists only if the third edge intersects one of the domains marked by (*). The family of trisecants intersecting e_i, e_{i+1}, e_j is homeomorphic to $[0,1]$; the family of trisecants intersecting e_i, e_{i+1}, e_k is homeomorphic to $[0,1)$.

Trisecants for a polygonal knot: skew edges

A triple of pairwise skew lines l_1, l_2, l_3 generates a unic conic surface, the ruled surface H (see Fig. 5). It is either hyperbolic paraboloid (if three lines are parallel to one plane) or one-sheeted hyperboloid. On the surface H there are two families of straight line generators. The lines l_1, l_2, l_3 belong to the same family of generators, and any line intersecting them belongs to the same family. Hence, the fourth linel_4 intersecting H gives rise to one or two lines intersecting l_1, l_2, l_3 and l_4.

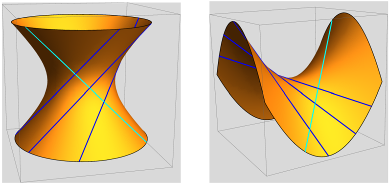

Figure 5: One-sheeted hyperboloid and hyperbolic paraboloid

The set of trisecants

Proposition 2.7

Let K be a non-trivial generic polygonal knot. Then the closure of the set of trisecants $\overline{\mathcal{T}}$ is a compact one-dimensional manifold with boundary piecewise-linearly embedded in K^3 so that $\mathcal{T} \subset K^3 \setminus \tilde{\Delta}$ $\partial\mathcal{T} \subset \Delta$. Moreover, each component $\mathcal{T}$ is either a simple closed curve or a simple open arc.

Proposition 2.8

Let K be a non-trivial generic polygonal knot. Then the projection π_{ij} $(1 \leq i < j \leq 3)$ gives rise to a piecewise smooth immmersion $\mathcal{T}$ to the set of secants S in such a way that $T = \pi_{ij}(\mathcal{T})$ has only double self-intersections.

Consider the closure of the set of secants S: $\overline{S} = (K^2 \setminus \tilde{\Delta}) \cup \tilde{\Delta}_+ \cup \tilde{\Delta}_-$, $\tilde{\Delta}_- = \{(a, b) \in K^2 \mid \gamma_{ab} = \overline{ab}\}$ and $\tilde{\Delta}_+ = \{(a, b) \in K^2 \mid \gamma_{ba} = \overline{ba}\}$.

The boundary of the set of trisecants

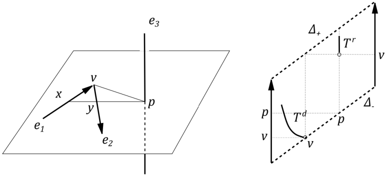

Figure 6: On the left, the interval of adjacent trisecants ends with a degenerate trisecant *vvp* or *pvv*. The corresponding trisecant intervals for S are on the right. The intervals in T^d and T^r correspond to trisecants with orders $e_1 e_2 e_3$ and $e_3 e_2 e_1$ respectively.

Lemma 2.9

Let K be a non-trivial generic polygonal knot. Then $\overline{T}^r \cap \tilde{\Delta}_- = \emptyset$ and $\overline{T}^d \cap \tilde{\Delta}_+ = \emptyset$.

The set of significant trisecants

To find a significant quadrisecant, let us consider trisecants abc, for which the segment bc is essential. Hence, $\mathcal{ET} = \pi_{23}^{-1}(ES) \cap \mathcal{T}$. Set $\mathcal{ET}^d = \mathcal{ET} \cap \mathcal{T}^d$ and $\mathcal{ET}^r = \mathcal{ET} \cap \mathcal{T}^r$.

Definition 2.10

Let $ET = \pi_{12}(\mathcal{ET})$ be the projection of the set of essential trisecants to the set of secants S; similarly, let us define $ET^d := \pi_{12}(\mathcal{ET}^d)$ $ET^r := \pi_{12}(\mathcal{ET}^r)$.

As the set T, the set ET is an immersion into S of a one-dimensional manifold and it can have only transverse double points.

Lemma 2.11

Let $ab \in ET^d \cap ET^r$ in S. Consequently, there exist c and d such that $abc \in \mathcal{ET}^r$ and $abd \in \mathcal{ET}^d$. Then $abcd$ or $abdc$ is an essential alternating quadrisecant.

Definition 2.12

A closed simple curve $\alpha : [0, 1] \to S$ *goes around S* one time, if its homotopy class is equal to 1 in $\pi_1(S) \equiv \mathbb{Z}$.

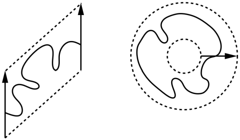

Figure 7: A curve winding 1 time around the set S

Lemma 2.13 (Pannwitz)

Let K be a non-trivial generic polygonal knot. Any curve which goes around S one time is a non-trivial generic polygonal knot. Each curve which goes around S once, intersects the set ET of essential trisecants.

Proof of Lemma 2.13

Let us first consider the case when the curve α does not contain trisecants. Let $\alpha = (x(s), y(s))$ be a parametrisation of the curve. Let us construct the rays $\overrightarrow{xy} \setminus \overline{xy}$, see Fig.. 23.

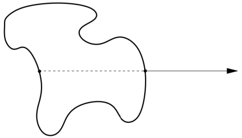

Figure 8: A ray $\overrightarrow{xy} \setminus \overline{xy}$.

The union of rays together with the point ∞ forms a disc D, whose boundary is the knot K. Then, by Dehn's lemma there exists an embedded disc with boundary K. Hence, the knot K is trivial. Contradiction completes the proof.
Now, let us consider the general case. One may assume that the intersection $\alpha \cap T$ is finite.

Proof of Lemma 2.13

For each non-essential trisecant on α by using a surgery one can remove an intersection between the spanning disc D and the knot K, as shown. 9.

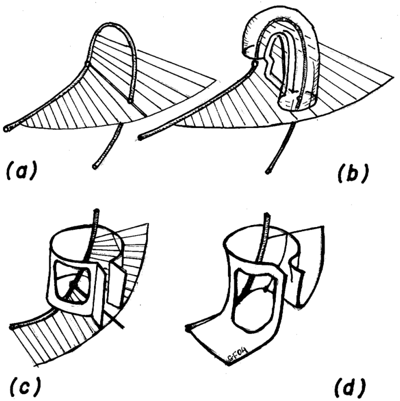

Figure 9: Surgery for a non-essential trisecant

A homological lemma

Proposition 2.14

Let A and B be closed subsets in the annulus S such that A does not intersect Δ_+ and B does not intersect Δ_-. If $A \cap B = \emptyset$, then there exists a curve passing once around S and not intersecting $A \cup B$.

Theorem 2.15

Each non-trivial generic polygonal knot in $\mathbb{R}^3$ has an essential quadrisecant.

Proof.

Assume that $ET^s \cap ET^d = \emptyset$ in S. According to Proposition 2.14 there exists a path passing once around S and not intersecting $ET = ET^s \cup ET^d$. This contradicts Lemma 2.13. Consequently, $ET^s \cap ET^d \neq \emptyset$. Then, by Lemma 2.11 there exists at least one alternating quadrisecant. □

Corollary 2.16

Ever smooth non-trivial knot in $\mathbb{R}^3$ has an essential alternating quadrisecant.

The total curvature of the curve

Definition 2.17

The *total curvature* of a closed broken line is equal to the difference between the sum of angles between all adjacent edges and πn, where n is the number of edges of the broken line.
The *total curvature* of a smooth curve is equal to the curvilinear first-type integral of the curvature.

Theorem 2.18

Any non-trivial knot in $\mathbb{R}^3$ *has total curvature strictly greater than* 4π*.*

Proof.

The knot K has an alternating quadrisecant. An alternating quadrisecant is a quadrilateral inscribed in K having total curvature 4π. Note that the addition of vertices to the broken line inscribed in K does not decrease the total curvature and increases the curvature if the added vertex belongs to one of the planes spanned by other vertices of the broken line. Hence, the total curvature of a non-trivial knot is strictly greater than 4π. □

The 2-hull

Definition 2.19

Let K be a closed curve in $\mathbb{R}^3$. Denote by *n-hull* $h_n(K)$ of the knot K the set of points $p \in \mathbb{R}^3$ such that K intersects each plane P passing through p at least $2n$ times.

Theorem 2.20

Any non-trivial knot has a non-empty 2-hull.

Proof.

We know that the knot K has an alternating essential quadrisecant $abcd$. Then each point t of the segment $\overline{bc}$ lies in the 2-hull of the knot K. □

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Later on, we shall consider long knots, i.e., embeddings f of the segment $\mathbb{I} = [0,1]$ into the cube $\mathbb{I}^3$ with fixed ends and tangent vectors at end points. Denote the space of embeddings by $\mathrm{Emb}(\mathbb{I}, \mathbb{I}^3)$.
For each long knot f consider the submanifold $\mathbf{Co}_i(f)$ and the subset $\mathrm{Int}(\Delta^3)$ consisting of triples of points $t_1 < t_2 < t_3$ such that $f(t_1)$, $f(t_2)$ and $f(t_3)$ are collinear and $f(t_i)$ lies on the straight line between the other points, see Fig. 10.

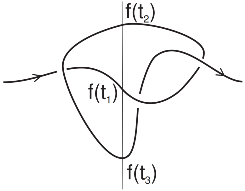

Figure 10: Collinear points on the knot giving rise to a point of $Co_1(f)$.

The self-linking index

Proposition 3.1

For each generic long knot parametrised by $f \in \mathrm{Emb}(\mathbb{I}, \mathbb{I}^3)$ *the closure* $\mathbf{Co}_i[f]$ *of the set* $\mathbf{Co}_i(f)$ *is a one-dimensional submanifold in* Δ^3*. Herewith the boundary of the manifold* $\mathbf{Co}_1[f]$ *belongs to the faces* $\Delta^3_{(1=2)} = \{t_1 = t_2\}$ *and* $\Delta^3_{(3=4)} = \{t_3 = 1\}$*, and the boundary of the manifolds* $\mathbf{Co}_3[f]$ *belongs to the faces* $\Delta^3_{(0=1)} = \{t_1 = 0\}$ *and* $\Delta^3_{(2=3)} = \{t_2 = t_3\}$*.*

Definition 3.2

Let us define the closure of the manifold with boundary $\mathbf{Co}_1[f]$ as an arbitrary piecewise-smooth 1-manifold $\overline{\mathbf{Co}_1[f]}$ such that $\overline{\mathbf{Co}_1[f]} \cap \mathrm{Int}(\Delta^3) = \mathbf{Co}_1(f)$ and $\overline{\mathbf{Co}_1[f]} \cap \partial(\Delta^3) \subset \Delta^3_{(1=2)} \cup \Delta^3_{(3=4)}$. The closure $\overline{\mathbf{Co}_3[f]}$ is defined analogously.

The self-linking index

Definition 3.3

For a long knot $\mathcal{K}$ parametrised by $f \in \mathrm{Emb}(\mathbb{I}, \mathbb{I}^3)$ we define the *self-linking invariant* as $\nu_2(\mathcal{K}) = lk(\overline{\mathbf{Co}_1[f]}, \overline{\mathbf{Co}_3[f]}) \in \mathbb{Z}$.

Theorem 3.4

The self-linking index ν_2 is an invariant of long knots.

Proof.

Since the sets $\Delta^3_{(1=2)} \cup \Delta^3_{(3=4)}$ and $\Delta^3_{(0=1)} \cup \Delta^3_{(2=3)}$ are contractible and have no common interior points, then $\nu_2(\mathcal{K})$ does not depend on the choice of closures $\overline{\mathbf{Co}_1[f]}$ and $\overline{\mathbf{Co}_3[f]}$.

A generic homotopy between two parametrisations f and g of the knot $\mathcal{K}$ gives rise to an oriented cobordism between manifolds $\mathbf{Co}_i[f]$ and $\mathbf{Co}_i[g]$. Hence, $lk(\overline{\mathbf{Co}_1[f]}, \overline{\mathbf{Co}_3[f]}) = lk(\overline{\mathbf{Co}_1[g]}, \overline{\mathbf{Co}_3[g]})$. □

An example

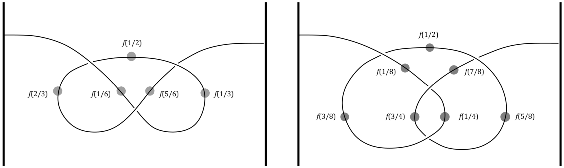

Figure 11: The trefoil and the figure eight. The (x_1, x_2)-projection. We mark points with $x_3 = 0$. On the arcs between marked points the coordinate x_3 has exactly one local maximum or minimum.

Example. Degenerate trisecants

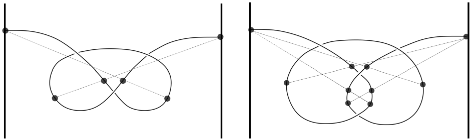

Figure 12: Boundary trisecants

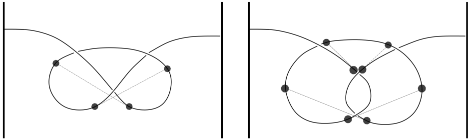

Figure 13: Tangent trisecants

Example. Quadrisecants

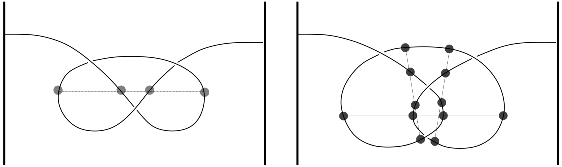

Figure 14: Quadrisecants of knots

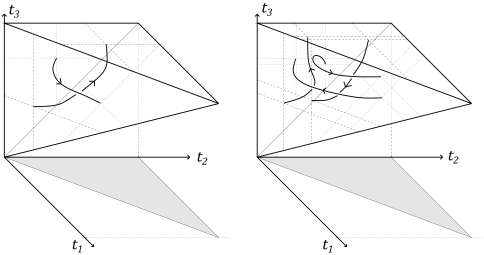

Figure 15: The sets of knot trisecants

In order to find the linking index ν_2, it suffices to count the intersection points between planar projections of manifolds $\mathbf{Co}_i[f]$. The following lemma turns out to be useful.

Lemma 3.5

Let $\rho\colon \Delta^3 \to \Delta^2$ be the orthogonal projection along t_1, and let f be the parametrisation of the knot $\mathcal{K}$. The intersection of projections $\mathbf{Co}_1[f]$ and $\mathbf{Co}_3[f]$ with respect to ρ corresponds to quadrisecants of the knot $\mathcal{K}$.

Proof.

The intersection of $\mathbf{Co}_3[f]$ and $\mathbf{Co}_1[f]$ corresponds to a triple of points $f(t_1^*), f(t_2^*), f(t_3^*)$ lying on the line L^* and to the triple $f(t_1'), f(t_2'), f(t_3')$ on the line L', where $t_2^* = t_2'$ and $t_3^* = t_3'$. Then $L^* = L'$, hence, the points $f(t_1'), f(t_1^*), f(t_2')$ and $f(t_3')$ are collinear. □

Theorem 3.6

The value of the invariant ν_2 for the trefoil is $+1$, and for the figure eight knot it is $\nu_2 = -1 - 1 + 1 = -1$.

The space $C_4(\mathbb{I}^3)$, consisting of collinear configurations of four points has twelve components. If (x_1, x_2, x_3, x_4) is a quadrisecant, let us orient the line from x_1 to x_2. The choice of orientation gives rise to a permutation of $\{1,2,3,4\}$: $\sigma(i) = j$ where i-th point on the line x_j. According to the choice of orientation $\sigma(2) > \sigma(1)$; this gives twelve permutations.

Definition 3.7

Let $\mathcal{C}_4$ denotes the subset of collinear configurations in $C_4(\mathbb{R}^3)$ corresponding to the 4-cycle (3142). Let the long knot $\mathcal{K} \subseteq \mathbb{I}^3$ be parametrised by the map $f\colon \mathbb{I} \to \mathbb{I}^3$. With each quadruple $x = (f(t_1), f(t_2), f(t_3), f(t_4)) \in \mathcal{C}_4$ we associate the sign ϵ_x equal to the sign of the determinant of the matrix:

$$\begin{bmatrix} |f(t_3) - f(t_2)| \cdot \det[v, f'(t_1), f'(t_3)] & |f(t_3) - f(t_1)| \cdot \det[v, f'(t_2), f'(t_3)] \\ |f(t_4) - f(t_2)| \cdot \det[v, f'(t_4), f'(t_1)] & |f(t_4) - f(t_1)| \cdot \det[v, f'(t_2), f'(t_4)] \end{bmatrix}$$

$$v = f(t_2) - f(t_1).$$

Using the sign notation, we may reformulate the definition of self-linking index as follows.

Proposition 3.8

Let $\mathcal{K} = im(f)$ be a generic long knot in $\mathbb{I}^3$. Then

$$\nu_2(\mathcal{K}) = \sum_{x \in C_4(\mathcal{K}) \cap \mathcal{C}_4} \epsilon_x$$

Proof.

The linking index $\nu_2(\mathcal{K}) = lk(\overline{\mathbf{Co}_1[f]}, \overline{\mathbf{Co}_3[f]})$ is equal to the sum of signs of projections $\rho(\mathbf{Co}_1[f])$ and $\rho(\mathbf{Co}_3[f])$, where $\rho(\mathbf{Co}_3[f])$ forms an overcrossing. By Lemma 3.5, these crossings bijectively correspond to quadrisecants of the type (3142). One readily checks that the crossing sign is ϵ_x. □

Theorem 3.9

The self-linking index ν_2 is a Vassiliev invariant of order 2.

Proof.

Let us show that the third derivative of this invariant is zero, i.e., for each knot $\mathcal{K}$ and a set of two crossing (changes) c_1, c_2, c_3 one has:

$$\sum_{\sigma\subset[3]} (-1)^{|\sigma|}\nu_2(\mathcal{K}_\sigma) = 0, \tag{1}$$

where $[3] = \{1, 2, 3\}$ $\mathcal{K}_\sigma$ denotes the knot obtained from $\mathcal{K}$ by changing the crossings c_i for $i \in \sigma$. One can assume that the crossing change is performed inside the ball $B_i, 1 \leq i \leq 3$, and that there is no line passing through these three balls. Then no quadrisecant l on one of the eight knots $\mathcal{K}_\sigma$ intersects all the three balls. Hence, the whole alternating sum of quadrisecants can be split into:

a) Sums corresponding to quadrisecants passing through the balls B_1, B_2

b) Sums corresponding to quadrisecants passing through the balls B_1, B_3

c) Sums corresponding to quadrisecants passing through the balls B_2, B_3

d) Sums corresponding to quadrisecants passing through no more than one of the balls B_1, B_2, B_3.

It is easy to see that each of these (alternating) sums is zero, which completes the proof. □

Corollary 3.10

The self-linking index $\nu_2(\mathcal{K})$ coincides with the coefficient $c_2(\mathcal{K})$ of the second degree in the Alexander-Conway polynomial.

Research Problems

1. One research problem about the self-linking number: how to make this invariant "non-commutative".
   This invariant is a certain COUNT of points in a certain configuration space. Is is possible to generalise this count (sum) to a product (as generators of a certain group)?
2. Construct the theory of quadrisecants for rectangular knots (i.e. polygonal knots with all edges parallel to one of the axes O_x, O_y, O_z).

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