

Alternating Quadriseccants of Knots

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ABSTRACT. It is known [Pann, Kup] that for every knotted curve in space, there is a line intersecting it in four places, a quadriseccant. Comparing the order of the four points along the line and the knot we can distinguish three types of quadriseccants; the alternating ones have the most relevance for the geometry of a knot. I show that every (nontrivial, tame) knot in \mathbb{R}^3 has an alternating quadriseccant. This result has applications to the total curvature, second hull and ropelength of knots.

1. Introduction

Throughout this paper a *knotted curve* will denote an oriented nontrivial tame knot in \mathbb{R}^3 . (By knot, we mean a homeomorphic image of S^1 in \mathbb{R}^3 , modulo reparametrizations. By tame, we mean the knot is ambient isotopic to a polygonal knot.) A secant line is a straight line which intersects the knot in at least two distinct places. Triseccant, quadriseccant and quintiseccant lines are straight lines which intersect a knot in at least three, four and five distinct places, respectively. It is clear that any closed curve has a 2-parameter family secants. A simple dimension count shows there is a 1-parameter family of triseccants and that quadriseccants are discrete (0-parameter family). Quintiseccants are a -1 -parameter family; meaning they exist only for a codimension 1 set of knots (Proposition 6).

Thus we expect knotted curves to have quadriseccants. Indeed, in 1933, E. Panwitz [Pann] first showed that every nontrivial generic polygonal knot in \mathbb{R}^3 has at least $2u^2$ quadriseccants, where u is the unknotting number of a knot. In 1980, H.R. Morton and D.M.Q. Mond [MM] independently proved every nontrivial generic knot has a quadriseccant and conjectured that a generic knot with crossing number n has at least $\binom{n}{2}$ quadriseccants. It was not until 1994 that G. Kuperberg [Kup] managed to extend the result and show that all (nontrivial tame) knots in \mathbb{R}^3 have a quadriseccant.

Quadriseccants come in three basic types. These are distinguished by comparing the orders of the four points along the knot and along the quadriseccant line. There are three relative orderings of an oriented quadriseccant and an unoriented knot. If

1991 *Mathematics Subject Classification*. Primary 57M25.

Key words and phrases. Knots, links, quadriseccants, total curvature.

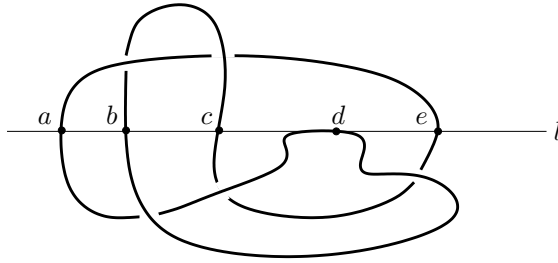
$abcd$ is the ordering along the quadriseccant, then the possible orderings along the knot are $abcd$, $abdc$ and $acbd$. I will show that every nontrivial tame knot has at least one quadriseccant with knot ordering $acbd$. This type of quadriseccant is called an alternating quadriseccant. This result refines the previous work about quadriseccants, giving greater geometric insight into knots and also providing several interesting applications. This paper contains the main results of my PhD thesis [Den] and I am grateful to my advisor J.M Sullivan for suggesting this problem to me. Recently, R. Budney *et al.* [BCSS] have shown that the finite type 2 Vassiliev invariant can be computed by counting alternating quadriseccants with appropriate multiplicity. While this result implies the existence of alternating quadriseccants for many knots, our Main Theorem shows existence for *all* (nontrivial tame) knots. On the other hand, our results provide no way to count alternating quadriseccants.

The existence of alternating quadriseccants provides new proofs to two previously known results about the geometry of knotted curves. For smooth closed curves, the total curvature can be thought of as the total angle through which the unit tangent vector turns (or the length of the tangent indicatrix). Equivalently, the total curvature of an arbitrary curve is the supremal total curvature of inscribed polygons. Around 1949 I. Fáry [Far] and J.W. Milnor [Mil] proved independently that the total curvature of a nontrivial tame knot in \mathbb{R}^3 is at least 4π . Note that the total curvature of an inscribed polygon is less than or equal to the total curvature of the curve it is inscribed in. The total curvature of an alternating quadriseccant is 4π . Thus if a knotted curve has an alternating quadriseccant, the total curvature will be greater than or equal to 4π , giving the Fáry-Milnor theorem another proof. In 1998, the Fáry-Milnor theorem was extended to knotted curves in Hadamard¹ manifolds by C. Schmitz [Schm] and S.B. Alexander and R.L. Bishop [AB].

The second new proof is for the theorem about the second hull of a knotted curve in \mathbb{R}^3 . Intuitively, the second hull is the part of space that the knotted curve winds around twice. In 2000, J. Cantarella *et al* [CKKS] proved that the second hull of a knotted curve in \mathbb{R}^3 is nonempty. This paper conjectured the existence of alternating quadriseccants for knotted curves in \mathbb{R}^3 as another way of proving that the second hull is nonempty. This is because the mid-segment bc of alternating quadriseccant $abcd$ is in the second hull.

Finally, quadriseccants may be applied to the ropelength problem. The ropelength problem asks to minimize the length of a knotted curve subject to maintaining an embedded tube of fixed diameter around the tube; this is a mathematical model of tying the knot tight in a rope of fixed thickness. The ropelength of a knot is the quotient of its length and its thickness. The thickness is the diameter of the largest embedded normal tube around the knot. The exact value of the minimum ropelength needed to tie any nontrivial knot is not currently known. In joint work with J.M. Sullivan and Y. Diao [DDS], essential alternating quadriseccants are used to improve the known lower bounds of ropelength from 12 to 15.66. Several independent simulations (see for instance [Pie, Sul]) have found a trefoil knot with

¹A Hadamard manifold is a complete simply-connected Riemannian manifold with non-positive sectional curvature.

FIGURE 1. A quintisecant line l and quintisecant $abcde$.

ropelength less than 16.374. This is presumably close to the minimizer, so the new bounds are quite sharp.

The proof of the main theorem will extend ideas taken from all of the previous papers, but in particular from [Kup, Pann, Schm]. At its core, the proof assumes that alternating quadriseccants do not exist, then uses this to create a contradiction to knottedness. A quadriseccant includes a number of triseccants, thus a large part of the proof will be dedicated to a detailed understanding the structure of the set of triseccants of a knot, both in K^3 and when projected to the set of secants $S = K^2 \setminus \Delta$.

Section 2 introduces terminology and explores the relationship between quadriseccants and triseccants. Lemma 1 shows any knotted curve has at least a 1-parameter family of triseccants and Lemma 2 shows that alternating quadriseccants occur when triseccants of same and different orders share common points. A detailed understanding of the structure of the set of triseccants is required. In particular, this set has very nice properties when the knot is a generic polygonal knot. In Section 3, the definition of a generic polygonal knot is given and the conditions for genericity are shown to be generic. In Section 4 the structure of the set of triseccants of generic polygonal knots is examined in detail, both in K^3 and $K^2 \setminus \Delta$. In $K^2 \setminus \Delta$, it is shown to be a piecewise immersed 1-manifold that intersects itself in double points. In the end we wish to prove the existence of alternating quadriseccants for any nontrivial tame knot. Any tame knot is the limit of a sequence of generic polygonal knots. Thus we first prove the existence of an essential alternating quadriseccant for nontrivial generic polygonal knots. The notion of essential is required so that quadriseccants do not degenerate to triseccants in the limit. Section 5 defines the notion of essential for secants, triseccants and quadriseccants and describes the structure of the set of essential triseccants. Section 6 gives the proof that any nontrivial generic polygonal knotted curve has an essential alternating quadriseccant. In Section 7, the Main Theorem (Theorem 27) is proved: every nontrivial tame knot in \mathbb{R}^3 has an alternating quadriseccant. This is strengthened in Corollary 29: every nontrivial knot of finite total curvature in \mathbb{R}^3 has an *essential* alternating quadriseccant. Finally, Section 8 provides applications of essential alternating quadriseccants to the total curvature, second hull and ropelength of a knotted curves.

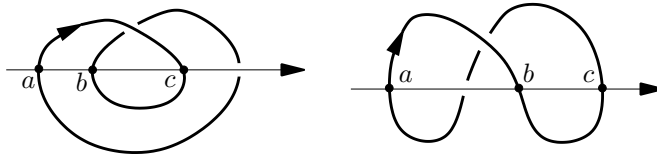


FIGURE 2. The points of each trisecant have order abc along the line. The left has *different* order acb along the knot and the right has *same* order abc along the knot.

2. Secants, trisecants and quadriseccants

Recall that a *knotted curve* is an oriented nontrivial tame knot in \mathbb{R}^3 .

DEFINITION. An n -*secant line* for a knotted curve K is an oriented line whose intersection with K has at least n components.

DEFINITION. An n -*secant* is an ordered n -tuple of points in K (no two of which lie in a common straight subarc of K) which lie in order on an n -secant line. A 2-secant will be called a secant, a 3-secant will be called a trisecant, a 4-secant will be called a quadriseccant and a 5-secant will be called a quintiseccant.

The line through an n -secant may intersect the knot in more than n places, but this does not affect the definition of an n -secant. Figure 1 shows a knot with quintiseccant line l and quintiseccant $abcde$. For example, the quintiseccant line l also includes quadriseccant $abcd$, trisecant abe , and secant bd .

In K^n , let $\tilde{\Delta}$ denote the set of n -tuples in which some pair of points lie in a common straight subarc of K . Also let Δ denote the big diagonal, the set of n -tuples in which some pair of points are equal. Then $\Delta \subset \tilde{\Delta}$. (The inclusion can be strict as for polygonal knots.) For a closed curve K , any two distinct points determine a straight line. These points are a secant if and only if they do not lie on a common straight subarc of K . Thus the set of secants $S = K^2 \setminus \tilde{\Delta}$ and is topologically an annulus. More generally, any n distinct points are not necessarily collinear, hence the set of n -secants is contained in $K^n \setminus \tilde{\Delta}$.

Consider the set of trisecants. Take a trisecant abc with intersection points in that order along the trisecant line. There are $|S_3| = 6$ possibilities for their ordering along K . Along K , the order is just a cyclic order. Thus there are $|S_3/C_3| = 2$ cyclic orderings of the oriented knot and oriented trisecant line. Picking the lexicographically least element in each coset, we see the order along the knot is abc or acb . These orderings are respectively called *same* and *different*. Figure 2 illustrates the two types of trisecant.

We may make a more general observation about the ordering of points along a knot. Let $abc \in K^3$ and assume that a , b and c are distinct. As K is oriented, moving from a to b to c along the K will either match the orientation of K or not. These are the two cyclic orderings discussed above. Let \mathcal{S} be the set of triples of K^3 where the triples have the *same* ordering along the knot and let \mathcal{D} be the set of triples of K^3 where the ordering of the triples *differs* from their ordering

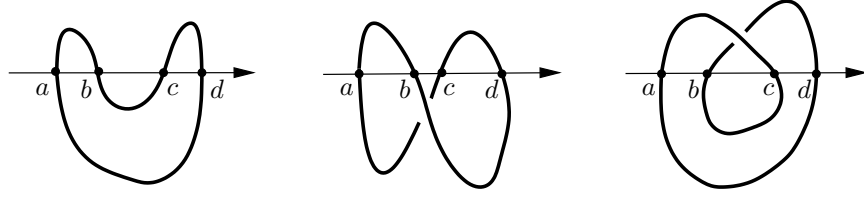


FIGURE 3. From left to right, quadriseccants $abcd$ are simple, flipped and alternating, because the dihedral order of the points along K is $abcd$, $abdc$ and $acbd$, respectively.

along the knot. Observe that \mathcal{S} and \mathcal{D} are the connected components of $K^3 \setminus \tilde{\Delta}$. Then \mathcal{S} and \mathcal{D} are disconnected as perturbing a , b and c a little will not change their ordering. The only way to change the order along the knot from abc to acb is for a , b and c to no longer be distinct. That is, either $a = b$, $b = c$ or $a = c$. This is precisely the big diagonal $\Delta \subset \tilde{\Delta}$.

DEFINITION. Let $\mathcal{T} \subset K^3 \setminus \tilde{\Delta}$ denote the set of all trisecants of a knot K . Let the closure of \mathcal{T} in K^3 be denoted by $\overline{\mathcal{T}}$ and let the boundary of \mathcal{T} be defined as $\partial\mathcal{T} := \overline{\mathcal{T}} \setminus \mathcal{T}$. Let $\mathcal{T}^s = \mathcal{T} \cap \mathcal{S}$ denote the set of trisecants where the ordering along the trisecant is the *same* as the ordering along the knot. Let $\mathcal{T}^d = \mathcal{T} \cap \mathcal{D}$ denote the set of all trisecants where the ordering along the trisecant is *different* to the ordering along the knot. Clearly $\mathcal{T}^s \cap \mathcal{T}^d = \emptyset$. Note that switching the orientation of either the knotted curve or the trisecant line interchanges \mathcal{T}^s and \mathcal{T}^d .

Just as with trisecants, we may compare the ordering of the intersection points of a quadriseccant line with their ordering along the knot. Given intersection points $abcd$ in that order along the quadriseccant line, there are $|S_4|$ possibilities for their ordering along K . Again, the order along K is only a cyclic order and ignoring the orientation of K is just a dihedral order. Thus there are $|S_4/D_4| = 3$ dihedral orderings of an oriented quadriseccant and unoriented knot. Let $abcd$ be the order of intersection points along the quadriseccant line, then the three equivalence classes or types of quadriseccants are represented by $abcd$, $abdc$ and $acbd$, where we have chosen the lexicographically least order as the name for each. Figure 3 illustrates these orderings.

DEFINITION. Quadriseccants of type $acbd$ are called *alternating quadriseccants*. Quadriseccants of types $abcd$ and $abdc$ are called *simple* and *flipped* respectively.

Alternating quadriseccants (see the right-most quadriseccant in Figure 3) are so named as the quadriseccant's points alternate from one end of the quadriseccant line to the other as the ordering along the knot is followed. Note that unlike trisecants, switching the orientation of the quadriseccant line does not change the type of quadriseccant.

We have already noted that each closed curve has a 2-parameter family of secants $S = K^2 \setminus \tilde{\Delta}$. It turns out that knotted curves also have (at least) a 1-parameter family of trisecants as seen in the following lemma originally due to Pannwitz [Pann].

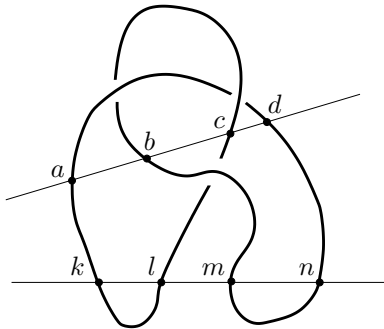


FIGURE 4. A trefoil knot with alternating quadriseccant $abcd$ and simple quadriseccant $klmn$. Other trefoil knots might not have the simple quadriseccant shown. However, the Main Theorem shows that every knotted curve has an alternating quadriseccant.

LEMMA 1. *Each point of a knotted curve K is the first point of at least one triseccant.*

PROOF. Suppose there is a point a of the knot which is not the start point of any triseccant. That is, no points $b, c \in K$ are collinear with a , with a as the start point. The union of all chords ab for $b \in K$ is a disk with boundary K . If two chords ab and ac intersect at a place other than a , then they overlap and one is a subinterval of another. They form a triseccant (abc or acb), contrary to the assumption. Thus the disk is embedded, contradicting the assumption of knottedness. \square

If we view the knotted curve K from a point $a \in K$ as in Lemma 1, then the triseccants occur where we see a crossing of K . Thus we expect that there will be a finite set of triseccants with first point a . In fact, Pannwitz [Pann] proved for generic polygonal knots that there are at least $2u^2$ triseccants from any point $a \in K$ (where u is the unknotting number of the knot). Schmitz [Schm] has shown Lemma 1 also holds for knotted curves in Hadamard manifolds. In Section 4 we will show that the set of triseccants is generically an embedded 1-manifold in K^3 .

What about higher order secants? In Section 3 we will see that knotted curves generically have no n -secants where $n \geq 5$. However, knotted curves do have quadriseccants. Figure 4 shows a trefoil with two quadriseccants. Quadriseccant $abcd$ is an alternating quadriseccant and quadriseccant $klmn$ is a simple quadriseccant. The trefoil knot, for example, does not necessarily have the simple quadriseccant shown. The rest of this work will be dedicated to proving the following theorem:

MAIN THEOREM. *Every knotted curve in \mathbb{R}^3 has at least one alternating quadriseccant.*

Observe that quadriseccants are formed when several triseccants share common points. Quadriseccant $abcd$ includes four triseccants (1) abc , (2) abd , (3) acd , (4) bcd . Pannwitz [Pann] showed quadriseccants exist by looking for pairs of triseccants like (1) abc and (3) acd . Here, the first and third points of triseccant abc are the

same as the first and second points of trisecant acd . Kuperberg [Kup] showed that quadriseccants exist by looking for pairs of trisecants like $(2)abd$ and $(3)acd$. Here, the first and third points of the trisecants are the same. Schmitz [Schm] nearly showed that alternating quadriseccants exist by looking at families $(1)abc$ and $(2)abd$, where the first and second points of the trisecants are the same, but in his proof, some quadriseccants may degenerate to trisecants. I will use Schmitz' approach, as it allows us to use the orderings of coincident trisecants to determine the ordering of the quadriseccant.

Let quadriseccant $abcd$ be an alternating quadriseccant as in Figure 3 (right). As we have seen it includes several trisecants: abc and bcd are trisecants of different ordering, and abd and acd are trisecants of same ordering. Given an alternating quadriseccant, there will be trisecants of same and different ordering which have common points. For example, $abd \in \mathcal{T}^s$ and $abc \in \mathcal{T}^d$ have the first two points in common. In fact the converse is true: if trisecants of same and different orderings have the first two points in common, then there is an alternating quadriseccant.

Define the projection $\pi_{12} : K^3 \rightarrow K^2$ by $\pi_{12}(xyz) = xy$ and let $T = \pi_{12}(\mathcal{T}) \subset S$ be the projected image of trisecants \mathcal{T} . A secant $ab \in S$ is in T if and only if it is the first two points of some trisecant abc . We further set $T^s := \pi_{12}(\mathcal{T}^s)$ and $T^d := \pi_{12}(\mathcal{T}^d)$.

LEMMA 2. *Let $ab \in T^s \cap T^d$ in S . This means that there exists c, d such that $abc \in \mathcal{T}^d$ and $abd \in \mathcal{T}^s$. Then either $abcd$ or $abdc$ is an alternating quadriseccant.*

PROOF. Trisecants abc and abd lie on a common line, which must be a quadriseccant line. (Points c and d do not lie on the same straight subarc of K . If they did, then trisecants abc and abd would have the same ordering.) The order of the intersection points along the quadriseccant line is either $abcd$ or $abdc$. First, assume the order is $abcd$. Using the definition of T^s and T^d , the order of the intersection points along the knot must be $acbd$. But this means that $abcd$ is an alternating quadriseccant. A similar argument shows $abdc$ is an alternating quadriseccant. \square

Thus to prove that any knotted curve has at least one alternating quadriseccant, it is sufficient to prove $T^s \cap T^d \neq \emptyset$ in S . To do this, we need to restrict our attention to generic polygonal knots (to be defined in the next section). We will then prove the existence of alternating quadriseccants for generic polygonal knots and will use limit arguments to extend the result to all knotted curves.

3. Generic polygonal knots

In order to clearly understand the structure of the set of trisecants we only consider generic polygonal knots. This section defines generic polygonal knots and looks at some implications of the definition for n -secants. The next section uses the conditions which define generic polygonal knots to help give a detailed description of the structure of the set of trisecants.

DEFINITION. Let K be a closed polygonal curve in \mathbb{R}^3 . Let the vertices of K be $v_1, \dots, v_n \in \mathbb{R}^3$. Then K is *non-degenerate* if no four v_i are coplanar and no three v_i are collinear.

An n -gon in \mathbb{R}^3 is determined by the position of its n vertices. Thus we identify the space of all n -gons with \mathbb{R}^{3n} with the usual topology.

PROPOSITION 3. *The set of non-degenerate n -gons is an open dense set in \mathbb{R}^{3n} .*

PROOF. When $n = 3$, the third vertex cannot lie on the straight line spanned by the other two vertices. Thus the set of degenerate configurations has codimension 1 in the set of all configurations.

For $n \geq 4$, the set of degenerate n -gons is the union of $\binom{n}{4}$ cubic hypersurfaces, each given as the locus where some particular quadruple of vertices is coplanar. This has codimension 1 in the set of all configurations. Note that the situation when three vertices are collinear is a subset of the codimension 1 cubic hypersurfaces, so a special case is not needed to cover this situation. (They will have even higher codimension.)

Thus the set of all degenerate configurations is a codimension 1 (or higher) algebraic surface in the set of all configurations. This is closed and nowhere dense. Hence the non-degenerate n -gons are open and dense in \mathbb{R}^{3n} . \square

DEFINITION. An edge e_k denotes a *closed* edge, including its endpoints. One vertex is *consecutive* to another vertex if they share a common edge. An edge is *adjacent* to another edge if they share a common vertex. Thus given a vertex v , there are two edges incident to it (and adjacent to each other). The *osculating plane* at a vertex v is the plane spanned by the two edges incident to v .

PROPOSITION 4. *Any non-degenerate polygon K has the following properties:*

- (1) *It is embedded.*
- (2) *The line connecting any two consecutive vertices only intersects K in their common edge. The line connecting any other two vertices does not intersect K again.*
- (3) *Two coplanar edges always share a common vertex.*
- (4) *A line intersecting the interiors of two adjacent edges does not hit any vertex of K .*
- (5) *The three points of a trisecant do not lie on three adjacent edges.*

PROOF.

(1) If K is not embedded, then K intersects itself. If two non-adjacent edges e_i, e_j intersect, then the four vertices of e_i and e_j are coplanar contradicting non-degeneracy. Note that two adjacent edges can not intersect (except at their common vertex) as they are not collinear.

(2) Any two consecutive vertices share a common edge e_k which determines a line E_k . Suppose E_k intersects K in another edge e_l . Then the vertices of e_k and e_l are coplanar, contradicting non-degeneracy. Now suppose the line connecting any two (non consecutive) vertices v_i and v_j intersects K in edge e_l . Then v_i, v_j and the vertices of e_l are coplanar, again contradicting non-degeneracy.

(3) Two coplanar edges always share a common vertex, else the plane contains four vertices of a non-degenerate polygon. In other words, the lines determined by any two non-adjacent edges of a non-degenerate polygon are skew.

(4) A line intersecting the interiors of two adjacent edges does not hit any vertex of K . If it did then the plane containing the edges also contains the vertex.

(5) The three points of a trisecant do not lie on three adjacent edges. Suppose trisecant t intersects e_1 , e_2 and e_3 (three adjacent edges). Then e_1 and e_2 have a vertex in common, as do e_2 and e_3 . Therefore e_1 and e_3 lie in the plane determined by t and e_2 and thus have vertex in common. But the three edges cannot form a triangle as each contains a point of t . \square

These conditions provide extra information about non-degenerate polygons, as well as n -secant lines. Condition (1) shows that a non-degenerate polygon is indeed a knot. Condition (2) shows that no two adjacent edges of K are collinear and that each component of intersection of an n -secant line with a non-degenerate polygon K is a single point. Condition (2) also tells us that there are no multi-vertex trisecant lines. Trisecant lines (and higher order secant lines) can intersect K in at most *one* vertex.

DEFINITION. A generic polygonal knot K has two kinds of trisecants (n -secants). A *vertex trisecant* (n -secant) includes one vertex of K and a *nonvertex trisecant* (n -secant) includes no vertices of K .

Recall that a doubly-ruled surface is a surface with two rulings of straight lines. Every line of one ruling intersects every line of the other ruling (maybe at infinity) and each point on the doubly-ruled surface lies on exactly one line from each ruling. It is a well known fact (see for instance [PW] Ch3 or [Otal]) that any three pairwise skew lines generate a doubly-ruled surface.

PROPOSITION 5. *A triple of pairwise skew lines E_1, E_2, E_3 determines a doubly-ruled surface, either a one-sheeted hyperboloid or a hyperbolic paraboloid. Moreover, the lines E_i will lie on one ruling of the surface and any line t intersecting all the E_i will lie on the other ruling. See Figure 5. There is an open interval or a circle of such lines t .* \square

DEFINITION. A non-degenerate polygon K is *generic* if it satisfies the following *genericity conditions*:

(G1) Given any three pairwise skew edges e_1, e_2, e_3 of K and the doubly-ruled surface H that they generate, no vertex of K (except endpoints of e_1, e_2, e_3) may lie in H .

(G2) There are no quintisecants (or higher order secants).

(G3) There are no vertex trisecant lines which lie in the osculating plane of the vertex. See Figure 6.

PROPOSITION 6. *The set of all generic n -gons is open and dense in \mathbb{R}^{3n} .*

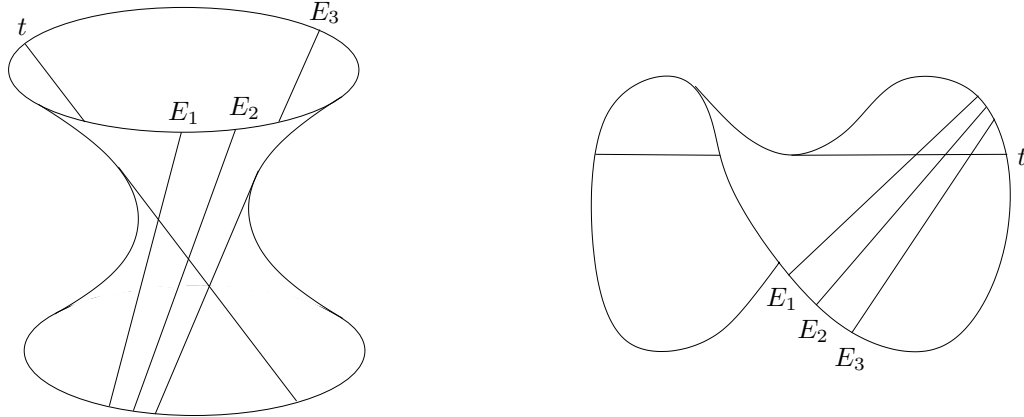


FIGURE 5. Three pairwise skew lines determine a doubly-ruled surface either a one sheeted hyperboloid (left) or a hyperbolic paraboloid (right). The three skew lines lie in one ruling and the lines t intersecting them lie in the other.

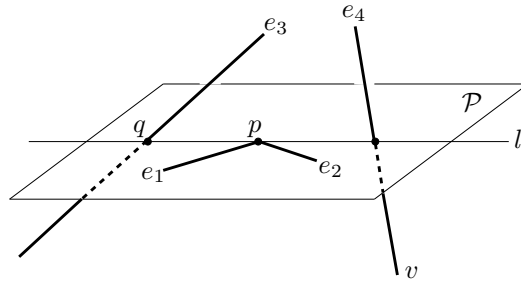


FIGURE 6. A non-generic trisecant line l lying in the osculating plane of vertex p , breaking genericity condition (G3) of the definition of a generic knot.

PROOF. The proof uses the same ideas as the proof of Proposition 3 so is sketched here. In each case, n -gons which fail to meet these conditions lie in an algebraic variety of positive codimension — that is non-generic configurations are closed and nowhere dense. These conditions are, in some sense, natural conditions to consider. If they are broken then the lines determined by the edges of the knot lie in degenerate configurations, as discussed in [BELSW].

(G1) Having fixed e_1, e_2, e_3 , the non-generic configurations are precisely the doubly-ruled surface that they generate: a codimension 1 condition. (G2) Repeating the discussion in the introduction, we see that secants form a 2-parameter family, trisecants a 1-parameter family, quadrisecants a 0-parameter family and quintisecants a -1 -parameter family. That is, quintisecants exist only for a codimension 1 set of n -gons. (Similarly only a codimension k set of n -gons would have $(k + 4)$ -secants for $k > 1$.) (G3) Here, non-generic configurations occur when the fourth edge intersects a line l determined by the the two adjacent edges and the third edge. That is, the plane spanned by l and one vertex of the fourth edge

must also contain the other vertex of the fourth edge. This is a codimension 1 condition. \square

4. The Structure of the Set of Trisecants

The assumption of generic conditions has strong implications for the structure of the set of trisecants \mathcal{T} in K^3 and in the set of secants S . **Unless stated otherwise, all the results of this section will apply to generic polygonal knotted curves.**

For a generic polygonal knot, no two adjacent edges of K are collinear. Thus K^3 is a union of (closed) cubes $\bigcup_{i,j,k}(e_i \times e_j \times e_k)$. The fattened big diagonal $\tilde{\Delta}$ in K^3 is a union of cubes along the big diagonal $\Delta := \{abc \in K^3 \mid a = b \text{ or } b = c \text{ or } a = c\}$. That is, $\tilde{\Delta} = \bigcup_i(e_i \times e_i \times K) \cup \bigcup_i(e_i \times K \times e_i) \cup \bigcup_i(K \times e_i \times e_i)$. As trisecants have three distinct points, no two of which lie on an edge of K , then $\mathcal{T} \subset K^3 \setminus \tilde{\Delta}$. Thinking of K^3 as a cubical complex, \mathcal{T} avoids the 1-skeleton as trisecants do not include more than one vertex of K . Recall that there are two types of trisecants: vertex and nonvertex. Vertex trisecants are just an intersection of \mathcal{T} with the 2-skeleton of the cubical complex. Moreover, as trisecant abc is an ordered tuple of points then abc only lies in the cube $e_1 \times e_2 \times e_3$, where $a \in e_1$, $b \in e_2$ and $c \in e_3$. Trisecant abc does not lie in $e_2 \times e_1 \times e_3$, nor in $e_3 \times e_2 \times e_1$. (However trisecant cba lies in $e_3 \times e_2 \times e_1$.) In a similar way, K^2 is a union of squares and the set of secants $S = K^2 \setminus \tilde{\Delta}$, a topological annulus.

DEFINITION. Recall that the osculating plane at a vertex v of a generic polygonal curve is the plane spanned by the two edges incident to v . A *degenerate trisecant* is a triple vvp or pvv in $\Delta \subset K^3$, where p is in the osculating plane of vertex v and the line determined by pv may not lie in the double wedge bounded by the lines determined by the edges incident to v . (In Figure 7, p may only lie in regions 1 or 3.)

DEFINITION. Let K be a generic polygonal curve and $\mathcal{C} = e_1 \times e_2 \times e_3$ be a cube in K^3 . Then we will see that the set $\mathcal{T} \cap \mathcal{C}$ of trisecants in that cube, if nonempty, is homeomorphic to an interval so we call it an *interval of trisecants*. There are no trisecants intersecting three adjacent edges, so either all edges e_i are pairwise skew and we call it an interval of *skew trisecants* or some e_i and e_j are adjacent and we call it an interval of *adjacent trisecants*.

LEMMA 7. *Let K be a generic polygonal curve. If nonempty, an interval of trisecants $\mathcal{T} \cap \mathcal{C}$ in a cube $\mathcal{C} = e_1 \times e_2 \times e_3$ in K^3 is homeomorphic to a closed or half-open interval. The endpoint or endpoints lie in (the interior of) distinct faces of \mathcal{C} , and the interior of the interval lies in the interior of \mathcal{C} . The half-open interval only occurs when e_2 is adjacent to either e_1 or e_3 ; here, the open end of the interval approaches a degenerate trisecant in the (interior of) an edge of \mathcal{C} . Along an interval of skew trisecants, each of the three points $p_i \in e_i$ moves monotonically and smoothly along the edge e_i . Along an interval of adjacent trisecants, if e_i is the nonadjacent edge, then p_i is constant, while the other p_j move smoothly and monotonically along e_j .*

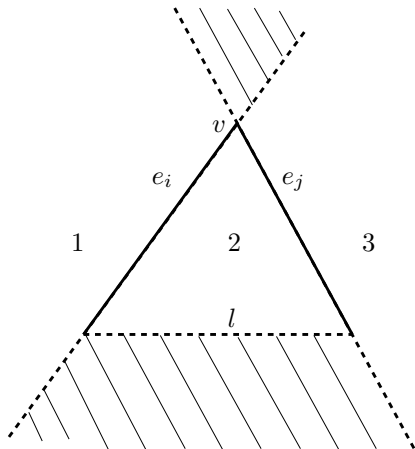


FIGURE 7. Regions in the plane \mathcal{P} spanned by edges e_i and e_j determine the type of interval of trisecants. If the third edge e_k intersects \mathcal{P} in the shaded regions there are no trisecant lines intersecting e_i, e_j, e_k . If e_k intersects \mathcal{P} in region 2, the set of trisecants is homeomorphic to $[0, 1]$. If e_k intersects \mathcal{P} in regions 1 or 3, the set of trisecants is homeomorphic to $[0, 1)$.

PROOF. We first consider where $\mathcal{T} \cap \mathcal{C}$ is an interval of adjacent trisecants. Let \mathcal{P} be the plane spanned by two adjacent edges e_i and e_j and let p be the unique point of intersection of the third edge e_k with \mathcal{P} . (Non-degeneracy implies that p is in the interior of e_k .) Figure 7 shows \mathcal{P} separated into regions divided by lines determined by e_i and e_j and the line l between the non-intersecting vertices of e_i and e_j . For trisecants to occur then point p must be in one of regions 1, 2 or 3 (and not the shaded regions). There is a 1-parameter family of such trisecants. If p is in region 2, then e_1 is adjacent to e_3 and trisecants are of the form apb , where $a \in e_1$ and $b \in e_3$. Clearly, the interval of trisecants is homeomorphic to a closed interval as in Figure 8 (left). If p is in either region 1 or 3, then e_2 is adjacent to either e_1 or e_3 and trisecants are of the form abp where $a \in e_1$ and $b \in e_2$ or pab where $a \in e_2$ and $b \in e_3$. Figure 8 (right) shows the first possibility. In either case, the interval of trisecants will be homeomorphic to a half-open interval and ends on the degenerate trisecant vpv (or pvv) where v is the common vertex of e_1 and e_2 (or e_2 and e_3). This point is not in the set of trisecants, vpv (or pvv) $\in \Delta \subset \tilde{\Delta}$.

We now consider where $\mathcal{T} \cap \mathcal{C}$ is an interval of skew trisecants. By Proposition 5, the triple of pairwise skew edges determines a doubly-ruled surface, either a one-sheeted hyperboloid or hyperbolic paraboloid. The edges e_i ($i = 1, 2, 3$) all lie in one ruling. A trisecant line is a line in the other ruling which intersects e_1, e_2 and e_3 . Let E_i be the line determined by e_i . From Proposition 5 we know there is an circle of lines intersecting E_1, E_2 and E_3 . Let I_1 be the closed sub-interval of these lines which intersects the edge e_1 and the lines E_2 and E_3 . As e_1 is an edge (closed), I_1 is homeomorphic to a closed interval. Let I_2 and I_3 be defined

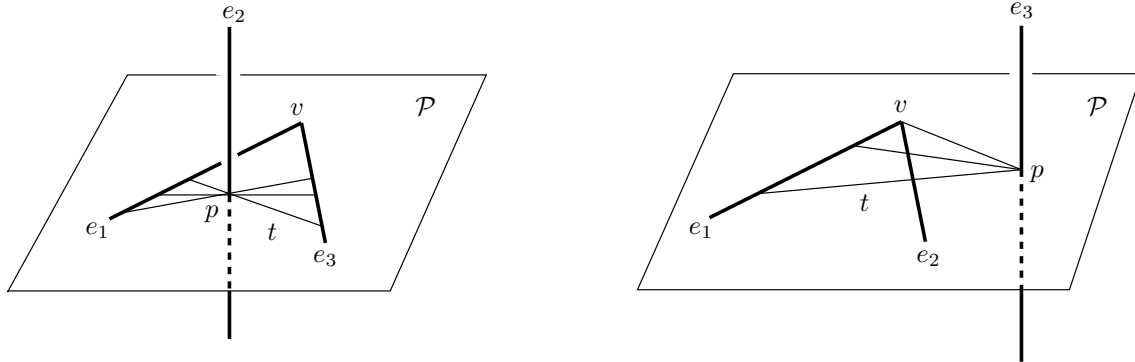


FIGURE 8. Intervals of adjacent trisecants. The left picture shows an interval homeomorphic to a closed interval, the right an interval homeomorphic to a half-open interval.

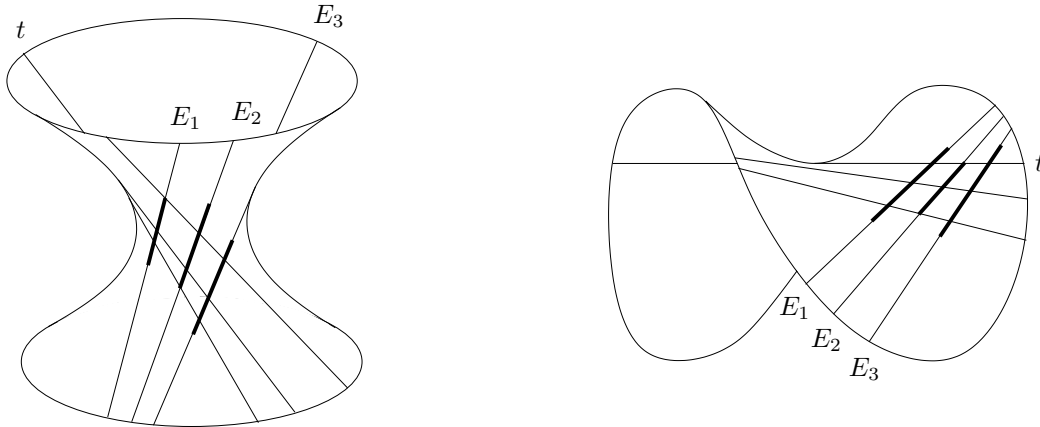


FIGURE 9. Two doubly-ruled surfaces, each with a closed interval of trisecant lines intersecting edges e_1 , e_2 and e_3 .

in a similar manner, they too are homeomorphic to a closed interval. Thus, the set of trisecants through e_1 , e_2 and e_3 is homeomorphic to the intersection of I_1 , I_2 and I_3 (three closed intervals in an open interval), which is either a closed interval or a point, see Figure 9. Note that this interval of trisecants must start and end with a vertex trisecant corresponding to one of the endpoints of the I_i . The interval cannot have zero length (be a point) nor start with a line intersecting two (or more) endpoints of I_i , as either of these cases corresponds to a trisecant with two (or more) vertices of the knot — contradicting non-degeneracy. Hence, if it is nonempty, the set of trisecants through e_1 , e_2 and e_3 is homeomorphic to a closed interval of non-zero length which starts and ends with vertex trisecants.

We may say more about the structure of $\mathcal{T} \cap \mathcal{C}$ for both intervals of adjacent and skew trisecants. In all cases the interior of the interval of trisecants lies in the interior of \mathcal{C} and as the endpoints correspond to vertex trisecants, they lie on the interior of a face of \mathcal{C} . Parameterize K with respect to arclength; along an interval of

skew trisecants, each of the three points $p_i \in e_i$ moves monotonically and smoothly along the edge e_i . Along an interval of adjacent trisecants, on the nonadjacent edge p is constant while on the adjacent edges, the points move monotonically and smoothly along their corresponding edges. (An explicit calculation for both cases may be found in [Den] in the Appendix.) Thus $\mathcal{T} \cap \mathcal{C}$ is smoothly embedded in \mathcal{C} . \square

PROPOSITION 8. *Let K be a generic polygonal curve. If nonempty, $\overline{\mathcal{T}}$ is a compact 1-manifold with boundary, embedded in K^3 in a piecewise smooth way with $\mathcal{T} \subset K^3 \setminus \tilde{\Delta}$ and $\partial\mathcal{T} \subset \Delta$. Moreover $\partial\mathcal{T}$ is exactly the set of degenerate trisecants.*

PROOF. The definition of a trisecant shows $\mathcal{T} \subset K^3 \setminus \tilde{\Delta}$. Consider a face \mathcal{F} between two cubes \mathcal{C} and \mathcal{C}' in K^3 . Suppose the interval of trisecants in \mathcal{C} ends at a particular vertex trisecant t in \mathcal{F} . There are three possibilities for the interval of trisecants in \mathcal{C}' : there is an interval of trisecants starting at t , there is an interval of trisecants starting at another vertex trisecant t' in \mathcal{F} or there are no intervals of trisecants which start on \mathcal{F} . In the last two cases, the intersection $\mathcal{T} \cap \mathcal{C}'$ is either a single point or an interval of trisecants and a single point. Lemma 7 rules out these cases. Thus in \mathcal{C}' , there must be an interval of trisecants starting at t . Hence \mathcal{T} is a 1-manifold embedded in K^3 in a piecewise smooth way.

To understand the claim about $\overline{\mathcal{T}}$, we must understand $\partial\mathcal{T}$. Recall that the half open intervals of trisecants occur when an interval of adjacent trisecants ends on a degenerate trisecant vvp (or pvv) where v is a vertex of K . Thus the limit point (what is added in $\overline{\mathcal{T}}$) is a degenerate trisecant and lies in Δ . Also note that no other interval of trisecants in K^3 can end at the degenerate trisecant vvp (or pvv). Degenerate trisecant vvp can be a common degenerate vertex trisecant to only two cubes in K^3 . Without loss of generality denote these cubes $e_1 \times e_2 \times e_3$ and $e_2 \times e_1 \times e_3$ where v is the common vertex of e_1 and e_2 . But trisecants are ordered tuples, so at most one of these has nonempty intersection with \mathcal{T} . Similarly for pvv . Hence $\overline{\mathcal{T}}$ is a compact 1-manifold with boundary embedded in a piecewise smooth way in K^3 . \square

We now wish to understand the structure of $T = \pi_{12}(\mathcal{T})$ the projection of the set of trisecants to the set of secants $S = K^2 \setminus \tilde{\Delta}$. Recall that a secant $ab \in S$ is only in T if it is the first two points of trisecant abc .

We first put a metric on S (and hence a topology) in the following way. The knot K has an orientation, is parameterized with respect to arclength and $d(a, a')$ is the shorter arclength between a and a' along K . We give K^2 the product metric such that $d((a, b), (a', b')) = d(a, a') + d(b, b')$. Now consider S as a metric space, not with the subset metric, but with the path metric $d_S((a, b), (a', b')) = \inf_{\gamma}(\text{len}(\gamma))$, where γ is any path in S from (a, b) to (a', b') and $\text{len}(\gamma)$ the length of γ (using the product metric from K^2). Let the completion of S in this metric be

$$\overline{S} = (K^2 \setminus \tilde{\Delta}) \cup \tilde{\Delta}_+ \cup \tilde{\Delta}_- .$$

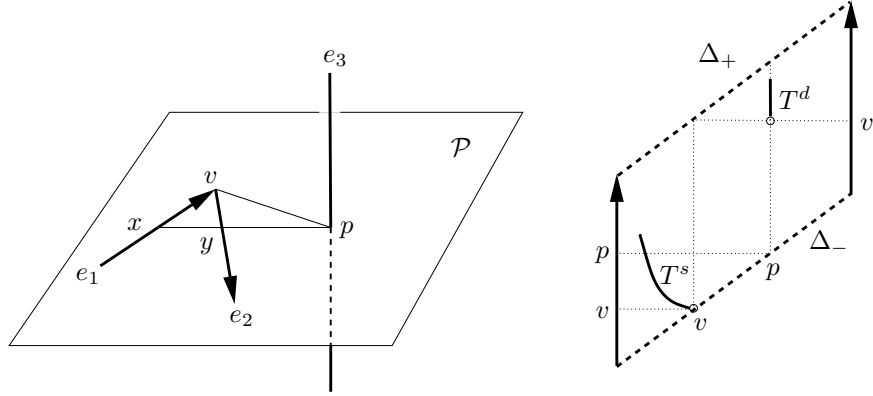


FIGURE 10. The left picture shows an interval of adjacent trisecants ending on degenerate trisecant vvp or pvv . The right picture shows the corresponding intervals of trisecants in S . The intervals of T^s and T^d correspond to trisecants with linear ordering $e_1e_2e_3$ and $e_3e_2e_1$ respectively.

Here $\tilde{\Delta}_-$ denotes the lower edge of S , that is approaching $\tilde{\Delta}$ in K^2 from above. That is $ab \in K^2$ and $b > a$: a and b are close together with b just after a in the order of K . Similarly $\tilde{\Delta}_+$ denotes the upper edge of S . Let Δ_- , Δ_+ denote the lower and upper big diagonals of S respectively. The topology of S is imposed by the metric. Hence the upper and lower edges of the annulus S are far from each other. The shortest path between them lies *in* S and by definition can *not* cross Δ .

Let \bar{T} be the projection of $\bar{\mathcal{T}}$ in \bar{S} , $\bar{T} := \pi_{12}(\bar{\mathcal{T}})$. Now π_{12} maps to K^2 and $\bar{S} \not\subseteq K^2$, thus to understand \bar{T} we must describe where $\pi_{12}(\partial T)$ lies in \bar{S} .

A half-open interval of trisecants occurs when an interval of adjacent trisecants ends on a degenerate trisecant. (See Figures 8 and 10.) Let the points of the adjacent trisecants lie on edges e_1 , e_2 and e_3 and let e_1 and e_2 be adjacent with common vertex v . Let e_3 intersect the plane spanned by e_1 and e_2 in the point p . Suppose the points of the trisecants lie on $e_1e_2e_3$ in that order, then the third point p is fixed. Thus the interval of trisecants ends at degenerate trisecant $vvp \in \Delta \subset K^3$ and in \bar{S} , the interval ends at either Δ_+ or Δ_- . Now suppose the points of the trisecant lie on $e_3e_2e_1$ in that order, then the first point p is fixed. Thus the interval of trisecants ends at degenerate trisecant $pvv \in \Delta \subset K^3$ and in \bar{S} , it ends at pv in the interior of S .

In fact we can say more. First consider the case where knot is oriented from e_1 to e_2 as illustrated in Figure 10 (left). Trisecants whose points lie on edges $e_1e_2e_3$ in that order are trisecants of same ordering T^s . Moreover, as the interval of trisecants abp ends at degenerate trisecant vvp , the first point of the trisecant ($a \in e_1$) moves with the parameterization of K and the second ($b \in e_2$) against. Thus in K^2 , $b > a$ and the interval of trisecants approaches the diagonal from above. Hence this part of T^s has negative slope and ends at $vv \in \Delta_-$. In this situation, $\pi_{12}(\partial T) \subset \Delta_-$. Trisecants whose points lie on $e_3e_2e_1$ in that order are trisecants

of different ordering \mathcal{T}^d . The first point of the trisecant is fixed and the second moves against the parametrization. Hence this interval of trisecants is a vertical line, decreasing and ending at the point $pv \in S$ and $\pi_{12}(\partial\mathcal{T}) \subset S$. Figure 10 (right) shows the two corresponding intervals of trisecants in S .

Now repeat these arguments but with the orientation of K reversed — from e_2 to e_1 . Trisecants whose points lie on edges $e_1e_2e_3$ in that order are now in T^d and the interval of trisecants has negative slope and ends on $vv \in \Delta_+$ and $\pi_{12}(\partial\mathcal{T}) \subset \Delta_+$. Trisecants whose points lie on edges $e_3e_2e_1$ in that order are in T^s and the interval of trisecants is a vertical line increasing and ending at $pv \in S$ and $\pi_{12}(\partial\mathcal{T}) \subset S$.

We have completely described the behavior of $\partial T := \pi_{12}(\partial\mathcal{T})$ in \bar{S} . Either ∂T lies in $S = K^2 \setminus \tilde{\Delta}$ as the limit point of a vertical interval of trisecants whose first point is fixed, or ∂T lies on Δ_+ or Δ_- as the limit point of a negatively sloped interval of trisecants whose third point is fixed.

DEFINITION. Recall $\bar{T} = \pi_{12}(\bar{\mathcal{T}})$ and $\partial T := \pi_{12}(\partial\mathcal{T})$. In a similar way, define $\bar{T}^s = \pi_{12}(\bar{\mathcal{T}}^s)$, $\bar{T}^d = \pi_{12}(\bar{\mathcal{T}}^d)$, $\partial T^s = \pi_{12}(\partial\mathcal{T}^s)$ and $\partial T^d = \pi_{12}(\partial\mathcal{T}^d)$.

In Lemma 14 we will show that if $\bar{T}^s \cap \bar{T}^d \neq \emptyset$, then $T^s \cap T^d \neq \emptyset$. Hence by Lemma 2 it will be sufficient to show that $\bar{T}^s \cap \bar{T}^d \neq \emptyset$ in \bar{S} in order to show that an alternating quadrisecant exists.

LEMMA 9. *Let K be a generic polygonal curve. The projection π_{12} is a piecewise smooth immersion of \mathcal{T} into S .*

PROOF. Let \mathcal{C} be a cube in K^3 . From Lemma 7, $\mathcal{T} \cap \mathcal{C}$ is both smooth and monotonic in at least two variables (corresponding to edges of \mathcal{C}). Thus the tangent vector is never vertical. The map $\pi_{12}(\mathcal{T})$ is just an orthogonal projection hence the projection π_{12} of $\mathcal{T} \cap \mathcal{C}$ in S is a smooth immersion. If nonempty, \mathcal{T} is an embedded 1-manifold in K^3 . Thus π_{12} is a piecewise smooth immersion of \mathcal{T} in S . \square

DEFINITION. Let K be a generic polygonal curve and let \mathcal{C} be a cube in K^3 . The set $\pi_{12}(\mathcal{T} \cap \mathcal{C})$, if nonempty, is homeomorphic to an interval. Thus we (again) call it an *interval of trisecants* or an *interval of T in S* .

The following two lemmas show that in S , T intersects itself only in double-points. This means that quadrisecants (if they exist) are isolated.

LEMMA 10. *Let K be a generic polygonal curve. In S , the intersection of two intervals of T is either be empty, one point, or two distinct points.*

PROOF. We assume that T is nonempty or the result is trivial. First assume that the two intervals of T in S do not share a common vertex. The two intervals have a point in common if and only if two trisecants have the same first and second points. That is, only if they have the same first and second edges. However, the third edges differ, or else there would not be two intervals of T intersecting in S . There are three cases to consider. Let the first two edges be denoted e_1 and e_2 and let the differing third edges be denoted by e_3 and e'_3 .

Case 1: Both intervals are intervals of adjacent trisecants. There is only one possible case, the first two edges are adjacent. (If e_1 and e_3 are adjacent and e_1

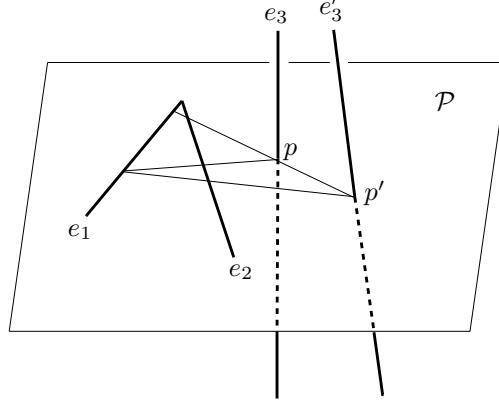


FIGURE 11. Trisecants with same two adjacent first edges can have at most one line in common.

and e'_3 are adjacent, then a quadrisequant whose points lie on $e_1e_2e_3e'_3$ includes a trisecant whose points lie on three adjacent edges $e_1e_3e'_3$, contradicting non-degeneracy.) Both intervals of trisecants lie in the plane \mathcal{P} spanned by e_1 and e_2 . Let e_3 intersect \mathcal{P} in the point p and e'_3 intersect \mathcal{P} in the point p' . Figure 11 illustrates this. The line through pp' is the only possible common trisecant line.

Case 2: One interval is an interval of adjacent trisecants and the other an interval of skew trisecants. Let the points of the skew trisecant lie on edges e_1, e_2, e_3 ; these determine a doubly-ruled surface. Let the points of the adjacent trisecants lie on edges e_1, e_2 and e'_3 . As e_1 and e_2 are pairwise skew, either e_1 and e'_3 are adjacent, or e_2 and e'_3 are adjacent. In either case, the plane spanned by the two adjacent edges intersects the doubly-ruled surface in some quadratic curve. However, as one of the edges lies in the doubly-ruled surface, the curve degenerates to a pair of straight lines, one of which includes edge e_1 (or e_2). The other straight line is the only possible common trisecant line and only if it is both a trisecant line on the doubly-ruled surface and the plane.

Case 3: Both intervals are intervals of skew trisecants. Let H be the doubly-ruled surface generated by the lines E_i determined by the polygonal edges e_i . Suppose e'_3 intersects H , it does so in one or two points. The trisecant line(s) through these point(s) will only correspond to a quadrisequant line if it is a trisecant through the edges e_1, e_2, e_3 , and through the edges e_1, e_2, e'_3 . In fact this is all that can happen. The edge e'_3 can never be contained in H . If it were then its vertices lie in H which contradicts genericity condition (G1). Thus in all three cases the intersection of two intervals of T is either empty, or one or two points.

Finally assume the two intervals of T have a common vertex. This corresponds to a vertex trisecant which joins two intervals of trisecants. In K^3 , these intervals of trisecants lie in adjacent cubes $\mathcal{C} = e_1 \times e_2 \times e_3$ and $\mathcal{C}' = e'_1 \times e'_2 \times e'_3$. As the cubes are adjacent, two of the edges are the same and the third differs. If $e_1 \neq e'_1$ (or $e_2 \neq e'_2$) then the first (or second) points of the intervals of trisecants lie on different edges. Thus the only possible common point in S is the projection of the common vertex trisecant. If the first two edges are the same and the third differs,

then the intervals of trisecants may intersect in S . There are three cases to consider as above. The details are omitted as they are similar to the previous arguments. \square

LEMMA 11. *Let K be a generic polygonal curve. No more than two points of \mathcal{T} can have the same image under π_{12} .*

PROOF. If there were three (or more) points of \mathcal{T} with the same image under π_{12} , this would correspond to a quintisecant (or higher order secant) in contradiction to genericity condition (G2). (See also Figure 13A.) \square

The structure of T described is sufficient to prove that nontrivial generic polygonal knotted curves have an alternating quadriseccant. However, with the addition of extra genericity conditions, it is possible to provide even more details about the structure of T . These genericity conditions and details are found in [Den].

Let $\pi_{ij}(K^3) \rightarrow K^2$ be projection onto the i th and j th coordinates (where $i < j$ and $i, j = 1, 2, 3$.) We have extensively studied the structure of $\pi_{12}(\mathcal{T})$ for generic polygonal curves and have shown that, if nonempty, it is the image of a piecewise smooth immersion and which intersects itself at double-points. We can also say the same thing about $\pi_{13}(\mathcal{T})$ and $\pi_{23}(\mathcal{T})$. Lemma 9 carries over immediately. Similarly Lemma 10 and Lemma 11 can be proved for $\pi_{13}(\mathcal{T})$ and $\pi_{23}(\mathcal{T})$ with only minor alterations. Thus the following proposition holds:

PROPOSITION 12. *Let K be a generic polygonal curve. In K^2 , the projection π_{ij} ($i < j$ and $i, j = 1, 2, 3$) is a piecewise smooth immersion of \mathcal{T} into S and $T = \pi_{ij}(\mathcal{T})$ intersects itself at double points.* \square

For the most part we are only interested in $T = \pi_{12}(\mathcal{T})$ in S . Recall that $\pi_{12}(\mathcal{T})$ allows us to capture the orderings of coincident trisecants which in turn enables us to find alternating quadriseccants. In Lemma 14 we will show that if $\overline{T}^s \cap \overline{T}^d \neq \emptyset$ in \overline{S} , then $T^s \cap T^d \neq \emptyset$ in S . Thus it is important to understand exactly where $\overline{T}^s \cap \overline{T}^d \neq \emptyset$ but T^s and T^d do not intersect. Recall $\overline{T} = \pi_{12}(\overline{\mathcal{T}}) = \pi_{12}(\mathcal{T} \cup \partial\mathcal{T})$. After Proposition 8 we showed that $\partial T = \pi_{12}(\partial\mathcal{T})$ lies on Δ_+ or Δ_- as the limit point of an interval of trisecants whose first two points both end on a vertex of K and whose third point is fixed. If the second and third points of the trisecants both end on a vertex of K and the first is fixed, then ∂T lies in S .

The following lemma shows that \overline{T}^s stays away from $\tilde{\Delta}_+$ and \overline{T}^d stays away from $\tilde{\Delta}_-$. Hence \overline{T}^s and \overline{T}^d cannot intersect on Δ_+ or Δ_- , but only in S .

LEMMA 13. *Let K be a generic polygonal knotted curve. In S , $d_S(\overline{T}^d, \tilde{\Delta}_-) \geq h$ and $d_S(\overline{T}^s, \tilde{\Delta}_+) \geq h$, where h is the minimum edge length of K .*

PROOF. As secant ab approaches $\tilde{\Delta}_-$ it is the same as approaching the diagonal in K^2 from above. Here, a and b are close together and b is just after a in the order of the knot. Now consider secant $ab \in T^d$. The order along the knot is acb (where c is the third point of the trisecant.) Thus a and b are separated by at least one edge as c must come in between them and by definition, distinct points of trisecants lie on distinct edges. In fact a and b must be separated by at least two edges. If they were just separated by one edge, then the points of the trisecant abc lie on three

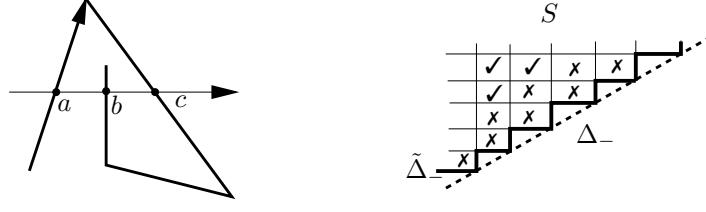


FIGURE 12. The first two points of trisecants of different order must be at least two edge lengths apart. This information is shown on part of the set of secants S on the right. The ticked boxes indicate where trisecants of different order may be.

consecutive edges contradicting non-degeneracy. Figure 12 (left) illustrates this and Figure 12 (right) shows this information recorded on S . Trisecants of different order may be in boxes marked with a tick, not a cross. They are at least one edge away from $\tilde{\Delta}_-$ (indicated by a bold line). Hence $d_S(T^d, \tilde{\Delta}_-) \geq h$, where h is the minimum edge length of K . A similar argument shows $d_S(T^s, \tilde{\Delta}_+) \geq h$. \square

LEMMA 14. *Let K be a generic polygonal curve. If $\overline{T^s} \cap \overline{T^d} \neq \emptyset$ in \overline{S} , then $T^s \cap T^d \neq \emptyset$ in S .*

PROOF. Lemma 13 shows $\overline{T^s}$ and $\overline{T^d}$ can only intersect in the interior of S . Suppose $t \in \overline{T^s}$ and $t' \in \overline{T^d}$ with $\pi_{12}(t) = \pi_{12}(t') \in S$. We want to show $t \in T^s$ and $t' \in T^d$. Assume, by way of contradiction that $t \in \partial T^s$ and consider two cases: $t' \in T^d$ and $t' \in \partial T^d$.

Case 1: Assume that $t \in \partial T^s$ and $t' \in T^d$ and $\pi_{12}(t) = \pi_{12}(t') \in S$. Figure 13B illustrates this case.

Let $pv = \pi_{12}(t) = \pi_{12}(t')$. As $t \in \partial T^s$, in S there is a vertical interval of trisecants which increases to and ends at secant pv . Thus pv is the projection of degenerate trisecant $pvv \in \mathcal{T}$. Let p belong to edge e_1 and vertex v belong to edges e_2 and e'_2 . The vertical open interval of trisecants corresponds to trisecants with first point p and second and third points on edges e_2 and e'_2 . But $pv = \pi_{12}(t')$, where $t' \in T^d$. Thus pv is also the first two points of a trisecant of different order pva , where a lies on edge e_3 . Secant pv , and hence trisecant pva lies in the plane spanned by e_2 and e'_2 . Thus trisecant pva lies in the osculating plane of vertex v , contradicting genericity condition (G3).

Case 2: Assume that $t \in \partial T^s$ and $t' \in \partial T^d$ and $\pi_{12}(t) = \pi_{12}(t') \in S$. Figure 13C illustrates this case.

Let $pv = \pi_{12}(t) = \pi_{12}(t')$. As $t \in \partial T^s$ and $t' \in \partial T^d$, pv belongs to two vertical intervals of trisecants which both end at secant pv in S . The trisecants of same and different order both have p as a starting point and both have points which lie on edges which have v as a common vertex. But there can only be two edges with v as a common vertex. This implies t and t' have the same first point and both have second and third points on the same two edges. This implies that the two intervals

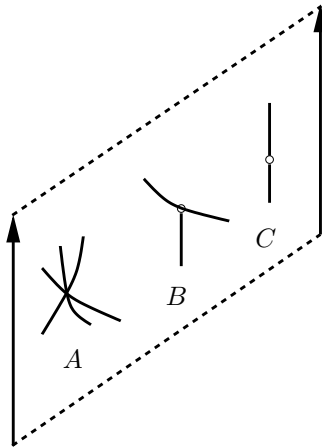


FIGURE 13. Intersections of T in S forbidden for a generic polygonal knot.

of trisecants are in fact the same (and hence have the same order). This contradicts the fact t and t' are trisecants of same and different order respectively. \square

In Lemma 2 we showed that an alternating quadriseccant exists when T^s and T^d have a common point in S . In Section 3 and Section 4 we examined the structure of the set of trisecants for generic polygonal knots. Through Lemma 14 we see that it is sufficient to prove $\overline{T^s}$ and $\overline{T^d}$ have a common point in S in order to show that an alternating quadriseccant exists. In fact, we will prove a stronger result than this in Section 6.

5. Essential Quadriseccants

Eventually we want to prove that all nontrivial tame knots have an alternating quadriseccant. To do this we will prove that nontrivial generic polygonal knots have an *essential* alternating quadriseccant and then use a limit argument. We will define essential shortly — this is the topological notion required so the limit argument works.

In Section 3 we showed that the set of generic polygonal knots is open and dense in the set of polygonal knots. For any knotted curve K , there is a sequence of generic polygonal knots $\{K_i\}_{i=1}^{\infty}$ which converge to K . In Lemma 28 we show how this sequence of K_i may be chosen so that K_i is ambient isotopic to K and converges in C^0 to K .

Thus we have $K_i \rightarrow K$. Theorem 26 shows that each generic polygonal knot K_i has an essential alternating quadriseccant $a_i b_i c_i d_i$. As each K_i is contained in some ball in \mathbb{R}^3 , eventually the entire sequence $\{K_i\}_{i=1}^{\infty}$ lies in a common compact subset of \mathbb{R}^3 . By taking a subsequence if necessary, there is a sequence of essential alternating quadriseccants which converge to some quadriseccant $abcd \in K^4$. We may also assume that the order of $a_i b_i c_i d_i$ along K_i is $a_i c_i b_i d_i$.

LEMMA 15. *Given the sequence of generic polygonal knots with converging alternating quadriseccants as described above. If b does not lie on the same subarc of K as c , then no two limit points lie on the same straight subarc of K and so $abcd$ is a quadriseccant.*

PROOF. Given two points $p, q \in K$, we denote $p \sim q$ to mean p and q lie on the same common straight subarc of K . Quadriseccant points lie on the same straight subarc of K in the limit if and only if they are next to each other in the ordering of *both* the quadriseccant line and the knot. Thus $b \approx c$ implies $a \approx b$ as the order along the knot is $a_i c_i b_i$. Similarly $b \approx c$ implies $c \approx d$ as the knot order is $c_i b_i d_i$ and $b \approx c$ implies $a \approx d$ as the knot order is $a_i c_i b_i d_i$. \square

In order to show the limit $abcd$ is a quadriseccant, we only have to worry about whether the middle two intersection points b and c merge together. Panwitz [Pann] proved that generic polygonal knots have a quadriseccant and Kuperberg [Kup] proved that all (nontrivial tame) knots have quadriseccants. He did this by using the limit argument outlined above. In order to show the limit was a quadriseccant, he introduced the notion of essential secants and quadriseccants (which he called “topologically nontrivial”). As we have encountered the same problem, we also use the notion of essential, but redefine and extend it to suit our purposes. We start by defining when an arc of a knot is essential, capturing part of the knottedness of K . Generically, a knot K together with a secant segment $S = \overline{ab}$ forms a knotted Θ -graph in space. To adapt Kuperberg’s definition, we consider such knotted Θ -graphs. A similar discussion is found in [Den, DDS].

DEFINITION. Suppose α, β and γ are three disjoint simple arcs from a to b , forming a knotted Θ -graph. Then we say that the ordered pair (α, β) is *inessential* if there is a disk D bounded by the knot $\alpha \cup \beta$ having no interior intersections with the knot $\alpha \cup \gamma$. (We allow self-intersections of D , and interior intersections with β , as will be necessary if $\alpha \cup \beta$ is knotted.)

An equivalent definition is illustrated in Figure 14: Let $X := \mathbb{R}^3 \setminus (\alpha \cup \gamma)$, and let δ be a parallel curve to $\alpha \cup \beta$ in X . Here by *parallel* we mean that $\alpha \cup \beta$ and δ cobound an annulus embedded in X . We choose δ so that it is homologically trivial in X (that is, so that δ has linking number zero with $\alpha \cup \gamma$). Let $h(\alpha, \beta) \in \pi_1(X)$ denote the (free) homotopy class of δ . Then (α, β) is *inessential* if $h(\alpha, \beta)$ is trivial. We say that (α, β) is *essential* if it is not inessential.

This notion is clearly a topological invariant of the (ambient isotopy) class of the knotted Θ -graph. We apply this definition to arcs of a knot K below. But first, some useful notation. Let $a, b \in K$. The arc from a to b following the orientation of the knot is denoted γ_{ab} . The arc from b to a , γ_{ba} is similarly defined. The secant segment from a to b is denoted \overline{ab} . We now define an essential arc of a knot following [DDS], rather than the stronger notion found in [Den].

DEFINITION. If K is a knotted curve and $a, b \in K$, let $S = \overline{ab}$. We say γ_{ab} is *essential* if for every $\epsilon > 0$ there exists some ϵ -perturbation of S (with endpoints

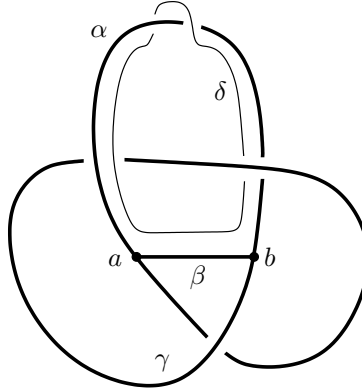


FIGURE 14. In the knotted Θ -graph $\alpha \cup \beta \cup \gamma$, the ordered pair (α, β) is essential. To see this, we find the parallel $\delta = h(\alpha, \beta)$ to $\alpha \cup \beta$ which has linking number zero with $\alpha \cup \gamma$ and note that is homotopically nontrivial in the knot complement $\mathbb{R}^3 \setminus (\alpha \cup \gamma)$. In this illustration, β is the straight segment \overline{ab} , so we may equally say that the arc α of the knot $\alpha \cup \gamma$ is essential.

fixed) to a tame curve S' such that $K \cup S'$ forms an embedded Θ in which (γ_{ab}, S') is essential.

Note that this definition is quite flexible. We will see that it ensures that the set of essential secants is closed in S . It also allows for the situation where K intersects S . We could allow perturbations only in this case, the combing arguments of [DS] show the resulting definitions are equivalent. Note also that we only require S' to be close to S in the C^0 sense. Thus S' could be locally knotted, however we only care about a homotopy class, not the isotopy class of h .

In [CKKS] it was shown that if K is an unknot, then any arc γ_{ab} is inessential. This follows immediately, because the homology and homotopy groups of $X := \mathbb{R}^3 \setminus K$ are equal for an unknot, so any curve δ having linking number zero with K is homotopically trivial in X . Dehn's lemma [Dehn, Pap, Rolf] is used to prove a converse statement.

THEOREM 16 ([DDS]). *If $a, b \in K$ and both γ_{ab} and γ_{ba} are inessential, then K is unknotted.* \square

DEFINITION. A secant ab of K is *essential* if both subarcs γ_{ab} and γ_{ba} are essential. Otherwise it is *inessential*. Let ES be the set of essential secants in S . Figure 15 illustrates an inessential and an essential secant of a trefoil knot.

To call a quadriseccant $abcd$ essential, we could follow Kuperberg [Kup] and require that secants ab , bc and cd all be essential. However the notion of essential is needed only for those secants whose endpoints are consecutive along the knot. The following definition is consistent with the more general definition found in [DDS].

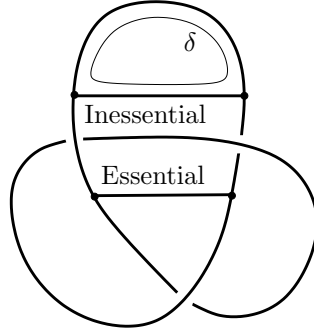


FIGURE 15. Examples of essential and inessential secants for the trefoil knot. The parallel curve δ is shown for the inessential secant.

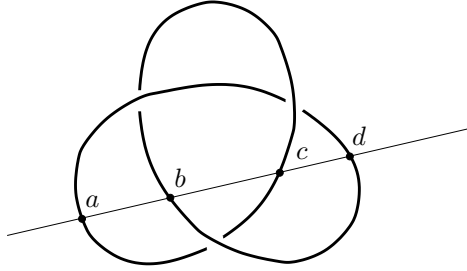


FIGURE 16. Essential alternating quadriseccant $abcd$.

DEFINITION. An *essential trisecant* abc is a trisecant which is essential in the second segment bc . An *essential alternating quadriseccant* $abcd$ is an alternating quadriseccant which is essential in the second segment bc . (See Figure 16.) Trisecants and alternating quadriseccants that are not essential are called *inessential*.

We show in the proof of Main Theorem that when taking the limit of essential secants, the points of the limit secant do not combine to form one point, nor lie on a common straight subarc of K . Thus when taking the limit of essential alternating quadriseccants, we may be assured of an alternating quadriseccant. We want to prove that any nontrivial generic polygonal knot has an *essential* alternating quadriseccant. Thus we need to understand the relationship between essential trisecants and essential quadriseccants.

DEFINITION. Let \mathcal{ET} be the set of essential trisecants in K^3 , really, $\mathcal{ET} = \pi_{23}^{-1}(ES) \cap \mathcal{T}$. Define \mathcal{ET}^s and \mathcal{ET}^d to be the sets of essential trisecants of same and different orderings in K^3 . Let $ET = \pi_{12}(\mathcal{ET})$ be the projection of the set of essential trisecants to the set of secants S and similarly define $ET^s := \pi_{12}(\mathcal{ET}^s)$ and $ET^d := \pi_{12}(\mathcal{ET}^d)$.

LEMMA 17. *Let $ab \in ET^s \cap ET^d$ in S . This means that there exists c and d such that $abc \in \mathcal{ET}^d$ and $abd \in \mathcal{ET}^s$. Then either $abcd$ or $abdc$ is an essential alternating quadriseccant.*

PROOF. That $abcd$ or $abdc$ is an alternating quadriseccant follows from Lemma 2. To show it is essential, the midsegment bc , respectively bd , must be essential. The order along the quadriseccant line is either $abcd$ or $abdc$. If it is $abcd$ then bc is essential as $abc \in \mathcal{ET}^d$. If it is $abdc$, then bd is essential as $abd \in \mathcal{ET}^s$. \square

Thus in order to show that a generic polygonal knot has at least one essential alternating quadriseccant, it is sufficient to prove that ET^s and ET^d have common points in S .

In the following Lemmas we describe the relationship between the set of essential triseccants and the set of triseccants. We use this relationship to completely understand the structure of ET in S .

LEMMA 18. *Let K be a generic polygonal knotted curve. In $S \setminus \pi_{13}(\mathcal{T})$, each connected component consists of either inessential or essential secants. Moreover, the set of essential secants is closed in S .*

PROOF. Recall that $\pi_{13} : K^3 \rightarrow K^2$ is projection onto the first and third coordinates. Proposition 12 showed that π_{13} is a piecewise smooth immersion of \mathcal{T} in S and $\pi_{13}(\mathcal{T})$ intersects itself in double points.

We wish to show that in S , secants change from inessential to essential *only* when there is a triseccant $t \in \pi_{13}(\mathcal{T})$. Let ab be an inessential (essential) secant. By perturbing $a, b \in K$ we can get to all nearby secants in S . As long as the chord \overline{ab} never touches K during the perturbation then the topological type of the knotted Θ -graph has not changed. Thus all secants near secant ab will have the same topological type in the knotted Θ -graph *unless* K intersects \overline{ab} . In this case, $ab \in \pi_{13}(\mathcal{T})$. Thus secants change from inessential to essential *only* when there is a triseccant $t \in \pi_{13}(\mathcal{T})$ and each connected component of $S \setminus \pi_{13}(\mathcal{T})$ consists of either essential or inessential secants. (For example, Figure 17 shows a triseccant abd changing from inessential to essential through quadriseccant $abcd$. Here secant bd changes from inessential to essential via triseccant bcd .)

Moreover, ES is closed in S . Take a sequence of essential secants $\{s_i\}_{i=1}^{\infty}$, and suppose $s_i \rightarrow s$ where $s \in \pi_{13}(\mathcal{T})$. First assume $s_i \notin \pi_{13}(\mathcal{T})$. Then for any $\epsilon > 0$, far enough along the sequence each s_i is an ϵ -perturbation of s . (By being within $\epsilon/2$ of an $\epsilon/2$ -perturbation of s .) Thus by definition s is essential. Now assume $s_i \in \pi_{13}(\mathcal{T})$. Then for each s_i there is a sequence $s_i^j \rightarrow s_i$, where s_i^j are essential secants and $s_i^j \notin \pi_{13}(\mathcal{T})$. For each i , there is a j_i such that the sequence $\{s_i^{j_i}\}_i$ of essential secants (not in $\pi_{13}(\mathcal{T})$) converges to s . Again, for any $\epsilon > 0$, far enough along the sequence each $s_i^{j_i}$ is an ϵ -perturbation of s and thus by definition s is essential. \square

COROLLARY 19. *In S , intervals of $\pi_{13}(\mathcal{T})$ between two connected components of essential secants are essential. Intervals of $\pi_{13}(\mathcal{T})$ between a component of essential and a component of inessential secants are essential.*

PROOF. Apply Lemma 18. \square

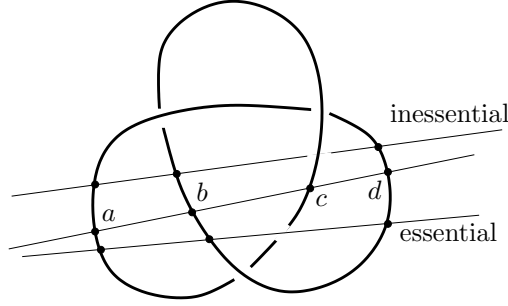


FIGURE 17. Secant bd changes from inessential to essential through trisecant bcd and so trisecant abd changes from inessential to essential through quadriseccant $abcd$.

Note that Lemma 18 says nothing about intervals of $\pi_{13}(\mathcal{T})$ between two connected components of inessential secants. For non-generic knots it is possible to have such isolated essential secants, but this case does not matter for us.

We use this information to describe exactly how the sets of trisecants and essential trisecants are related. Let $\overline{\mathcal{ET}}$ denote the closure of \mathcal{ET} in K^3 and let $\overline{ET} = \pi_{12}(\overline{\mathcal{ET}})$. In a similar way $\overline{ET}^s = \pi_{12}(\overline{\mathcal{ET}}^s)$ and $\overline{ET}^d = \pi_{12}(\overline{\mathcal{ET}}^d)$.

LEMMA 20. *Let K be a generic polygonal knot. Then \mathcal{ET} is a closed subset of \mathcal{T} and $\overline{\mathcal{ET}}$ is a finite union of closed circles and closed intervals in $\overline{\mathcal{T}}$.*

PROOF. Trisecant abc is inessential when the second segment bc is essential. By definition, $\mathcal{ET} = \pi_{23}^{-1}(ES) \cap \mathcal{T}$. Now ES is closed in S and $\pi_{23}^{-1}(ES)$ is closed in K^3 . Thus \mathcal{ET} is a closed subset of \mathcal{T} . By Lemma 18, connected components of $S \setminus \pi_{13}(\mathcal{T})$ are either inessential or essential secants. By examining where $\pi_{23}(\mathcal{T})$ lies in relation to the connected components of essential secants, we may determine which trisecants are essential or inessential.

Let $e_i \times e_j$ be the subset of S consisting of all secants with points lying on edges e_i and e_j in that order. In $e_i \times e_j$, $\pi_{13}(\mathcal{T})$ and $\pi_{23}(\mathcal{T})$ are intervals of trisecants. By modifying the arguments found in Lemma 10, these intervals either do not intersect or intersect in one or two (distinct) points. It is only at these points of intersection, that $\pi_{23}(\mathcal{T})$ can change from inessential to essential. Thus in each $e_i \times e_j \subset K^2$, $\pi_{23}(\mathcal{T})$ either remains inessential, remains essential or changes from inessential to essential at most *twice*. Hence $\overline{\mathcal{ET}}$ is a *finite* union of closed circles and closed intervals in $\overline{\mathcal{T}}$. \square

LEMMA 21. *Let K be a generic polygonal knotted curve. Projection π_{12} is a piecewise smooth immersion of \mathcal{ET} to S . In $\overline{\mathcal{T}}$, \overline{ET} is a finite union of closed intervals and closed circles and in S , ET intersects itself in double points.*

PROOF. As \mathcal{ET} is closed in \mathcal{T} , Lemma 9 shows projection π_{12} is a piecewise smooth immersion of \mathcal{ET} to S . Thus ET inherits the properties of \mathcal{T} found in Lemmas 10 and 11. As $\overline{\mathcal{ET}}$ is also closed in $\overline{\mathcal{T}}$, $\pi_{12}(\overline{\mathcal{ET}})$ is a finite union of closed intervals and circles. \square

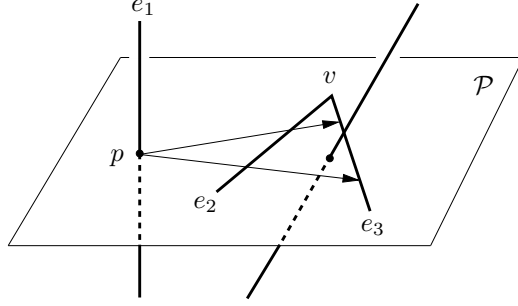


FIGURE 18. The interval of trisecants through pe_2e_3 must eventually be inessential as it ends at degenerate trisecant pvv . Close to vertex v the second segment through e_2e_3 is inessential. Trisecants through e_3e_2p may be essential until they end at degenerate trisecant vvp as the second segment e_2p may be essential.

LEMMA 22. *Let K be a generic polygonal knotted curve. In S , $d_S(\overline{ET}^d, \tilde{\Delta}_-) \geq h$ and $d_S(\overline{ET}^s, \tilde{\Delta}_+) \geq h$, where h is the minimum edge length of K .*

PROOF. As \overline{ET} is a subset of \overline{T} , Lemma 13 gives the result. \square

Thus \overline{ET}^s and \overline{ET}^d can only intersect in $S = K^2 \setminus \tilde{\Delta}$. As ET is closed in T , the boundary of ET will contain points other than $\overline{ET} \setminus ET$ and can be found anywhere in $\overline{S} = S \cup \tilde{\Delta}_+ \cup \tilde{\Delta}_-$. The content of the next lemma is that the points in $\overline{ET} \setminus ET$ must occur on Δ_+ or Δ_- .

LEMMA 23. *In \overline{S} , $\pi_{12}(\overline{\mathcal{E}T} \cap \partial T) \subset \Delta_-$ or Δ_+ .*

PROOF. In \overline{S} , we know that $\partial T = \pi_{12}(\partial T)$ occurs when an interval of adjacent trisecants ends on a degenerate trisecant pvv or vvp (see Lemma 7). From discussion in Section 4, ∂T occurs in S when the degenerate trisecant is of the form pvv . Let the points of the trisecant lie on edges e_1 , e_2 and e_3 in that order and let e_2 and e_3 be adjacent edges with common vertex v . Let e_1 intersect the plane \mathcal{P} spanned by e_2 and e_3 at point p . See Figure 18. Trisecants with points on pe_2e_3 must eventually be inessential. This is because the second segment through e_2 and e_3 must eventually be inessential. As the trisecant converges to pvv , there will be an embedded disk in \mathcal{P} spanned by parts of e_2 , e_3 and the part of the trisecant line between e_2 and e_3 .

On the other hand, the part of ∂T that occurs on Δ_+ or Δ_- might contain \overline{ET} . Use the same set up as in the previous paragraph and in Figure 18, but assume the points of the trisecant lie on $e_3e_2e_1$ in that order. Here the interval of adjacent trisecants end on degenerate trisecant vvp . The second segment e_2p of the trisecant may be essential. Thus there is nothing to prevent an interval of essential trisecants ending at degenerate trisecant vvp and $\pi_{12}(vvp) = vv \in \Delta_+ \cup \Delta_-$. \square

Thus unlike T , ET (and also ET^s , ET^d) are closed in S . Moreover, as points of $\overline{ET} \setminus ET$ must occur on Δ_- or Δ_+ , we need not be concerned about degenerate

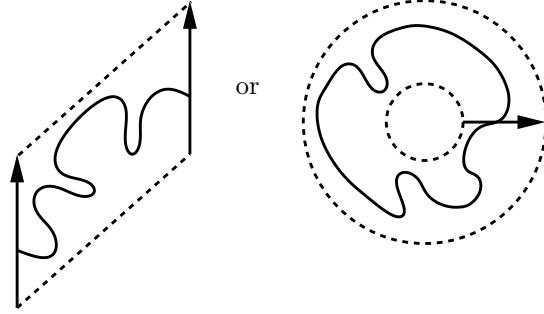


FIGURE 19. A curve winding once around S is a simple closed curve homotopic to the generator 1 in $\pi_1(S) \cong \mathbb{Z}$.

triseccants when considering $ET^s \cap ET^d$. Hence there is no need for a result similar to Lemma 14 in order to show that generic polygonal knotted curves have an essential alternating quadriseccant. We have all the information needed.

6. Essential Alternating Quadriseccants for Generic Polygonal Knots

Lemma 17 and Lemma 23 showed that $ET^s \cap ET^d \neq \emptyset$ in S is all that is needed to prove that an essential alternating quadriseccant exists for generic polygonal knotted curves.

The underlying motivation for the proofs in this section is that idea that every curve going from left to right across a square is intersected by every curve going from the top to the bottom. In our case, we don't have a square, but an annulus. This may be thought of as a square with the left and right edges identified. Thus every curve winding once around the annulus is intersected by every curve going across (from top to bottom).

In Lemma 24 we show that all curves winding once around S (defined below) must be intersected by ET . Thus intuitively ET “goes across” the annulus S . This together with the fact that ET^s avoids the top of S and ET^d avoids the bottom of S , implies that $ET^s \cap ET^d \neq \emptyset$ in S .

DEFINITION. A closed curve winding once *around* S is a simple (continuous) curve $\alpha : [0, 1] \rightarrow S$ which is homotopic to 1 in $\pi_1(S) \cong \mathbb{Z}$. See Figure 19.

LEMMA 24. (*Pannwitz*) *Let K be a nontrivial generic polygonal knotted curve. Every closed curve winding once around S has nonempty intersection with the set of essential triseccants.*

PROOF. This proof contains ideas from each of [Pann, Schm, Kup]. Assume, by way of contradiction, that there is a curve α winding once around S with *empty* intersection with the set of essential triseccants. As α avoids the set of essential triseccants, each point of α is the first two points of a secant or an *inessential* triseccant. To simplify the argument we also assume that the curve α intersects $\pi_{12}(\mathcal{T} \setminus \mathcal{ET})$ transversely away from self-intersections. As the set of essential secants is closed in S , curves avoiding ET are in an open set. Any curve which is not

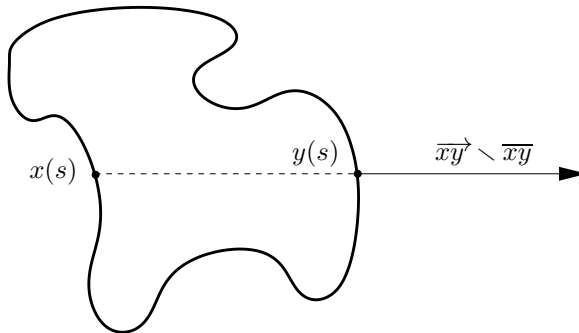


FIGURE 20. An example of a ray $\overrightarrow{xy} \setminus \overline{xy}$. The union of such rays and the point at infinity form a disk with the knot as its boundary.

transverse to $\pi_{12}(\mathcal{T} \setminus \mathcal{ET})$ has another curve arbitrarily close to it which is. Also any curve which passes through a self-intersection of $\pi_{12}(\mathcal{T} \setminus \mathcal{ET})$ has another curve arbitrarily close to it which avoids that self-intersection. Once we have proved the theorem for curves which intersect $\pi_{12}(\mathcal{T} \setminus \mathcal{ET})$ transversely away from self-intersections, we have proved it for *all* curves winding once around S .

Let us first assume each point of α is a secant. We will deal with the case where a point of α is the first two points of an inessential trisecant later. Let $\alpha = (x(s), y(s))$ where we have parameterized $[0, 1]$ with respect to arclength. Construct all (geodesic) rays $\overrightarrow{xy} \setminus \overline{xy}$. This is the collection of rays that start at $y(s)$ (the second point of the secant) and extend to infinity in the direction of the vector from $x(s)$ to $y(s)$. Such a ray is illustrated in Figure 20.

With the point at infinity, the union of the rays forms a spanning disk D with K as its boundary. This is because of the continuity of the curve α and the fact that α is a closed curve homotopic the generator 1 in $\pi_1(S)$. Thus both $x(s)$ and $y(s)$ have degree 1 on the annulus S , and hence traverse the knot once. The assumption that each point on α is a secant means that the rays do not intersect the knot again. However, the disk may have self-intersections. In particular there may be self-intersections on the boundary of the disk as $y(s)$ is not necessarily strictly monotonic around K .

The aim is to apply Dehn's Lemma. To do so, we must find a disk bounded by K which is the image of a PL-map and whose self-intersections do not occur on the boundary. We do this by altering the disk D constructed above. Construct an ϵ -neighborhood N around K . (This is the set of all points within ϵ of the knot.) We take ϵ small enough so that this a regular neighborhood. If necessary perturb N slightly so that D intersects the boundary of N , ∂N , transversely. The disk D intersects ∂N in several kinds of closed curves. Most are homologous to 0 on ∂N , but one will be homologous to a curve γ of homotopy class $(1, m)$. We wish to replace D with a disk that only intersects ∂N in γ .

The curves homologous to 0 on ∂N are disjoint, thus abstractly, these intersection curves are many families of nested circles. To remove these curves, take the outermost intersection circle of a family of nested circles which has some points in

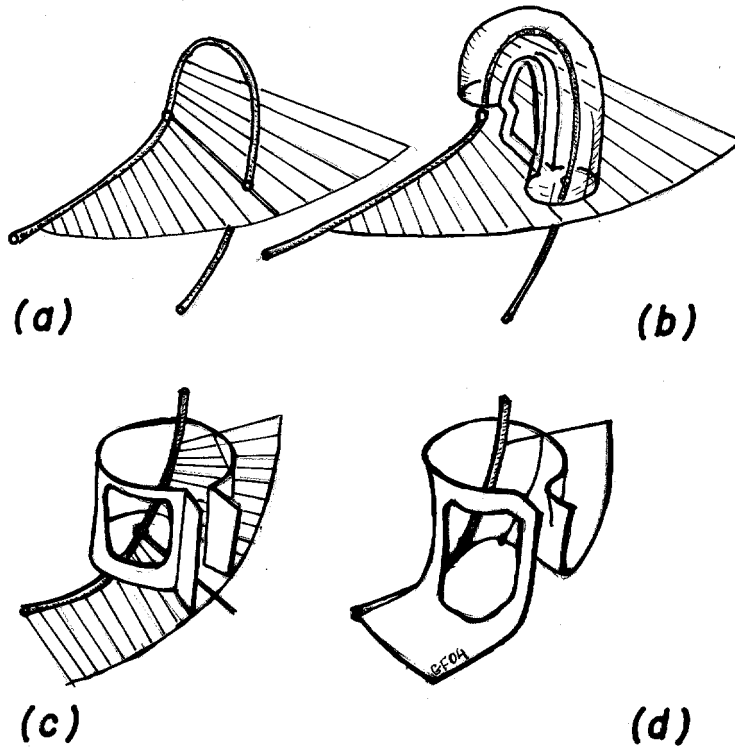


FIGURE 21. Figure illustrating the surgery described in Lemma 24.

its interior lying inside N . The circle bounds a disk A on D and a disk B on ∂N . Replace A by B and push B slightly off ∂N to simplify the intersection. Repeating this process for all other families of nested circles eliminates all curves homologous to 0. Call the new disk so constructed D' and note that D' intersects ∂N only in the curve γ .

We wish to replace the part of D' outside ∂N with a PL disk. Approximate all of D' by a PL disk and use this to approximate γ very closely by a PL curve γ_0 outside ∂N . (Note γ_0 is of the same homology class as γ in the solid toroidal shell they both lie on.) Remove the part of the PL disk inbetween γ_0 and K (this is mostly inside N). Thus we have a PL-disk with γ_0 as its boundary. We would like to join K to γ_0 with an embedded PL collar, but can't as γ_0 might have self intersections. Now γ_0 is homologous to a curve of homotopy class $(1, m)$. Put a smaller regular ϵ -neighborhood N_1 inside N . On ∂N_1 put an embedded $(1, m)$ curve denoted γ_1 . Thus γ_0 and γ_1 live on the boundary of a solid toroidal shell where they are homologous. We use a PL-homotopy in this shell to join γ_0 and γ_1 and then join γ_1 to K by an embedded PL-collar. This creates a new PL-disk with boundary K and with no boundary intersections. By Dehn's Lemma, we may replace this new disk by an embedded one, giving a contradiction to knottedness.

Now on the original curve α there may be a point which is the first two points of an inessential trisecant abc . We assumed that α intersects $\pi_{12}(\mathcal{T})$ transversely away from self-intersections, thus such a point is isolated. In order to apply Dehn's lemma, we perform surgery on the disk D so that the third point c of the trisecant no longer intersects the disk. As the second segment \overline{bc} of trisecant abc is inessential, segment \overline{bc} and one arc of the knot spans a disk \mathcal{D} whose interior avoids the knot. (See Figure 21(a)².) The spanning disk \mathcal{D} intersects the disk D in the segment \overline{bc} . Take a small embedded ϵ -neighborhood of the knot, this intersects both the original disk D and the spanning disk \mathcal{D} . Now make two copies of the spanning disk \mathcal{D} and move them apart. Remove the parts of the copies of \mathcal{D} inside the ϵ -neighborhood. Also remove the parts of D inside the two copies of \mathcal{D} (this is a small strip about the inessential segment \overline{bc}). (See Figure 21(b).) After smoothing things out, the original disk D has been altered so that it does not intersect the third point of the trisecant. However there is a self-intersection near the second point b of the trisecant. This is illustrated in Figure 21(c) and (d). This surgery may be repeated for any other points of the curve α which are the first two points of an inessential trisecant. Thus we obtain a disk D whose interior avoids K and we apply Dehn's Lemma as before to get a contradiction to knottedness. \square

PROPOSITION 25. *Let A and B be two closed subsets of the annulus S , such that A lies outside some neighborhood of Δ_+ and B lies outside some neighborhood of Δ_- . If $A \cap B = \emptyset$, then there is a curve winding once around S avoiding $A \cup B$.*

PROOF. Look at the Mayer-Vietoris sequence:

$$\dots \rightarrow H_1(S \setminus (A \cup B)) \rightarrow H_1(S \setminus A) \oplus H_1(S \setminus B) \rightarrow H_1(S) \rightarrow \dots$$

Using the assumption that A lies outside some neighborhood U of Δ_+ , construct a path α in $U \subset S \setminus A$ which winds once around S . Similarly, as B lies outside some neighborhood V of Δ_- , construct a path β in $V \subset S \setminus B$ which winds once around S with reverse orientation. These paths represent homology classes in $H_1(S \setminus A)$ and $H_1(S \setminus B)$ and the image of these classes in $H_1(S)$ is homologous to $+1$ and -1 respectively. The map $f : H_1(S \setminus A) \oplus H_1(S \setminus B) \rightarrow H_1(S)$ takes a pair (p, q) to the sum of the image of p and the image of q in $H_1(S)$. Thus the image of (α, β) under f is 0. Now the map $g : H_1(S \setminus (A \cup B)) \rightarrow H_1(S \setminus A) \oplus H_1(S \setminus B)$ takes a class γ to the pair $(p, -q)$ where p is the image of γ in $H_1(S \setminus A)$ and q is the image of γ in $H_1(S \setminus B)$. Therefore, by exactness of the sequence at $H_1(S \setminus A) \oplus H_1(S \setminus B)$, there is a γ in $H_1(S \setminus (A \cup B))$ which maps to (α, β) . This γ may be represented by a path in $S \setminus (A \cup B)$. As γ maps to α , γ winds once around S and avoids $A \cup B$. \square

THEOREM 26. *Every nontrivial generic polygonal knotted curve in \mathbb{R}^3 has an essential alternating quadrisecant.*

PROOF. Lemma 22, together with ET closed in S , shows Proposition 25 may be applied to ET^s and ET^d in S . If $ET^s \cap ET^d = \emptyset$ in S , then there exists a path winding once around S avoiding $ET = ET^s \cup ET^d$. This is a contradiction

²Thanks to G. Francis for permission to use Figure 21 and Figure 22.

to Lemma 24 (Pannwitz). Hence in S , $ET^s \cap ET^d \neq \emptyset$, which by Lemma 17 shows that there is at least one essential alternating quadriseccant. \square

7. Main Result

We now extend Theorem 26 by using a limit argument to show that essential alternating quadriseccants exist for *all* nontrivial tame knots K in \mathbb{R}^3 . First, we define the kind of closed curves and the kind of convergence that we need for the limit to make sense.

Milnor [Mil] defined the total curvature of an arbitrary curve as the supremal total curvature of inscribed polygons. Curves of finite total curvature are exactly those rectifiable curves whose unit tangent vectors have bounded variation. Any curve of finite total curvature has well-defined one-sided tangent vectors everywhere; these are equal and opposite except at countably many *corners*. (See also discussion of curves of finite total curvature in CITE.) When considering limits of knots of finite total curvature, we will need to have control over the behavior of tangent vectors. Because knots of finite total curvature have corners, we now define a notion of convergence similar to, but more general than C^1 -convergence.

DEFINITION. Suppose $\{K_i\}_{i=1}^\infty$ is a sequence of (closed) curves of finite total curvature. Then we say K_i *converges in a C^1 sense* to a limit curve K if there is a set of parameterizations of K_i such that for all $\epsilon > 0$ there is an I such that once $i > I$ then for all t , $K_i(t)$ is within ϵ of $K(t)$. Also for all t , left and right unit tangent vectors of $K_i(t)$ and $K(t)$ are within ϵ of each other.

THEOREM 27 (Main Theorem). *Every knotted curve in \mathbb{R}^3 has an alternating quadriseccant.*

PROOF. This uses ideas from [Kup, Schm] that have been altered and extended to suit this case. Given a knotted curve K , since it is tame, there is a homeomorphism $h : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that $h(K)$ is smooth. Furthermore h may be chosen so that h is smooth on $\mathbb{R}^3 \setminus K$ (see [Kup]).

Use Lemma 28 to construct a sequence of generic polygonal knots $\{K_i\}_{i=1}^\infty$ such that K_i converges to K and with the following properties:

- (1) $h(K_i) \cap h(K) = \emptyset$
- (2) $h(K_i)$ converges in a C^1 sense to $h(K)$.

By construction each generic polygonal K_i has the same isotopy type as K , thus each K_i is nontrivial. By Theorem 26 we know that each K_i has an essential alternating quadriseccant $a_i b_i c_i d_i$. By picking a subsequence if necessary we may assume that the original sequence converges to $abcd \in K^4$. We also assume the order of $a_i b_i c_i d_i$ along the knot is $a_i c_i b_i d_i$. From Lemma 15, it is sufficient to show that b and c do not lie on the same straight subarc of K in order to show $abcd$ is a quadriseccant. Thus we examine the limiting behavior of $b_i c_i$. Recall that each (generic) K_i has no quintiseccants, hence $K_i \cap \overline{b_i c_i} = \{b_i, c_i\}$ only.

Let S_i be the essential segment $\overline{b_i c_i}$, let $\gamma_{S_i} = \gamma_{b_i c_i}$ and suppose, by way of contradiction, that b and c lie on the same straight subarc of K . Let $S = \overline{bc}$. Then $\gamma_{bc} = S$ (thus is contained in K) and $h(S) = h(\gamma_{bc}) \subset h(K)$. Take a small

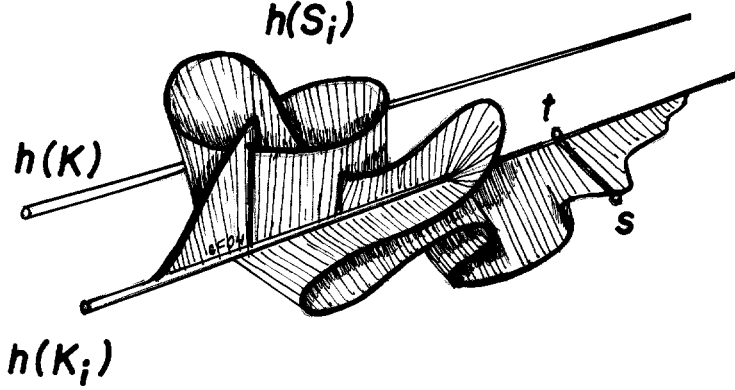


FIGURE 22. Disk constructed in the proof of Theorem 27. The boundary of D is $h(S_i) \cup h(\gamma_{S_i})$ and D is constructed as the union of line segments from $h(S_i)$ to $h(\gamma_{S_i})$.

neighborhood of $h(\gamma_{bc})$ that lies inside the embedded normal tubular neighborhood about $h(K)$. As $h(\gamma_{bc}) = h(S)$, far enough down the sequence $h(S_i)$ and $h(\gamma_{S_i})$ both lie inside this neighborhood. For each point $s \in h(S_i)$ take the unique point $t \in h(K_i)$ lying in the same normal disk to $h(K)$. Take the union of line segments from s to t , as illustrated in Figure 22. This forms a spanning disk D of $h(\gamma_{S_i}) \cup h(S_i)$. But $h(K_i)$ does not intersect D° , the interior of D , else $h(\gamma_{S_i})$ is not a section. (In Figure 22 we see $h(S_i)$ intersects D° and D° has intersections, as allowed in the definition of inessential.) Thus for the knotted theta $\Theta_i = h(K_i) \cup h(S_i)$, the pair $(h(\gamma_{S_i}), h(S_i))$ is inessential. As essential is a topological notion for a knotted theta, the pair (γ_{S_i}, S_i) is inessential and hence secant $b_i c_i$ is inessential, a contradiction. \square

All that remains is to construct the desired sequence of generic polygonal knots.

LEMMA 28. *Let K be a knotted curve and let h be the homeomorphism $h : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ described in [Kup] such that $h(K)$ is smooth and h is smooth on $\mathbb{R}^3 \setminus K$. Then there is a sequence of generic polygonal knots $\{K_i\}_{i=1}^\infty$ such that K_i converges to K with the following properties:*

- (1) $h(K_i) \cap h(K) = \emptyset$
- (2) $h(K_i)$ converges in a C^1 sense to $h(K)$.

PROOF. As $h(K)$ is smooth, there is a sequence of smooth knots $\{K_i^s\}_{i=1}^\infty$ such that $h(K_i^s) \cap h(K) = \emptyset$ and $h(K_i^s)$ converges in C^1 to $h(K)$. For each smooth K_i^s , there is a family of inscribed polygons which are ambient isotopic to K_i^s (see for instance [CF]). We now show that these polygonal knots may be perturbed to give a sequence $\{K_i\}_{i=1}^\infty$ of generic polygonal knots whose images under h are “close” to each $h(K_i^s)$ and hence converge in a C^1 sense to $h(K)$.

By assumption each $h(K_i^s)$ is disjoint from $h(K)$ and so lies outside an open normal embedded tube of radius ϵ_i about $h(K)$. In fact $h(K_i^s)$ can be chosen to lie

in a toroidal shell between two tubes of radius ϵ_i and $2\epsilon_i$ respectively about $h(K)$. The map h is only smooth on the complement of K and its derivatives may become unbounded at K (if K is not smooth for example). However, as a continuous map is bounded on a compact set, the derivatives of h are bounded in the toroidal shell that $h(K_i^s)$ lies inside. So the ratio of the maximum to minimum values of both the first and the second derivatives of h is bounded by some constant. Because these derivatives are bounded, it is a straightforward matter to find a generic polygonal K_i sufficiently close to K_i^s , so that if two corresponding points and tangent vectors are very close to one another in distance and angle respectively, then the image of the points and vectors under h will be very close as well. As $h(K_i^s)$ converges in C^1 to $h(K)$, then $h(K_i)$ converges in a C^1 sense to $h(K)$. Thus $\{K_i\}_{i=1}^\infty$ is the desired sequence of generic polygonal knots. \square

In essence Theorem 27 showed that the limit of essential secants is still a secant. In [DDS] we showed (with some extra assumptions) that for a nontrivial $C^{1,1}$ knot a limit of essential secants is still essential. In fact this result can be generalized to nontrivial knots of finite total curvature in one of two ways. The first is to generalize the arguments found in [DDS]. The second is to recall that being essential is a topological property of a knotted Θ -graph. So in order to show that a limit of essential secants remains essential, it is sufficient to show that nearby knotted Θ s are isotopic. This approach has been pursued in another paper [DS]. We also conjecture that any nontrivial (tame) knot has essential quadriseccants, but in the meantime we claim the following.

COROLLARY 29. *Every nontrivial knot of finite total curvature in \mathbb{R}^3 has an essential alternating quadriseccant.*

PROOF. Given a knot of finite total curvature, there is a family of inscribed polygons which are ambient isotopic to it (see for instance [AB]). By perturbing these slightly, we find a sequence of generic polygonal knots converging in a C^1 sense to K . By Theorem 27 these have essential alternating quadriseccants. Some subsequence of these quadriseccants converges to an alternating quadriseccant, which is essential (as proved in [DS]). \square

8. Corollaries to the Main Theorem

The Main Theorem has two immediate applications. It is used to give alternate proofs to two previously known theorems about the geometry of knotted curves. The existence of alternating quadriseccants for knotted curves captures the intuition that a space curve must loop around at least twice to become knotted. This intuition is also reflected in results about total curvature and second hull of knotted curves.

Recall that for polygonal closed curves the total curvature is the sum of the exterior angles and Milnor [Mil] defined the total curvature of an arbitrary curve as the supremal total curvature of inscribed polygons. In 1929, M. Fenchel [Fen] proved that the total curvature of a closed curve in \mathbb{R}^3 is greater than or equal to 2π , equality holding only for plane convex curves. In 1947, K. Borsuk [Bor] extended this result to \mathbb{R}^n and conjectured the following:

THEOREM 30. *A knotted curve in \mathbb{R}^3 has total curvature greater than 4π .*

This result was first proved around 1949 by both Milnor [Mil] and I. Fáry [Far]. It has since become known as the Fáry-Milnor theorem. In his proof, Milnor used the idea of bridge number³, making the observation that for a knotted curve, there are planes in every direction which cut the knot at least four times. More recently in [CKKS], the theorem was proved using the existence of second hull for a knotted curve (defined later). Here we give a new proof using the existence of alternating quadrisecants.

PROOF. A knotted curve K has an alternating quadrisecant by Theorem 27. An alternating quadrisecant is an inscribed polygon in K with total curvature 4π . By definition, its total curvature is less than or equal to the total curvature of the knotted curve it is inscribed in. Therefore $\kappa(K) \geq 4\pi$. To get a strict inequality, observe that a knot is not coplanar. By repeatedly adding vertices to an alternating quadrisecant, the newly formed inscribed polygon eventually has four vertices non-coplanar, giving it, and hence the knot, total curvature strictly greater than 4π . (See Lemma 31 from [Mil] below.) \square

LEMMA 31. *Adding a new vertex to a closed polygon cannot decrease its total curvature. The total curvature must increase if the new vertex a_j and three adjacent vertices $(a_{j-1}, a_{j+1}, a_{j+2})$ are not coplanar.* \square

The convex hull of a connected set K in \mathbb{R}^3 is characterized by the fact that every plane through a point in the hull must intersect K . If K is a closed curve, then a generic plane must intersect K an even number of times. Thus every plane through each point of the convex hull is cut by K at least twice.

This idea may be generalized as in [CKKS]:

DEFINITION. Let K be a closed curve in \mathbb{R}^3 . Its n th hull $h_n(K)$ is the set of points $p \in \mathbb{R}^3$ such that K cuts every plane P through p at least $2n$ -times. If the intersections are transverse, then (thinking of P as horizontal, and orienting K) there are equal numbers of upward and downward intersections. To handle non-transverse intersections, we again orient K and adopt the following conventions. First, if $K \subset P$, we say K cuts P twice (once in either direction). If $K \cap P$ has infinitely many components, then we say K cuts P infinitely often. Otherwise, each connected component of the intersection is preceded and followed by open arcs in K , with each lying to one side of P . An *upward intersection* will mean a component of $K \cap P$ preceded by an arc below P or followed by an arc above P . (Similarly, a *downward intersection* will mean a component preceded by an arc above P or followed by an arc below P .) A glancing intersection, preceded and followed by arcs on the same side of P , thus counts twice, as both an upward and a downward intersection.

³The bridge number $b(K)$ of a knot K is the minimum number of bridges (or overpasses) occurring in a diagram of the knot, where the minimum is taken over all possible diagrams of K .

Alternating quadriseccants give the geometric intuition that a knot must travel twice around some point in space. Milnor observed that for a knotted curve, there are planes in every direction which cut the knot four times. More generally, there are points through which every plane cuts the knots four times. But these points are precisely the second hull. In other words:

THEOREM 32. *A knotted curve has nonempty second hull.*

This result was originally proved in [CKKS]. This paper conjectured the existence of alternating quadriseccants as another way of proving that the second hull of knotted curves is nonempty.

PROOF. By Theorem 27 we know that a knotted curve K has an alternating quadriseccant $abcd$. Let t be a point on the midsegment \overline{bc} . We claim that t is in the second hull.

Project the knot radially to the unit sphere about t . Let the points a, b be at the north pole (N) and c, d be at the south pole (S) of the sphere. As $abcd$ is an alternating quadriseccant we see that the projected knot visits the poles in the order $NSNS$. To show that t is in the second hull, it suffices to show that the knot projection intersects any great circle at least four times. The same conventions given in Definition 8 apply to counting intersections of K with great circles.

There are two cases. Either the great circles are meridians (passing through the north and south pole) or they are not. Suppose the great circles are not meridians. Then as the projected curve visits the poles in order $NSNS$, it must cut such a great circle at least four times. Now suppose the great circle is a meridian. We divide the projected knot into four arcs, each arc is the part of K between a north and south pole. If all of these arcs lie on the meridian then this counts as an infinite number of intersections. If one of these arcs deviates from the meridian, this counts as two intersections. Thus if two or more arcs leave the meridian, then there are at least four intersections. Suppose that only one arc deviates from the meridian. Such a curve stays in the plane determined by the meridian most of the time and goes to one side once. This gives a knot of bridge number 1, which by [Mil] is the unknot. Thus for a knotted curve, the projected curve intersects all great circles at least four times and the second hull is non-empty. \square

The final application uses the second Main Theorem about essential alternating quadriseccants to improve the known lower bounds of the ropelength of knotted curves from 12 to 15.66.

DEFINITION. Ropelength is the (scale invariant) quotient of length over thickness.

The thickness $\tau(K)$ of a space curve is defined [GM] to be twice the infimal radius $r(x, y, z)$ of a circle passing through any three distinct points of K . J. Cantarella, R. Kusner and J.M. Sullivan [CKS] showed that $\tau(K) = 0$ unless K is $C^{1,1}$. If K is C^1 we can define normal tubes around K , and then $\tau(K)$ is the supremal radius such that this tube remains embedded. Cantarella, Kusner and

Sullivan also proved that any (tame) knot or link has a ropelength minimizer and gave lower bounds for the ropelength of links [CKS]. These are sharp for certain simple cases, where each component of the link is planar. However, these examples are the only known ropelength minimizers, it seems difficult to describe explicitly the shape of any tight knot.

Numerical experiments [Pie, Sul] suggest that the minimum ropelength for a trefoil is slightly less than 16.4 and that there is another tight trefoil with different symmetry and ropelength 18.7. The best lower bound in [CKS] was 10.726, this was improved by Diao [Dia] who showed that any knot has ropelength more than 12.

In [DDS] we use the idea of essential quadrisecants to get better lower bounds for ropelength. For each of the three types of quadrisecants we use geometric arguments to obtain a lower bound for the ropelength of the knot having a quadrisecant of that type. The worst of these three cases is 13.936. However, from Corollary 29 we know that any nontrivial $C^{1,1}$ knot has an essential alternating quadrisecant. This result together with the ropelength bound for a knot with an essential alternating quadrisecant shows that any nontrivial knot has ropelength at least 15.66.

Acknowledgements

The author would like to acknowledge J.M. Sullivan for suggesting this problem and for his endless support and advice. Thanks also to S.B. Alexander, R.L. Bishop and J. Canteralla for their corrections and many helpful suggestions. Finally thanks to G. Francis for the use of two of his pictures and for his help with the other figures. This work was partially supported by a Bourgain Fellowship whilst the author was a graduate student at the University of Illinois at Urbana-Champaign.

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