2020-11-13 Kähler geometry

Prop If $K_X + \Delta$ is $\mathbb{R}$-Cartier on Toric cone then (X, Δ) is klt.

$C = \mu(X)$

$v_1, \dots, v_d$ normal primitive vectors

$l_a = \langle v_a, \cdot \rangle$

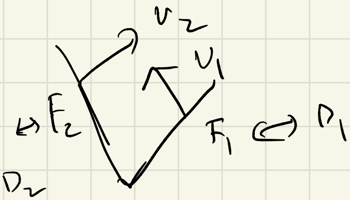
$\pi : Y \to X$ resolution

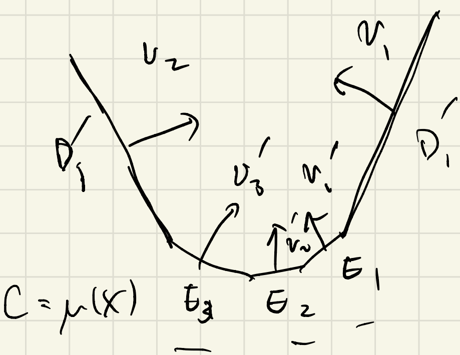
$\mathrm{exc}(\pi) = \bigcup E_i$

$E_i = \mu^{-1}(l(v_i', \cdot) = 0)$

$\sigma(1) = \{v_1, \dots, v_d\}$

$\Sigma(1) = \sigma(1) \cup \{v_i'\}$





$\Delta = \sum (1 - \beta_a) D_a$

$K_X + \Delta$ is $\mathbb{R}$-Cartier

$\Downarrow$ by previous prop

Let $p \in C \cap t^*$ s.t. $l_a(p) = \beta_a$

I will come back here in a moment.

Let us forget above for the moment. ②

Assume $K_X + \sum (1-\beta_a) D_a$ is $\mathbb{R}$-Cartier

(we proved last time)

Put $\Delta = \sum (1-\beta_a) D_a$

and call the pair (X, Δ) a <u>log pair</u>

($\Longleftrightarrow$ $K_X + \Delta$ is $\underset{\mathbb{Q}}{\mathbb{R}}$-Cartier).

Given a proper birational morphism $\pi : Y \to X$, the <u>exceptional set</u> $\mathrm{exc}(\pi) \subset Y$ is the smallest subset in Y such that

$$\pi : Y \setminus \mathrm{exc}(\pi) \to X \setminus \pi(\mathrm{exc}(\pi))$$

is an isomorphism.

For a log pair (X, Δ) a log resolution is a proper birational morphism $\pi : Y \to X$ such that Y is smooth and that

$$\pi^{-1}(\Delta) \cup \mathrm{exc}(\pi)$$

is a divisor with simple normal crossing support. ($\exists$ by Hironaka)

Def Δ' is proper transform of Δ if (3)

$$\Delta' = \sum_a (1-p_a) D_a'$$

$$D_a' = \overline{\pi^{-1}(D_a \cap X^{reg})} \quad \leftarrow \text{closure}$$

$$X^{reg} = X \setminus \{0\}.$$

Write $Exc(\pi) = \bigcup E_i$

$$K_Y + \Delta' + \sum e_i E_i = \pi^*(K_X + \Delta)$$

Def (X, Δ) is klt (Kawamata log terminal) if $e_i < 1$ for $\forall i$.

Another description

$$K_Y + \Delta' = \pi^*(K_X + \Delta) + \sum a_i E_i$$

$$(\text{thus } a_i = -e_i)$$

a_i are called the discrepancies.

(X, Δ) klt $\Longleftrightarrow$ $a_i > -1$

Rm $a_i - 1$ are called the log discrepancies.

We go back to page 1.

$$\pi^*(K_X+\Delta) = -\sum_{u\in\Sigma} \langle p, u\rangle D_u$$

$$= K_Y + \sum_{u\in\Sigma} (1-\langle p,u\rangle) D_u \quad (\text{toric})$$

$$= K_Y + \Delta' + \sum_{v'\in E_i} (1-\langle p, v'\rangle) D_{v'}$$

where $\Delta' = \sum_{a\in\sigma(1)} (1-\langle p,a\rangle) D_a$ is the proper (strict) transform of $\Delta = \sum_a (1-\beta_a) D_a$

$v' = \sum \lambda_a v_a \qquad \lambda_a \geq 0 \qquad (\lambda_a > 0 \text{ for } \exists a)$

$p \in C \qquad \langle p, v_a' \rangle > 0$

$(1-\langle p, v'\rangle) < 1 \qquad \therefore$ klt.

$\boxed{\text{Summary}}$

$$\beta\in\mathcal{B} \overset{\text{previous prop}}{\iff} K_X+\Delta \ \mathbb{R}\text{-Cartier}$$

$$\Longrightarrow (X,\Delta) \text{ klt}$$

$$\Longleftarrow \text{def of klt}$$

$(1) \iff (4)$

Next, we want to show (1) $\Longleftrightarrow$ (5).

To see ~~this~~ we need Abreu-Guillemin theory "Legendre transform".

$X = C(S)$ Toric Kähler cone of dim n $= m+1$.

(compact Toric Kähler case is similar.)

$$\mu : X \longrightarrow C \quad \text{moment map}$$
$$\cup$$
$$F_a \quad \text{facet}$$

$\mu^{-1}(\mathrm{Int}(C))$ is a $(\mathbb{C}^*)^n$-orbit.

$(w^1, \dots, w^n) \in (\mathbb{C}^*)^n$

Write $w^i = e^{z^i}$, $z^i = x^i + \sqrt{-1}\,\theta^i = \log w^i$

logarithmic holomorphic coordinates.

F Kähler potential i.e. $\qquad P_3 = \left\{ \frac{1}{2} \langle 3, \rangle \right\}$

$$\omega = 2i\partial\bar{\partial} F$$

(for the cone case $X = C(S)$, $F = \frac{1}{4} r^2$)

the metric is T^n-invariant, so is F. (6)

$\therefore \quad F = F(x^1, \cdots, x^n)$ does not depend on $\theta^1, \cdots, \theta^n$.

For $X \in \mathfrak{t}$, $\quad i(X)\omega = -d\mu_X$

$$X = \frac{\partial}{\partial \theta^i}, \quad \mu_X =: y_i$$

$$\therefore \quad \omega = -\sum d\theta^i \wedge dy_i = \sum dy_i \wedge d\theta_i$$

$$\omega = 2i\partial\bar{\partial}F = -2i\bar{\partial}\partial F \qquad \frac{1}{2}\left(\frac{\partial}{\partial x^i} - i\frac{\partial}{\partial \theta^i}\right) \times(-i)$$

$$X = \underbrace{X' + \overline{X'}}_{X''} = \frac{1}{2}\left(\frac{\partial}{\partial \theta^i} + i\frac{\partial}{\partial x^i}\right) + \frac{1}{2}\left(\frac{\partial}{\partial \theta^i} - i\frac{\partial}{\partial x^i}\right)$$

$$i(X)\omega = 2i\bar{\partial}(X'F) - 2i\partial(X''F)$$

$$= -\bar{\partial}\frac{\partial F}{\partial x^i} - \partial\frac{\partial F}{\partial x^i} = -d\left(\frac{\partial F}{\partial x^i}\right) \quad F_i$$

$$\therefore \quad y_i = \mu_X = \frac{\partial F}{\partial x^i} = F_i$$

$(y_1, \cdots, y_n, \theta_1, \cdots, \theta_n)$ is called the action-angle coordinates.

From $(x^1, \dots, x^n, \theta^1, \dots, \theta^n)$ log hol coord

to $(y_1, \dots, y_n, \theta_1, \dots, \theta_n)$ action angle coord.

is called the Legendre transform.

$$y_i = F_i$$

$$i\left(\frac{\partial}{\partial \theta_i}\right) \omega = -dy_i$$

$$\omega = \sum_i dy_i \wedge d\theta_i = \sum_{i,j} \frac{\partial F_i}{\partial x_j} dx_j \wedge d\theta_i$$

$$= \sum_{i,j} F_{ij}\, dx^i \wedge d\theta^j$$

$$g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \omega\left(\frac{\partial}{\partial x^i}, J\frac{\partial}{\partial x^j}\right) = \omega\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial \theta^j}\right)$$

$$= F_{ij}$$

Put $G_{ij} \overset{\text{def}}{=} g\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}\right)$. Then

$$F_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = g\left(\frac{\partial y_k}{\partial x^i} \frac{\partial}{\partial y_k}, \frac{\partial y_l}{\partial x^j} \frac{\partial}{\partial y_l}\right)$$

$$= F_{ik} G_{kl} F_{lj}$$

$$F'' = F'' G'' F'' \qquad \therefore F'' G'' = \underline{1}$$

$$\therefore (G_{ij}) = (F_{ij})^{-1}$$

Write as a convention

$$(G^{ij}) = (G_{ij})^{-1} = (F_{ij})$$

So $G^{ij} = F_{ij}$

Lemma If we put $G = \sum_{i=1}^{n} x_i y_i - F$ (*)

then

$$G_{ij} = \frac{\partial^2 G}{\partial y_i \partial y_j}.$$

Def $G(y_1, \dots, y_n)$ is called the symplectic potential (as a funct of $(y_1, \dots, y_n)$.)

Proof of Lemma

same

$\because$ (*) $\Longrightarrow$ $\dfrac{\partial G}{\partial x^k} = y_k + \sum x_i \dfrac{\partial y_i}{\partial x^k} - F_k$

$$= \sum x_i \frac{\partial y_i}{\partial x^k}$$

$$y_i = \frac{\partial F}{\partial x_i}$$

On the other hand

$$\frac{\partial G}{\partial x_k} = \sum_i \frac{\partial G}{\partial y_i} \frac{\partial y_i}{\partial x_k}$$

$$\therefore \; x_i = \frac{\partial G}{\partial y_i}$$

(9)

$$\therefore \left(\frac{\partial^2 G}{\partial y_i \partial y_j}\right) = \left(\frac{\partial x_i}{\partial y_j}\right) = \left(\frac{\partial y_i}{\partial x_j}\right)^{-1}$$

$$= \left(F_{ij}\right)^{-1} = \left(G_{ij}\right)$$

Summary : $F + G = \sum_i x_i y_i$

$$\frac{\partial G}{\partial y_j} = x_j \quad , \quad \frac{\partial F}{\partial x_i} = y_i$$

$$J \frac{\partial}{\partial \theta^i} = -\frac{\partial}{\partial x^i} = -\frac{\partial y_k}{\partial x_i} \frac{\partial}{\partial y_k} = -\sum_k F_{ik} \frac{\partial}{\partial y_k}$$

Multiply J both sides.

$$F_{jk} J \frac{\partial}{\partial y_k} = \frac{\partial}{\partial \theta_i}$$

$$\therefore J \frac{\partial}{\partial y_k} = \sum G_{ik} \frac{\partial}{\partial \theta_i}$$

So (G_{ij}) represents J w.r.T

action angle coordinates.

$$g\left(\frac{\partial}{\partial \theta^i}, \frac{\partial}{\partial \theta^j}\right) = g\left(J\frac{\partial}{\partial \theta^i}, J\frac{\partial}{\partial \theta^j}\right)$$

$$= g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = F_{ij} = G^{ij}$$

Recall $G_{ij} = g\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right)$, so

$$\boxed{g = \sum G_{ij}\, dy^i dy^j + \sum G^{ij}\, d\theta_i\, d\theta_j}$$