Combinatorics, Lecture 3, 2022/05/17

Recall

§3. Exponential generating function

Def. The exponential generating function for the sequence $\{a_0, a_1, a_2, \cdots\}$ is the power series

$$\boxed{f(x) = \sum_{n=0}^{+\infty} a_n \frac{x^n}{n!}}$$

Problem 1. Let $S_n = \#$ selections of n letters from $\{a, b, c\}$ such that both of the numbers of a's and b's are even.

$$\Rightarrow \quad \boxed{S_n = \sum_{\substack{x_1+x_2+x_3=n \\ x_1, x_2 \in \{0,2,4,\cdots\} \\ x_3 \geq 0}} 1}$$

By the previous fact, $S_n = [x^n] f(x)$,

where $f(x) = \left(\sum_{i \in \{0,2,4,6,\cdots\}} x^i\right)^2 \left(\sum_{i \geq 0} x^i\right)$

Problem 2 Let $T_n = \#$ arrangements of n letters from $\{a, b, c\}$ such that both of the numbers of a's and b's are even.

Solution

$$\boxed{T_n = \sum_{\substack{x_1+x_2+x_3=n \\ x_1, x_2 \in \{0,2,4,6,\dots\} \\ x_3 \geq 0}} \frac{n!}{x_1!\, x_2!\, x_3!}}$$

Let $g(x) = \left(\sum_{i \in \{0,2,4,\dots\}} \frac{x^i}{i!} \right)^2 \left(\sum_{i \geq 0} \frac{x^i}{i!} \right)$

We proved $\boxed{[x^n]\, g = \frac{T_n}{n!}}$

By Taylor series,

$$\begin{cases} e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!} \\ e^{-x} = \sum_{n=0}^{+\infty} (-1)^n \frac{x^n}{n!} \end{cases}$$

$$\frac{e^x + e^{-x}}{2} = \sum_{i \in \{0,2,4,\dots\}} \frac{x^i}{i!}$$

$$\Rightarrow g(x) = \left(\frac{e^{x}+e^{-x}}{2}\right)^2 \cdot e^{x}$$

$$= \frac{e^{3x}+2e^{x}+e^{-x}}{4}$$

$$= \sum_{i=0}^{+\infty} \left(\frac{3^i+2+(-1)^i}{4}\right)\frac{x^i}{i!}$$

$$\Rightarrow T_n = \frac{3^n+2+(-1)^n}{4}. \quad \blacksquare$$

Ordinary G.F. is often used to find the number of <u>selections</u> of some combinatorial objects, while exponential G.F. is often used to count the number of <u>arrangements</u> or some combinatorial objects <u>involving ordering</u>

<u>总结</u>:

<u>Fact 1</u>. Let $f(x) = \prod_{j=1}^{n} f_j(x)$.

Then
$$[x^k] f = \sum_{\substack{i_1+i_2+\dots+i_n=k \\ i_j \geq 0}} \left(\prod_{j=1}^{n} [x^{i_j}] f_j \right)$$

Fact 2 Let $f(x) = \prod_{j=1}^{n} f_j(x)$ and

Let $f_j(x) = \sum_{k=0}^{+\infty} a_k^{(j)} \frac{x^k}{k!}, \ \forall j \in [n],$

Then
$$f(x) = \sum_{k=0}^{+\infty} A_k \frac{x^k}{k!},$$

where
$$A_k = \sum_{\substack{i_1+i_2+\dots+i_n=k \\ i_j \geq 1}} \frac{k!}{i_1! i_2! \cdots i_n!} \left(\prod_{j=1}^{n} [x^{i_j}] f_j \right)$$

Lastly, let us mention the following formula.

Lagrange inversion formula: Let $f(x)$

be analytic (i.e., convergent power series)

in a neighborhood of $z=0$ and $f(0)\neq 0$.

If $\boxed{w=\frac{z}{f(z)}}$, then z can be

expressed as a power series

$$z=\sum_{k=1}^{+\infty} c_k w^k$$ with a positive

radius of convergence, where

$$c_k=\frac{1}{k!}\left\{\left(\frac{d}{dz}\right)^{k-1}(f(z))^k\right\}_{z=0}$$ ✓

§2. Basic on graphs

see notes

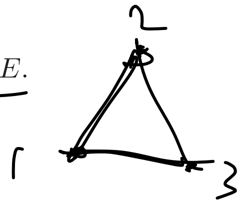
2 Basic of Graphs

In this second part of our course, we will discuss many interesting results in graph theory. We first introduce several basic definitions about graphs.

Definition 2.1. *A graph $G = (V, E)$ consists of a vertex set V and an edge set E, where the elements of V are called vertices and the elements of $E \subseteq \binom{V}{2} = \{\{x, y\} : x, y \in V\}$ are called edges.*

• Definition 2.1 provides the definition of a simple undirected graph, which is the very common graph we concern in this course. The word "undirected" means that the edge set E contains unordered pairs. Otherwise, G is called a directed graph. A graph is simple if it has no loops or multiple edges. A loop is an edge whose endpoints are equal. Multiple edges are edges having the same pair of endpoints.

• We say vertices x and y are adjacent if $\{x, y\} \in E$, write $x \sim_G y$ or $x \sim y$ or $xy \in E$.



• We say the edge xy is incident to the endpoints x and y.

• Let $e(G)$ be the number of edges in G, i.e., $e(G) = |E(G)|$.

• The degree of a vertex v in G, denoted by $d_G(v)$, is the number of edges in G incident to v.

• The neighborhood of a vertex v is the set of vertices that are adjacent to v, i.e., $N_G(v) = \{u \in V(G) : u \sim v\}$. Thus we have $d_G(v) = |N_G(v)|$.

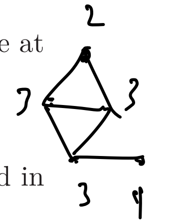
• A graph $G' = (V', E')$ is a subgraph of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E \cap \binom{V'}{2}$, i.e., $G' \subseteq G$.

• A subgraph $G' = (V', E')$ of $G = (V, E)$ is induced, if $E' = E \cap \binom{V'}{2}$, write $G' = G[V']$.

Definition 2.2. *Two graphs $G = (V, E)$ and $G' = (V', E')$ are isomorphic if there exists a bijection $f : V \to V'$ such that $i \sim_G j$ if and only if $f(i) \sim_{G'} f(j)$.*

• A graph on n vertices is a complete graph (or a clique), denoted by K_n, if all pairs of vertices are adjacent. So we have $e(K_n) = \binom{n}{2}$.

• A graph on n vertices is called an independent set, denoted by I_n, if it contains no edge at all.

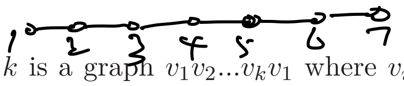


• Given a graph $G = (V, E)$, its complement is a graph $\overline{G} = (V, E^c)$ with $E^c = \binom{V}{2} \backslash E$.

• The degree sequence of a graph $G = (V, E)$ is a sequence of degrees of all vertices listed in a non-decreasing order.

3, 3, 3, 2, 1

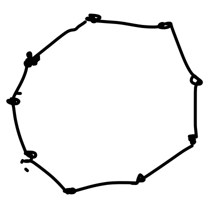
• The path P_k of length $k - 1$ is a graph $v_1v_2...v_k$ where $v_i \sim v_{i+1}$ for $i \in [k-1]$ and $v_j \neq v_l$ for any $j \neq l \in [k]$. Note that the length of a path P (denoted by $|P|$) is the number of edges in P.



P_7 has 6 edges

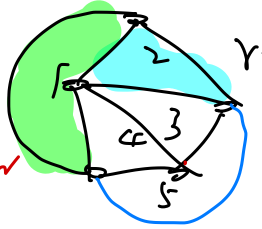
• A cycle C_k of length k is a graph $v_1v_2...v_kv_1$ where $v_i \sim v_{i+1}$ for $i \in [k]$, $v_{k+1} = v_1$, and $v_j \neq v_l$ for any $j \neq l \in [k]$.

• Let G be a simple graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G) = \{e_1, \ldots, e_m\}$. The adjacency matrix of G, denoted by $A(G)$, is the n-by-n matrix in which entry $a_{i,j}$ is the number of edges in G with endponts $\{v_i, v_j\}$. The incidence matrix $M(G)$ is the n-by-m matrix in which entry $m_{i,j}$ is 1 if v_j is an endpoint of e_j and 0 otherwise.



K_5 –

not planer



$r = 6$

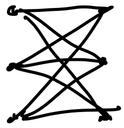
$v = 5$

$e = 9$

• A graph G is planar, if we can draw G on the plane such that its edges intersect only at their endpoints.

Remark 2.3 (Euler's Formula). *Let $G = (V, E)$ be a connected planar graph with v vertices and e edges, and let r be the number of regions in which some given embedding of G divides the plane. Then $v - e + r = 2$.*

Exercise 2.4. *Show that K_4 is planar but K_5 is not.*



$K_{3,3}$

Exercise 2.5. *Show that $K_{3,3}$ is not planar.*

The following Handshaking Lemma is the most basic lemma in graph theory.

Lemma 2.6 (Handshaking Lemma). *In any graph $G = (V, E)$,*

$$\sum_{v \in V} d_G(v) = 2e(G).$$

double counting

Proof. Let $F = \{(e, v) : e \in E(G), v \in V(G)$ such that v is incident to $e\}$. Then

$$\sum_{e \in E(G)} 2 = |F| = \sum_{v \in V} d_G(v).$$

∎

Corollary 2.7. *In any graph G, the number of vertices with odd degree is even.*

Proof. Let $O = \{v \in V(G) : d(v)$ is odd$\}$ and $\mathcal{E} = \{v \in V(G) : d(v)$ is even$\}$. Then by Lemma 2.6,

$$2e(G) = \sum_{v \in O} d_G(v) + \sum_{v \in \mathcal{E}} d_G(v).$$

Thus we have $\sum_{v \in O} d_G(v)$ is even, moreover we have $|O|$ is even. ∎

Corollary 2.8. *In any graph G, if there exists a vertex with odd degree, then there are at least two vertices with odd degree.*

§3. Double counting

The basic setting of the double counting technique is as follows. Suppose we have two finite sets A and B, and a subset $S \subseteq A \times B$. If $(a,b) \in S$, then we say a and b is <u>incident</u>.

For $\forall\, a \in A$, let N_a be the number of elements $b \in B$ satisfying $(a,b) \in S$;

For $\forall\, b \in B$, let N_b be 〃 〃 〃

〃 $a \in A$ satisfying $(a,b) \in S$.

Then
$$\boxed{\sum_{a \in A} N_a = |S| = \sum_{b \in B} N_b}$$

<u>Thm 1</u>. Let $H(n) = \sum_{i=1}^{n} \frac{1}{i}$ be the n^{th} Harmonic number. Let $T(j)$ be

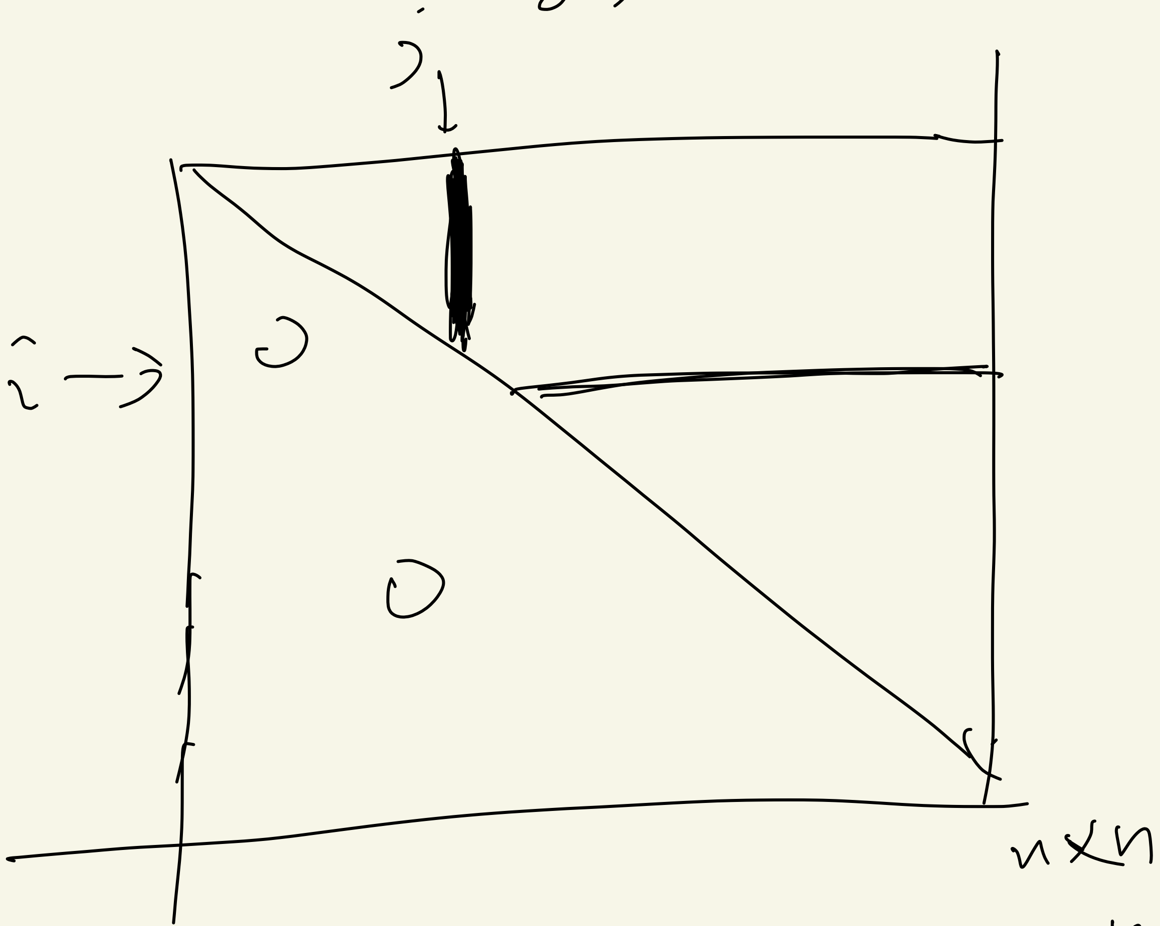
the number of divisions of the integer j.

Let $\overline{T(n)} = \frac{1}{n} \sum_{j=1}^{n} T(j)$.

Then $\left| H(n) - \overline{T(n)} \right| < 1$

<u>Pf</u>: Define a table $X = (x_{ij})$,

where $x_{ij} = \begin{cases} 1, & i \mid j \\ 0, & \text{otherwise} \end{cases}$



Then $\sum_{j=1}^{n} T(j) = \sum_{1 \le i \le j \le n} x_{ij} = \sum_{i=1}^{n} \left\lfloor \frac{n}{i} \right\rfloor$

$$\Rightarrow \quad \overline{T(n)} = \frac{1}{n}\sum_{j=1}^{n} T(j) = \frac{1}{n}\sum_{i=1}^{n}\left\lfloor \frac{n}{i} \right\rfloor$$

$$\Rightarrow \quad \left|\overline{T(n)} - H(n)\right| = \left|\frac{1}{n}\sum_{i=1}^{n}\left\lfloor \frac{n}{i}\right\rfloor - \sum_{i=1}^{n}\frac{1}{i}\right|$$

$$< 1 \qquad \blacksquare$$

§4. Sperner's Thm

Def: Let $\mathcal{F} \subseteq 2^{[n]}$ be a family of subsets of $[n]$. We say $\mathcal{F}$ is an independent system (or independent), if for any two subsets $A, B \in \mathcal{F}$, we have $A \nsubseteq B$ and $B \nsubseteq A$.

In other words, there is NO "containment" relationship between any two sets of $\mathcal{F}$.

e.g. $\forall$ fixed k, $\binom{[n]}{k}$ is independent.

Sperner's Theorem For any independent system $\mathcal{A}$ of $[n]$, we have $|\mathcal{A}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Def. A chain of subsets of $[n]$ is a sequence of distinct subsets

$$A_1 \subsetneq A_2 \subsetneq A_3 \subsetneq \cdots \subsetneq A_k.$$

A maximal chain is a chain such that NO other subsets of $[n]$ can be inserted into it to find a longer chain.

We have the following observations.

Any maximal chain looks like

$$\emptyset \subset \{x_1\} \subset \{x_1, x_2\} \subset \{x_1, x_2, x_3\} \subseteq \cdots \subset [n]. \quad (*)$$

Claim: There are exactly $n!$ maximal chains.

Why? This is because that every such a

maximal chain defines a unique permutation

$\pi : [n] \to [n]$, where $\pi(i) = x_i$. $\boxed{\Sigma}$

Pf of Sperner's thm (1st proof, double-counting)

Let $\mathcal{F}$ be an indep. system of $[n]$.

Now we count the number of pairs

$(\mathcal{C}, A)$ satisfying the following conditions:

(1) $\mathcal{C}$ is a maximal chain of $[n]$,

(2) $A \in \mathcal{C} \cap \mathcal{F}$

We have

$$(*) \quad \sum_{\mathcal{C}} N_{\mathcal{C}} = \#\text{pairs } (\mathcal{C}, A) = \sum_{A \in \mathcal{F}} N_A ,$$

where $N_{\mathcal{C}}$ is the number of subsets $A \in \mathcal{F} \cap \mathcal{C}$,

and N_A " " — of maximal chains $\mathcal{C}$ with $A \in \mathcal{C}$.

It is key to observe that

- $N_e \leq 1 \quad \forall\, \ell$
- $N_A = |A|!\,(n-|A|)!$

$$\underbrace{\emptyset \subsetneq \cdots \subsetneq \{x_1, \ldots}_{|A|!}\overset{A}{\uparrow}\underbrace{, x_a\} \subsetneq \cdots \subsetneq [n]}_{(n-|A|)!}$$

$\Rightarrow$ We have

$$n! = \sum_{e} 1 \geq \sum_{e} N_e = \sum_{A \in \mathcal{F}} N_A$$

$$= \sum_{A \in \mathcal{F}} |A|!\,(n-|A|)! = \sum_{A \in \mathcal{F}} \frac{n!}{\binom{n}{|A|}}$$

$$\geq \sum_{A \in \mathcal{F}} \frac{n!}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} = \frac{n!}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} |\mathcal{F}|$$

$$\Rightarrow \quad |\mathcal{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}. \qquad \boxed{\Sigma}$$

Remark: Sperner's Thm is tight.

Def. A chain is symmetric, if it consists of subsets of sizes $k, k+1, \cdots, n-k$ for some $k \geq 0$.

e.g. $n=4$, $\{1\} \subset \{1,2\} \subseteq \{1,2,4\}$ ✓

$\{1\} \subseteq \{2,3,4\}$ ✗

Thm 2. The family $2^{[n]}$ can be partitioned into a disjoint union of symmetric chains.

Pf 1. We can prove by induction on n.

Base case: $n=1$, $2^{[1]} = \{\phi, \{1\}\}$ ✓

Now we may assume that $2^{[n]}$ can be partitioned into a disjoint union of symmetric chains, say $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_t$.

Consider $2^{[n+1]}$.

For any $\mathcal{C}_i = \{P_k \subseteq P_{k+1} \subseteq \dots \subseteq P_{n-k}\}$

define two new symmetric chains for $2^{[n+1]}$:

$$\mathcal{C}_i' = \{P_{k+1} \subseteq P_{k+2} \subseteq \dots \subseteq P_{n-k}\}$$

and
$$\mathcal{C}_i'' = \{P_k \subseteq P_k \cup \{n+1\} \subseteq P_{k+1} \cup \{n+1\} \subseteq \dots \subseteq P_{n-k} \cup \{n+1\}\}$$

We assert that

$\bigcup_i \{\mathcal{C}_i', \mathcal{C}_i''\}$ is a disjoint

union of symmetric chains for $2^{[n+1]}$. $\blacksquare$