2020-10-30 Kähler geometry

Last time. Berman, ...

<u>Thm</u> (Chen-Donaldson-Sun, Tian)

A Fano mfd has a K-E metric if and only if it is K-polystable.

Other proofs.

- Aubin's continuity method (Datar-Szekelyhidi)
- Ricci-flow proof (Chen-Sun-Wang)
- Variational method (Berman-Boucksom-Jonsson)

K-stability Yaxin Liu 11/24, 27, 12/1, 4

Toric Kähler geometry.

<u>Def</u> A compact Kähler manifold (X^n, ω) is called Toric if $\exists$ effective, isometric, holomorphic action of T^n. ($(\mathbb{C}^*)^n$-action)

$\omega = g(\cdot, J\cdot)$. $\mathfrak{t} = \mathrm{Lie}(T^n)$.

Assume there is a moment map $\mu : X \longrightarrow \mathfrak{t}^*$ (or equivalent the action is Hamiltonian.)

For $v \in \mathfrak{t}$ (this is identified with a vector field on X.), $\exists\ \mu_v \in C^\infty(X)$ s.t. $i(v)\omega = -d\mu_v$.

$p \in X$, $\langle \mu(p), v \rangle = \mu_v(p)$.

Thm (Atiyah, Guillemin-Sternberg).

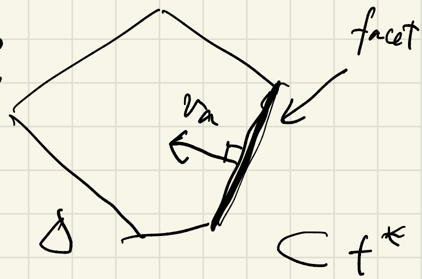
The image of μ is a convex polytope Δ.

$$\Delta = \{ p \in \mathfrak{t}^* \mid l_a(p) \geq \lambda_a \quad \text{for} \quad a = 1, \dots, d \}$$

$d = \#$ of facets.

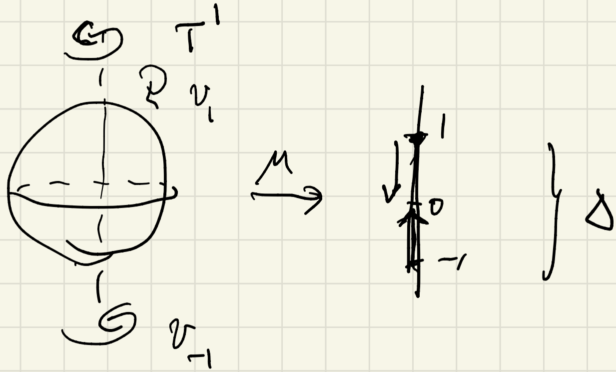
$l_a(p) = \langle v_a, p \rangle$, $\lambda_a \in \mathbb{R}$

$v_a \in \mathfrak{t}$



$$F_a = \{ p \in \Delta \mid l_a(q) = \lambda_a \}$$ facet.

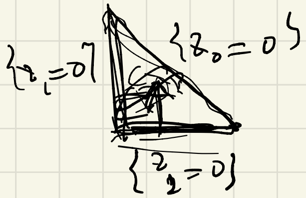
$\mathbb{CP}^1 = S^2$



($\mathbb{C}^*$-action)

F_a = the fixed point set of S^1-action generated by v_a

$\mathbb{C}P^2$ $\quad (z_0 : z_1 : z_2) \quad T^2$

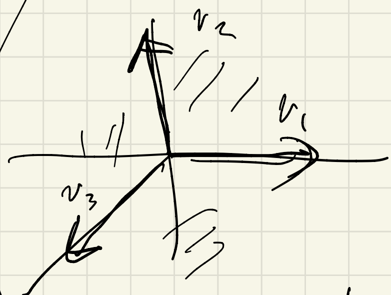


$$\left(e^{i\theta_1} z_0,\ e^{-(i\theta_1+i\theta_2)} z_1,\ e^{i\theta_2} z_2\right)$$

$\theta_1, \theta_2 \in \mathbb{R}$.

$\theta_1 + \theta_2 = 0 \qquad \theta_2 = -\theta_1$

On $\mathring{\Delta}$, $\exists\ (\mathbb{C}^*)^2$-action



t

"fan"

Algebraic geometry

Tadao Oda
Fulton
....

In general

$v_a \in \mathbb{Z}^n$ primitive. $\left(\Leftrightarrow \text{ if } l v_a \in \mathbb{Z}^n \text{ then } l = 1\right)$

Lemma $\quad X$ smooth Toric

$(*)$ If $F_{a_1} \cap \cdots \cap F_{a_n} = pt \in$ vertex of Δ.

then $v_{a_1}, \ldots, v_{a_n}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

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Delzant construction.

Given such (Δ, F_a, v_a) with ($*$) then $\exists$ smooth Toric Kähler manifold.

(Obtained by symplectic reduction (Marsden-Weinstein reduction) from a flat $\mathbb{C}^d$ (maybe).)

Homework. (Read Victor Guillemin's book. combinatorial ....)

M: Toric Fano manifold.

Thm (XuJia Wang - Xiaohua Zhu).

A Toric Fano manifold alway has a Kähler-Ricci soliton

$$\mathrm{Ric} - \omega = i\partial\bar\partial F \qquad \mathrm{grad}\, F \ \text{hol}$$

In particular

A toric Fano has a K-E metric $\Leftrightarrow$ $\mathrm{Fut} = 0$

original

$\Leftrightarrow$ K-poly stable (for Fano) de Borbon - Legendre. arXiv 2005

Geometry of Riemannian cone manifold.

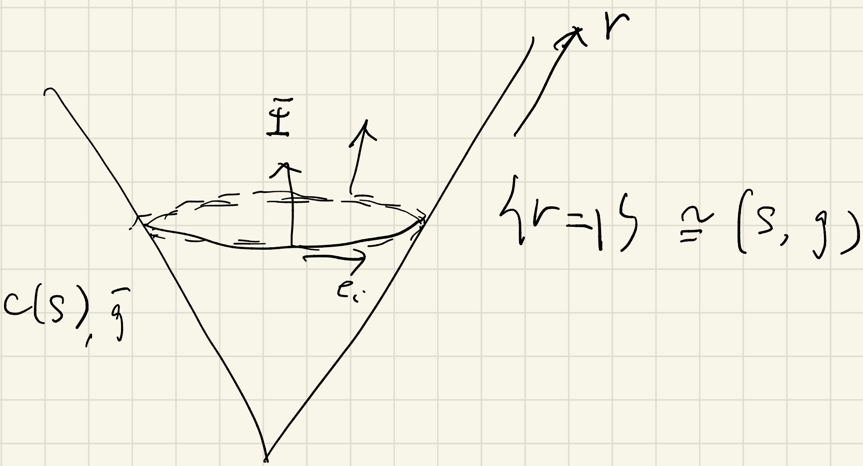
Def Let (S, g) be a Riemannian manifold with Riemannian metric g; $\dim S = k$.
The Riemannian cone $(C(S), \bar{g})$ is a Riemannian manifold diffeomorphic to

$$C(S) \cong \mathbb{R}^{+} \times S$$

with

$$\bar{g} = dr^2 + r^2 g$$

r the standard coordinate of $\mathbb{R}^{+}$.



S is often identified with the submanifold $\{r=1\}$ "cross section of $C(S)$"

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Prop (S,g) Einstein $\Leftrightarrow$ $(C(S), \bar{g})$ is Ricci-flat

$Ric = (k-1) g$ $\qquad$ ($Ricci \equiv 0$).

Proof Let R' be the curvature of $C(S)$, and h be the second fundamental form of $S = \{r = 1\}$. $\boxed{h = -g}$ totally umbilical. (Exercise)

Then Gauss equation

$$R'(X,Y,Z,W) = R(X,Y,Z,W) - h(X,Z)h(Y,W) + h(Y,Z)h(X,W).$$

If $e_1, \dots, e_k$ are orthonormal basis of T_pS,

$$\sum_{i=1}^{k} R'(X,e_i,Z,e_i) = \sum_{i=1}^{k} R(X,e_i,Z,e_i) - k\, g(X,Z) + \underbrace{\sum g(e_i,Z)\, g(X,e_i)}_{g(Z,X)}$$

$$= \underline{\sum_{i=1}^{k} R(X,e_i,Z,e_i)} - (k-1)\, g(X,Z)$$

Thus $\underline{Ric} = (k-1) g$ $\Leftrightarrow$ RHS $= 0$.

$$\therefore \sum_{i=1}^{k} R'(X,e_i,Z,e_i) = 0$$

If Φ is an orthonormal vector to S then

$$R'(X, \Phi, Z, \Phi) = 0 \qquad \forall X, Z$$

(exercise, use

$$\nabla'_X Y - \nabla_X Y = h(X, Y) = -g(X, Y)$$
)

Thus $R'_{ic} = 0$.

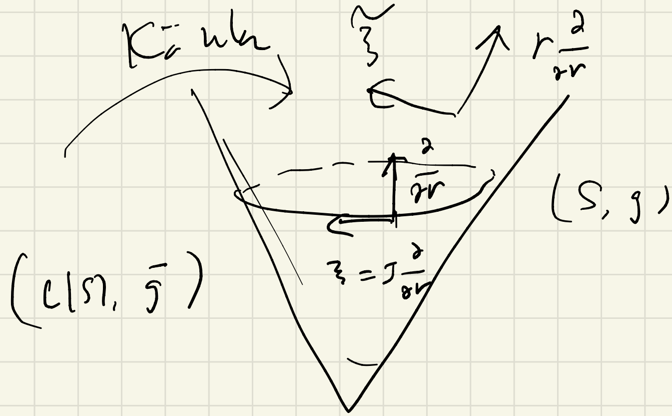
Sasakian geometry. (Boyer - Galicki)

Boyer-Galicki 3-Sasakian manifolds. Surveys in Differential Geometry vol 6. 1999, 128-184.

But just read section 2.1. 3-pages.

<u>Def</u> A Riemannian manifold (S, g) of dim $2m+1$ is called a Sasaki (Sasakian) manifold if its Riemannian cone $(C(S), \bar{g})$ is Kähler.

$$\dim_{\mathbb{C}} C(S) = m+1.$$



$r\frac{\partial}{\partial r}$ is called the radial vector field.

Let J be the complex structure of $C(S)$. Then

$\tilde{\xi} = J r \frac{\partial}{\partial r}$ is called the Reeb vector field.

Restricted to $S = \{r = 1\}$, put

$\xi = J\frac{\partial}{\partial r}$ Tangent to S.

is called the Reeb vector field.

Prop ξ is a Killing vector field i.e.

$$\nabla_i \xi_j + \nabla_j \xi_i = 0. \iff L_\xi g = 0$$

Proof

$$g(\nabla_X \xi, Y) = \bar{g}(\nabla_X J\partial_r, Y) \overset{\nabla J = 0}{=} \bar{g}(J \underbrace{\nabla_X \partial_r}_{X}, Y)$$

$= \bar{g}(JX, Y) \leftarrow$ skew in X, Y.

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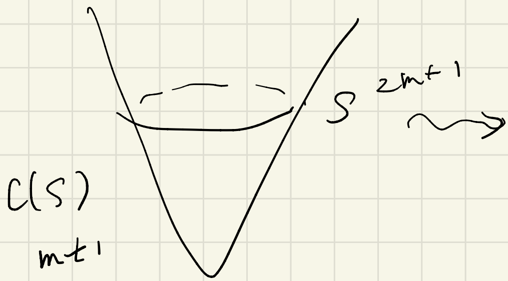
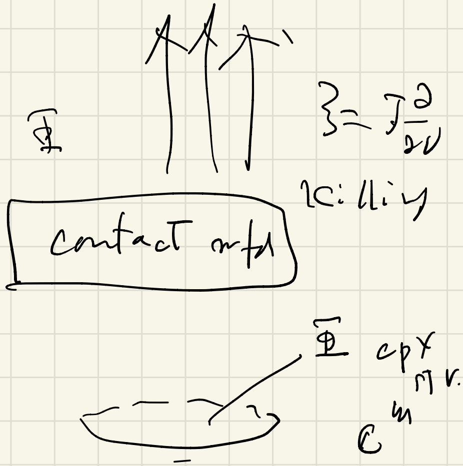
(exercise $\nabla_X \partial r = X$.)





We will see this orbit space is Kähler

Prop 1-form η (eta) on $S = \{r = 1\}$ defined by

$$\eta(Y) = g(\xi, Y)$$

is a contact form. In fact

$$d\eta(X, Y) = 2g(\Phi X, Y)$$

where $\Phi(X) = \nabla_X \xi$. Proof omitted.

Convincing example.

$$S = S^{2n+1} \subset \underline{\mathbb{C}}^{n+1} \cong \mathbb{R}^{2n+2}$$

$\uparrow$ standard sphere

$$\xi = \sum -y^i \frac{\partial}{\partial x^i} + x^i \frac{\partial}{\partial y^i}.$$

Euler.

orbit space $\mathbb{C}P^n$.