

Reference

• [P85] Pansu, "Pincement des variétés à courbure négative", 1985-1986.

• [KM62] Kervaire, Milnor, "Group of homotopy spheres I", 1962

• [L83] Levine, "Lectures on groups of homotopy spheres", 1983.

Identify $G(\sqrt{2})$ with $\text{Isom}(\mathbb{H}^n)$ as follows:

$$\mathbb{H}(\sqrt{2}) \xrightarrow[\cong]{f} \mathbb{H} = \{y \in \mathbb{R}^{n+1} \mid \psi_n(y) = y_1^2 + \dots + y_n^2 - y_{n+1}^2 = -1, y_{n+1} > 0\}.$$

$$\{x \in \mathbb{R}^{n+1} \mid \phi_n(x) = x_1^2 + \dots + x_n^2 - \sqrt{2} x_{n+1}^2 = -1, x_{n+1} > 0\}$$

$$(x_1, \dots, x_n, x_{n+1}) \mapsto (x_1, \dots, x_n, \sqrt[4]{2} x_{n+1}).$$

$$\begin{array}{ccc} G(\sqrt{2}) & \xrightarrow[\cong]{\downarrow} & \text{Isom}(\mathbb{H}^n) \\ \cap & & \cap \\ O(\mathbb{R}^{n+1}, \phi_n) & & O(\mathbb{R}^{n+1}, \psi_n) \\ A & \longmapsto & f \circ A \circ f^{-1}. \end{array}$$

To obtain a compact hyp. n -mfd $\mathbb{H}^n/\Gamma$, it suffices to find a discrete, torsion-free, cocompact subgroup Γ of $G(\sqrt{2})$ $\xrightarrow{\quad} \mathbb{H}^n/\Gamma \text{ compact}$
 $\Updownarrow$
 $\Gamma \text{ cocompact}$

Fact: let Γ be a discrete subgroup of $\text{Isom}^+(\mathbb{H}^n)$.
 Then $\Gamma \curvearrowright_{\text{free}} \mathbb{H}^n \iff \Gamma$ is torsion free.

cf. [BP, Lemma E.1.8].

$$\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\} \rightarrow \text{ring of integers in the field } \mathbb{Q}(\sqrt{2}).$$

$$\Gamma_n \triangleq G(\sqrt{2}) \cap GL(n+1, \mathbb{Z}[\sqrt{2}])$$

(Note that $G(\sqrt{2}) \subset GL(n+1, \mathbb{R})$).

Fact: Γ_n is a discrete, cocompact subgroup of $G(\sqrt{2})$.
cf. [BP, P.188-190]

Now it suffices to find a torsion-free subgroup of Γ_n with finite index

Let p be a prime. $\mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z}$.

For $a \in \mathbb{Z}$, let a_p be the image of a under $\mathbb{Z} \rightarrow \mathbb{Z}_p$.

$$\mathbb{Z}_p[\sqrt{2}] \cong \{c + d\sqrt{2} \mid c, d \in \mathbb{Z}_p\}.$$

$$\text{Consider } \pi_p: GL(n+1, \mathbb{Z}[\sqrt{2}]) \rightarrow GL(n+1, \mathbb{Z}_p[\sqrt{2}])$$

$$(a_{ij} + b_{ij}\sqrt{2}) \mapsto (a_p^i + b_p^i\sqrt{2})_{ij}.$$

$\Rightarrow \ker(\pi_p)$ is normal subgroup of finite index in $GL(n+1, \mathbb{Z}[\sqrt{2}])$.

Fact: $\ker(\pi_p)$ is torsion-free for all but finitely many p .
(cf. [BP, Lemma E.3.4]).

For each p s.t. $\ker(\pi_p)$ torsion-free

$$\Gamma_n^p \triangleq \Gamma_n \cap \ker(\pi_p) = G(\sqrt{2}) \cap \ker(\pi_p).$$

is torsion-free and has finite index in Γ_n .

$\Rightarrow \mathbb{H}^n / \Gamma_n^p$ is a closed hyp. mfd.

These are ϕ_n -mfd's in [GT87].

Re: $\mathbb{H}^n / \Gamma_n^p$ has injectivity radius $\gg 0$ if $p \gg 0$.

Prop ([GT87]) For every $n \in \mathbb{Z}_{\geq 2}$ and every $R > 0$,
 $\exists$ oriented ϕ_n -mfd N and a totally geodesic oriented

submfd V of codim 2 in N s.t.

(i) $\text{Rad}^\perp(V) \geq \mathbb{R}$

(ii) $[V] = 0 \in H_{n-2}(N)$; moreover V bounds a connected oriented totally geodesic hypersurface $W \subseteq N$.

$\Downarrow$

The construction of branched coverings N_s can be performed on these ϕ_n -mfd N .

Certain ones among these N_s do not have homotopy type of locally symmetric space ($\Rightarrow$ do not support a hyp. metric).

cf. [GT 87]

[P85, p.85. Section 3].

From [A00] we also have

Th 3.17 Given $\alpha > 0$, let N be a ϕ_n -mfd and let V be a totally geodesic oriented submfd of codim 2 in N such that $\text{Rad}(N) \geq 10\alpha$, and $\text{Rad}^\perp(V) \geq 5\alpha$. Then $\exists$ metric g_s on N_s s.t. ($K(g_s)$ denotes sectional curvature) $\rightarrow$ injectivity radius

(1) $K(g_s) < 0$ in a tubular neighborhood $T_R(V)$ of radius $R < \alpha$.

(2) $K(g_s) \equiv -1$ in $N_s - T_R(V)$

(3) $\exists p \in N_s - T_R(V)$ s.t. $d(p, T_R(V)) > 3\alpha$

Fact 3.18 : For $n \geq 4$ and $\alpha > 0$,

$\exists N^n$ s.t. the assumption of Th 3.17 holds and $N_s \not\cong$ locally symmetric space for all $s > 1$.

Prop 3.19 Let N be a closed, oriented real hyp. mfd and $V^{n-2} \subset W^{n-1} \subset N$

where V, W are totally geodesic submfd in M .
 Let $N_s \xrightarrow{f} N$ be s -sheeted cover branched at V as described before. Then the total
rational Pontryagin class $p(N_s) = 1$ → cf. [A00, Chapter 4].

Rk: For any closed real hyp. mfd $M \Rightarrow p(M) = 1$.

(Reason: By Sullivan's result, $\exists$ finite cover
 $\hat{M} \xrightarrow{g} M$ s.t. $T\hat{M} \oplus \varepsilon^1 \cong \varepsilon^{n+1}$
 $\Rightarrow p(\hat{M}) = 1$ and $H^*(M; \mathbb{Q}) \xrightarrow{g^*} H^*(\hat{M}; \mathbb{Q})$ monic cf. Hatcher's Alg. Top. P. 321
 $\Rightarrow g^* TM = T\hat{M} \Rightarrow g^* p(M) = p(\hat{M}) = 1$
 $\Rightarrow p(M) = 1$.)

pf Th 3.14: By Remark 3.18, for $n = 4k-2, k > 1$

and $\alpha \gg 0$ (How large will be apparent later),
 take N s.t. Th 3.17 holds and

$N_s \neq$ locally symmetric space for all $s > 1$.

$\Rightarrow \exists$ negatively curved metric g_s on N_s and

a hyperbolic ball $B(p, 3r) \subset N_s$

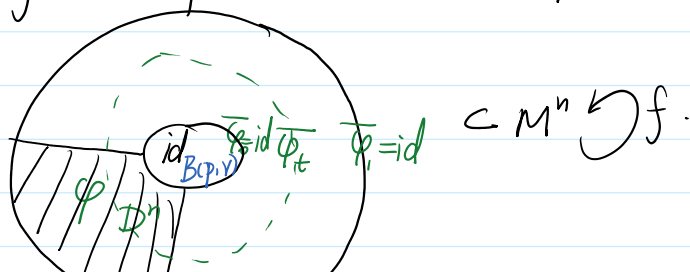
$$(M, g) \stackrel{\cong}{=} (N_s, g_s).$$

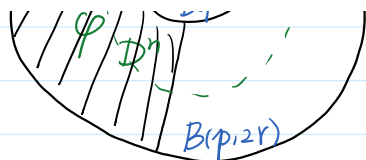
Strategy: find $f \in \text{Diff}(M)$ s.t. (1) (2) holds

$$\begin{aligned} (1) \quad \pi_0 \text{Diff}(M) &\longrightarrow \pi_0 R^{\leq 0}(M) \\ [f] &\longmapsto [f_* g] = [g] \end{aligned}$$

$$(2) \quad [f] \in \pi_0 \text{Diff}(M).$$

Construct f as before: $[\varphi] \in \pi_0 \text{Diff}(D^n, \partial) = \Theta_{n+1}$





By tapering process on the region

$$B(p, 3r) \setminus \bar{B}(p, 2r) \subset M$$

as before

$\Rightarrow \exists$ a path in $R^2(M)$ connecting g

and f_*g . i.e. (1) holds

It remains to show (2), i.e. $[f] \in \pi_0 \text{Diff}_0(M)$.

Suppose $[f] = 0$, i.e. f isotopic to id

$$\Rightarrow (M \times I)_f = (M \times I)_{\text{id}} = M \times S^1$$

$$\parallel \Delta \quad \boxed{M \times [0,1] / (x,0) \sim (f(x),1)} \rightarrow \text{mapping torus of } f$$

we have seen. $M \times S^1 \# \Sigma$ with $\pi_0 \text{Diff}(D^n, \partial) \xrightarrow{\cong} \Theta_{n+1}$
 $[f] \mapsto [\Sigma]$.

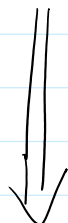
Prop 3.20 $\exists [\Sigma] \in \Theta_{n+1}$ s.t.
 $M \times S^1 \# \Sigma \not\cong M \times S^1$

$\Downarrow$
 $\exists \varphi \in \text{Diff}(D^n, \partial) \rightsquigarrow f \in \text{Diff}_0(N)$
 s.t. $[f] \neq 0$, i.e. (2) hold.

□.

pf Prop 3.20: By Prop. 3.19, $p(M) = 1$

$$\begin{aligned} T(M \times S^1) &= \text{pr}_1^* TM \oplus \text{pr}_2^* TS^1 \\ &= \text{pr}_1^* TM \oplus \mathbb{R}^1 \end{aligned}$$



$$\Downarrow$$

$$p(M^{4k+2} \times S^1) = 1.$$

Hence Prop 3.20 is a corollary of

Prop. 3.21 Let $k \geq 2$, M^{4k+1} be an oriented closed mfd with $p(M) = 1$. Then $\exists \Sigma \in \Theta_{4k+1}(\partial\pi)$ s.t.
 $M \# \Sigma \neq M$
 where $\Theta_{4k+1}(\partial\pi) = \{ [\Sigma] \in \Theta_{4k+1} \mid \Sigma = \partial W \text{ with } W \text{ a parallelizable mfd} \}$.
 Δ
 Θ_{4k+1}

} cf. [A00].

To show Prop 3.21, recall

$$a_k \equiv \begin{cases} 1 & \text{if } k \text{ even} \\ 2 & \text{if } k \text{ odd} \end{cases}$$

Th 3.22 $\Theta_{4k+1}(\partial\pi) = \mathbb{Z}_{t_k}$ where

$$t_k = a_k 2^{2k-2} (2^{2k-1} - 1) \text{num} \left(\frac{B_k}{4k} \right).$$

Moreover, $\exists$ a complete set of representatives

$$\{ S^{4k+1} = \Sigma_0, \Sigma_1, \dots, \Sigma_{t_k-1} \}.$$

in $\Theta_{4k+1}(\partial\pi)$ s.t. if W is parallelizable mfd with $\partial W = \Sigma_i$, then

$$\frac{\sigma(W)}{8} \bmod t_k \equiv i \bmod t_k.$$

} cf. [KM 62]
[L 83].

Notation:

• For prime $p \mid t_k$, if $[\Sigma] = [\Sigma_i] \in \Theta_{4k+1}(\partial\pi)$

s.t. $p \nmid i$, we say Σ is divisible by p , denoted by $[\Sigma] \in \langle p \rangle$.

• M^n, N^n : oriented mfd.

If M is diffeomorphic to N by an orientation

preserving diffeomorphism, then write
 $M \cong N$

Prop 3.23 For each $k \geq 2$, $\exists$ prime $p = p_{ck}$ ($\neq 2$)
 s.t. the following (1) (2) hold:

- (1) $p | \# \mathbb{Z}_k \rightarrow$ the order of $\mathcal{O}_{4k-1}(\partial \Sigma)$
- (2) If M^{4k-1} is an oriented closed mfd with
 $p(M) = 1$ and if $[\Sigma] \in \mathcal{O}_{4k-1}(\partial \Sigma)$ s.t. $M \# \Sigma \cong M$
 then $\Sigma \in \langle p \rangle$

cf. [A00, Corollary 3.7].

pf Prop 3.21 by assuming Prop 3.23 is true:

For $k \in \mathbb{Z} \geq 2$, let $p = p_{ck}$ be the prime in Prop 3.23
 and let $[\Sigma] \in \mathcal{O}_{4k-1}(\partial \Sigma) = \mathbb{Z}_{\# \mathbb{Z}_k}$ s.t. $[\Sigma] \notin \langle p \rangle$

$$\Rightarrow [-\Sigma] \notin \langle p \rangle$$

$$\xrightarrow{\text{Prop 3.23}} M \# \Sigma \not\cong M, M \# (-\Sigma) \not\cong M.$$

it suffices to show

Claim: Either $M \# \Sigma \not\cong M$
 or $M \# (-\Sigma) \not\cong M$

pf of Claim: Suppose
 $M \# \Sigma \cong M \cong M \# (-\Sigma)$.

Since $M \# \Sigma \not\cong M$, $M \# (-\Sigma) \not\cong M$

$\Rightarrow \exists$ orientation reversing diffeomorphism

$$f: M \# \Sigma \longrightarrow M$$

$$g: M \# (-\Sigma) \longrightarrow M.$$

$\Rightarrow g^+_{of}: M \# \Sigma \xrightarrow{\cong} M \# (-\Sigma)$ is orientation preserving

$$\Rightarrow M \# \Sigma \cong M \# (-\Sigma)$$

$$\Rightarrow M \# 2\Sigma \cong M \# (-\Sigma) \# \Sigma \cong M$$

Since $(-\Sigma) \# \Sigma \cong S^{4k-1} = \Sigma_0$. cf. [KM 62].

Prop 3.23 $\Rightarrow [2\Sigma] \in \langle p \rangle \xrightarrow{p \neq 2} [\Sigma] \in \langle p \rangle$. contradiction with the assumption that $[\Sigma] \notin \langle p \rangle$ $\square$

pf Prop 3.23: Let W be a parallelizable mfd with $\Sigma = \Sigma_i = \partial W$

$$\text{Let } f: M \xrightarrow[\text{orientation preserving}]{\cong} M \# \Sigma$$

$$\text{Note } \partial(M \times [0,1] \natural W) = M \sqcup -(M \# \Sigma).$$

$$Q^k \triangleq (M \times [0,1] \natural W)_f$$

$$\triangleq M \times [0,1] \natural W \quad \begin{array}{l} \alpha \sim f(\alpha), \\ \alpha \in M \end{array}$$

is oriented, closed, smooth mfd.

Fact 1: $\sigma(W) = \sigma(Q)$

Fact 2: Suppose $p(M)=1$. Then all decomposable Pontrjagin numbers of Q are zero, i.e.
 $p_{i_1} \cdots p_{i_e} [Q] = 0$ for all $e > 1$.

